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THE MINIMAL TIME FUNCTION FOR THE NONLINEAR DIFFUSION EQUATION

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Abstract. One proves that the minimal time function associated with the nonlinear equation $y_t - \Delta\beta(y) = u$ is continuous on $L^1(\Omega)$ under suitable conditions on β.

1. Introduction. We shall study here the minimum time problem for the nonlinear system

$$
(1.1)\quad
\begin{aligned}
&\frac{\partial y}{\partial t} - \Delta\beta(y) = u && \text{in } \Omega\times(0,\infty)\\
&y(x,0) = y_0(x) && (\forall)\ x\in\Omega\\
&\beta(y) = 0 \quad \text{in } \partial\Omega\times(0,\infty)
\end{aligned}
$$

where Ω is a bounded and open subset of R^N and β is a continuous, nondecreasing function on R such that $\beta(0) = 0$. The input function u belongs to constraint set

$$
(1.2)\quad \mathcal{U} = \{u\in L^\infty(\Omega\times(0,\infty));\ |u(x,t)| \le \rho \ \text{ a.e. } (x,t)\in\Omega\times(0,\infty)\}.
$$

Equations of this type arise in mathematical description of a flow of gas in porous media, population dynamics and other physical problems. Moreover, the classical two phase Stefan problem can be rewritten in the form (1.1) where β is the enthalpie function

$$
\beta(r) = \alpha_1(r+\gamma) \ \text{ for } r\le -\gamma;\quad \beta(r) = 0 \text{ for } -\gamma \le r \le 0,
$$
$$
\beta(r) = \alpha_2 r \quad \text{for } r > 0.
$$

It is well known that the operator $A_0 : L^1(\Omega) \to L^1(\Omega)$ defined by

$$
(1.3)\quad
\begin{aligned}
&A_0 y = -\Delta\beta(y) \qquad (\forall)\ y\in D(A_0)\\
&D(A_0) = \{y\in L^1(\Omega);\ \beta(y)\in W_0^{1,1}(\Omega),\ \Delta\beta(y)\in L^1(\Omega)\}
\end{aligned}
$$

is m-accretive and densely defined on $L^1(\Omega)$ and so, for every $y_0 \in L^1(\Omega)$, problem (1.1) has a unique mild solution $y \in C([0,T];L^1(\Omega))$ (see [5],[11],[12]). In other words, for each $\varepsilon > 0$, there exists an ε-discretization of (1.1) on $[0,T]$, $0 = t_0 \le t_1 \le \dots \le t_n \le T$, $u_1,u_2,\dots,u_n \in L^1(\Omega)$ and $v_0,v_1,\dots,v_n \in D(A_0)$ such that $t_i - t_{i-1} \le \varepsilon$, $T - t_n \le \varepsilon$, $\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \| u(s)-u_i \| ds \le \varepsilon$ and $\|v(t)-y(t)\| \le \varepsilon$ for $t \in [0,T]$, where $v(0) = v_0$, $v(t) = v_i$ for $t \in [t_{i-1},t_i]$

$$(1.4) \qquad \frac{v_i - v_{i-1}}{t_i - t_{i-1}} + A_0 v_i \ni u_i \quad \text{for } i=1,2,\dots,n.$$

Here $\|\cdot\|$ is the usual L^1-norm in $L^1(\Omega)$. It is readily seen that any mild solution y satisfies (1.1) in the sense of distributions i.e.,

$$(1.5) \qquad -\int_0^T \int_\Omega (y\psi_t + \beta(y)\Delta\psi)dx\, dt = \int_0^T \int_\Omega u\psi\, dx\, dt + \int_\Omega y_0 \psi(x,0)dx$$

for all $\psi \in C^{2,1}(\bar{\Omega} \times [0,T])$ such that $\psi(x,T) = 0$ on Ω and $\psi = 0$ on $\partial\Omega \times (0,T)$. Moreover, if $\lim_{|r| \to \infty} \frac{1}{|r|} \int_0^r \beta(s)ds = +\infty$ and $y_0 \in H^{-1}(\Omega)$ then $y \in C([0,T];H^{-1}(\Omega))$ and $\frac{\partial y}{\partial t} \in L^2(\delta,T);H^{-1}(\Omega))$, $\beta(y) \in L^2(\delta,T;H_0^1(\Omega))$ for every $\delta \in (0,T)$ (see [7] and [2] p.206).

Let us denote by $y(t,y_0,u)$ the mild solution to Eq.(1.1) and define

$$T(y_0,u) = \inf\{t;\ y(t,y_0,u) = 0\} \le +\infty$$

where $T(y_0,u) = 0$ if $y(t,y_0,u) \ne 0$ for all $t \ge 0$. The minimal time function associated with system (1.1) is defined by

$$(1.6) \qquad \Phi(y_0) = \inf\{T(y_0,u);\ u \in \mathcal{U}\}.$$

Here we shall study some continuity properties of the minimal time function. There is an extensive literature on the minimal time function associated with finite differential systems (see for instance [4] and the bibliography given there) and its relationship with theory of viscosity solutions for the Bellman equation. For some infinite differential control systems related results have been proved in [2],[8],[9].

2. <u>A controllability result.</u> We will prove here that each $y_0 \in L^\infty(\Omega)$ can be steered in finite time into origin by a certain input function u given in feedback form. To this purpose consider the feedback law $u = -\rho\,\text{sign } y$ and the corresponding closed loop system

$$(2.1)\quad \begin{aligned} &\frac{\partial y}{\partial t} - \Delta\beta(y) + \rho\,\text{sign } y \ni 0 && \text{in } \Omega \times (0,\infty) \\ &y(x,0) = y_0(x) && (\forall)\ x \in \Omega \\ &\beta(y) = 0 && \text{in } \partial\Omega \times (0,\infty) \end{aligned}$$

where $\text{sign } y = y/|y|$ for $y \neq 0$, $\text{sgn } 0 = [-1,1]$. In the space $X = L^1(\Omega)$ consider the operator

$$(2.2)\quad Ay = -\Delta\beta(y) + By \qquad (\forall)\ y \in D(A)$$

where

$$D(A) = \{y \in L^1(\Omega);\ \beta(y) \in W_0^{1,1}(\Omega), \Delta\beta(y) \in L^1(\Omega)\}$$

and

$$By = \{\xi \in L^\infty(\Omega),\ \xi(x) \in \text{sign } y(x),\ \text{a.e. } x \in \Omega\}.$$

By Theorem 1 in [8] we know that A is m-accretive in $L^1(\Omega)$ and so for every $y_0 \in L^1(\Omega)$, Eq.(2.1) has a unique mild solution $y \in C([0,\infty);L^1(\Omega))$.

We will prove now that there is a finite extinction time for the solutions y to Eq.(2.1).

LEMMA 1. <u>For each</u> $y_0 \in L^\infty(\Omega)$ <u>there exists</u> $T \le \rho^{-1}\|y_0\|_{L^\infty(\Omega)}$ <u>such that</u> $y(t) = 0$ <u>for</u> $t \ge T$.

<u>Proof.</u> By the Crandall-Ligget convergence theorem (see e.g. [2], p.104) $y(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t)$ in $L^1(\Omega)$ uniformly on compact intervals of $[0,\infty)$ where y_ε is the solution to difference equations

$$\begin{aligned} &y_\varepsilon(t) + \varepsilon A y_\varepsilon(t) = y_\varepsilon(t-\varepsilon) && \text{for } t \ge \varepsilon \\ &y_\varepsilon(t) = y_0 && \text{for } t \le 0 \end{aligned}$$

or equivalently

$$\text{(2.3)}\quad \begin{aligned} &y_\varepsilon(x,t) - \varepsilon\Delta\,\beta(y_\varepsilon(x,t)) + \rho\varepsilon\,\text{sign}\,y_\varepsilon(x,t) \ni y_\varepsilon(x,t-\varepsilon), && (\forall)\ x\in\Omega,\ t\geq\varepsilon \\ &y_\varepsilon(x,t) = y_0(x) && (\forall)\ x\in\Omega,\ t\leq 0 \\ &\beta(y_\varepsilon) = 0 && \text{in } \partial\Omega\times(0,\infty). \end{aligned}$$

We note that $\beta(y_\varepsilon(t))\in L^\infty(\Omega)\cap H_0^1(\Omega)$ for all t. We set $w(x,t) = \|y_0\|_{L^\infty(\Omega)} - \rho t$ and note that for all $t\in[0, \rho^{-1}\|y_0\|_{L^\infty(\Omega)}]$ we have

$$\text{(2.4)}\quad \begin{aligned} &w(x,t) - \varepsilon\Delta\beta(w(x,t)) + \varepsilon\rho\,\text{sign}\,w(x,t) \ni w(x,t-\varepsilon), \quad (\forall)\ x\in\Omega,\ t\geq\varepsilon \\ &w\geq 0 \quad \text{in } \partial\Omega\times(0,\infty);\ w(x,t) = \|y_0\|_{L^\infty(\Omega)} \quad (\forall)\ x\in\Omega,\ t\leq 0. \end{aligned}$$

To conclude the proof we will show that

$$\text{(2.5)}\quad |y_\varepsilon(x,t)| \leq w(x,t) \qquad (\forall)\ x\in\Omega,\ t\in[0, \rho^{-1}\|y_0\|_{L^\infty(\Omega)}).$$

To this end we subtract Eqs.(2.3),(2.4), multiply the result by $(\beta(y_\varepsilon)-\beta(w))^+$ and integrate on Ω. We get

$$\text{(2.6)}\quad \int_\Omega (y_\varepsilon(t)-w(t))(\beta(y_\varepsilon(t)) - \beta(w(t)))^+dx \leq$$

$$\leq \int_\Omega (y_\varepsilon(t-\varepsilon)-w(t-\varepsilon))(\beta(y_\varepsilon(t)) - \beta(w(t)))^+dx, \quad (\forall)\ t\geq 0$$

because $(\beta(y_\varepsilon) - \beta(w))^+\in H_0^1(\Omega)$ and β is monotonically increasing. Keeping in mind that $y_\varepsilon(t-\varepsilon) - w(t-\varepsilon) \leq 0$, for $0\leq t\leq\varepsilon$ it follows by (2.6) that

$$\text{(2.7)}\quad (y_\varepsilon(t) - w(t))(\beta(y_\varepsilon(t) - \beta(w(t)))^+ = 0 \text{ for } t\in[0,\varepsilon].$$

For $t\in[0,\varepsilon]$ we set $E_t = \{x\in\Omega;\ y_\varepsilon(x,t) > w(t)\}$. Then by (2.7) we see that $\beta(y_\varepsilon(x,t)) = \beta(w(t))$ on E_t and so by (2.3) we infer that

$$y_\varepsilon(x,t) = y_0(x) - \rho\varepsilon \leq w(t) \quad (\forall)\ x\in E_t.$$

Hence $y_\varepsilon \leq w$ in $\Omega\times[0,\varepsilon]$ and reiterating the process we conclude that $y \leq w$ on $\Omega\times[0,T]$ where $T = \rho^{-1}\|y_0\|_{L^\infty(\Omega)}$ as claimed.

In particular, it follows by Lemma 1 that the feedback control $u = -\rho\,\text{sign}\,y$ transfers $y_0\in L^\infty(\Omega)$ into origin in finite time. More precisely we have

THEOREM 1. *For every* $y_0 \in L^\infty(\Omega)$ *there are* $T \in [0, \rho^{-1}\|y_0\|_{L^\infty(\Omega)}]$ *and* $u \in \mathcal{U}$ *such that* $y(T,y_0,u) = 0$. *If in addition* $\beta \in C^1(R \setminus 0)$ *and*

$$\beta'(r) \geq C|r|^{\alpha-1} \qquad (\forall)\ r \in R \tag{2.8}$$

where $\alpha > \frac{N-2}{N}$ *then every* $y_0 \in L^1(\Omega)$ *can be steeread into origin by an input function* $u \in \mathcal{U}$.

Proof. If $y_0 \in L^\infty(\Omega)$ the conclusion of Theorem 1 follows by Lemma 1. If $y_0 \in L^1(\Omega)$ and (2.8) holds then $S(t)y_0 \in L^\infty(\Omega)$ for all $t > 0$ (see e.g. [6], [13]) and

$$\|S(t)y_0\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2+N(\alpha-1)}}\|y_0\|_{L^1(\Omega)}^{\frac{2}{2+N(\alpha-1)}} \qquad (\forall)\ t > 0 \tag{2.9}$$

where C is independent of y_0. Here S(t) is the nonlinear semigroup generated on $L^1(\Omega)$ by A_0, i.e., $S(t)y_0 = y(t,y_0,0)$. Then by the first part of theorem there are $u \in \mathcal{U}$ and $T = \varepsilon + \rho^{-1}\|S(\varepsilon)y_0\|_{L^\infty(\Omega)}$ such that $y(T,y_0,u) = 0$ as claimed.

In terms of minimal time function φ, Theorem 1 amounts to saying that

$$\varphi(y_0) \leq \rho^{-1}\|y_0\|_{L^\infty(\Omega)} \qquad (\forall)\ y_0 \in L^\infty(\Omega) \tag{2.10}$$

and $\varphi(y_0) < \infty$ for all $y_0 \in L^1(\Omega)$ if β satisfies condition (2.8).

3. *Existence of the optimal time controllers.* An input function $u \in \mathcal{U}$ is called optimal time controller for system (1.1) if $y(\varphi(y_0),y_0,u) = 0$.

THEOREM 2. *Under assumption* (2.8) *for every* $y_0 \in L^1(\Omega)$ *there is at least one optimal time controller.*

Proof. Let $y_0 \in L^1(\Omega)$ be arbitrary but fixed and let $\{u_n\} \subset \mathcal{U}$, $\{t_n\} \subset R^+$ be such that $t_n \to T = \varphi(y_0)$ and $y(t_n,y_0,u) = 0$. We set $y_n = y(t,y_0,u_n)$. Multiplying formally the equation

(3.1) $\frac{\partial y_n}{\partial t} - \Delta\beta(y_n) = u_n$ in $\Omega \times (0,\infty)$

$y_n(x,0) = y_o(x)$ in Ω ; $\beta(y_n) = 0$ in $\partial\Omega \times (0,\infty)$

by t y_n and integrating on $\Omega \times (0,t)$ we get the estimate

$$t \int_\Omega y_n^2(x,t)dx + \int_0^t \int_\Omega |y_n(x,s)|^{\alpha-1} |\nabla y_n(x,s)|^2 dx\, ds \leq C \int_0^t \int_\Omega y_n^2(x,s)dx\, ds, \quad (\forall)\ t>0$$

and this yields

(3.2) $$t \int_\Omega y_n^2(x,t)dx + \int_0^t \int_\Omega |y_n(x,s)|^{\alpha+1} dx\, ds \leq C \quad (\forall)\ t>0$$

where C is independent of y_o. (The justification of this formal calculus can be done approximating u_n and y_o by smooth functions.) On the other hand, since according to a result due to Baras [1](see [14], p.95), the set $\{y_n\}$ is relatively compact in $C([0,T];L^1(\Omega))$ we may assume that $y_n(t) \to y(t)$ uniformly on $[0,T]$ in the strong topology of $L^1(\Omega)$ and $u_n \to u$ weak star in $L^\infty(\Omega \times (0,T))$. To conclude the proof, it remains to show that (y,u) satisfies Eq.(1.1), i.e. $y = y(t,y_o,u)$. To this end fix $\delta \in (0,T)$ and note that on the interval $[\delta, T]$ y_n is a strong solution in $H^{-1}(\Omega)$ of the equation

$$y_n'(t) + \partial\phi(y_n(t)) = u_n(t) \quad \text{a.e. } t \in [\delta,T]$$

where $\phi(y) = \int_\Omega (\int_0^{y(x)} \beta(r)ds)dx$ and $\partial\phi(y) = -\Delta\beta(y)$ (see e.g. [2], p.67). Inasmuch as $y_n(t) \to y(t)$ strongly in $L^1(\Omega) \cap H^{-1}(\Omega)$ for $t \geq \delta$ we infer by standard device that $y \in W^{1,2}([\delta,T];H^{-1}(\Omega))$, $\beta(y) \in L^2(\delta,T;H_0^1(\Omega))$ and

$$y'(t) + \partial\phi(y(t)) = u(t) \quad \text{a.e. } t \in [\delta,T] .$$

Hence y is a mild solution to Eq.(1.1) on every interval $[\delta,T]$ and since $y \in C([0,T];L^1(\Omega))$ we conclude that y is a mild solution to Eq.(1.1) on the whole interval $[0,T]$.

4. The continuity of the minimal time function. We shall prove here the following theorem.

. THEOREM 3. If condition (2.8) holds then the minimal time function Hölder continuous on $L^1(\Omega)$.

Proof. Let y_0 and z_0 be arbitrary but fixed in $L^1(\Omega)$. As seen above there is $u_0 \in U$ such that $y(\varphi(z_0),z_0,u_0) = 0$. We set $z(t) = y(t,z_0,u_0)$ and note that by Lemma 1 and the obvious inequality

(4.2) $$\varphi(y) \le \varepsilon + \varphi(S(\varepsilon)y) \qquad \forall y \in L^1(\Omega),\ \varepsilon > 0$$

we have

$$\varphi(y_0) \le \varepsilon + \varphi(S(\varepsilon)y_0) \le \varepsilon + \varphi(z_0) + \varphi(S(\varepsilon)y(\varphi(z_0),y_0,u_0)) \le \varepsilon + \varphi(z_0) + \rho^{-1}\| S(\varepsilon)y(\varphi(z_0),y_0,u_0)\|_{L^\infty(\Omega)}$$

and by estimate (2.9) this yields

(4.3) $$\varphi(y_0) \le \varepsilon + \varphi(z_0) + C\,\varepsilon^{-\frac{N}{2+N(\alpha-1)}} \| y(\tau(z_0),y_0,u_0)\|_{L^1(\Omega)}^{\frac{2}{2+N(\alpha-1)}}$$

where C is independent of ε, y_0 and z_0. On the other hand, we have

$$\| y(\varphi(z_0),y_0,u_0) - y(\varphi(y_0),y_0,u_0)\|_{L^1(\Omega)} \le \|y_0 - z_0\|_{L^1(\Omega)}$$

and substituting in (4.3)

$$\varphi(y_0) - \varphi(z_0) \le \varepsilon + C\,\varepsilon^{-\frac{N}{2+N(\alpha-1)}} \|y_0 - z_0\|_{L^1(\Omega)}^{\frac{2}{2+N(\alpha-1)}}$$

If choose $\varepsilon = \|y_0-z_0\|_{L^1(\Omega)}^{\delta}$ where $\delta = 2/(N\alpha+2)$, we get

(4.4) $$|\varphi(y_0) - \varphi(z_0)| \le C\|y_0 - z_0\|_{L^1(\Omega)}^{\frac{2}{N\alpha+2}}, \quad \forall y_0, z_0 \in L^1(\Omega)$$

as claimed.

REMARK 1. A result of the nature of Theorem 3 can be established for the minimal time function ψ corresponding to control system

(4.5) $$\frac{\partial y}{\partial t} - \Delta y + \beta(y) \ni u \quad \text{in } \Omega \times (0,\infty)$$
$$y(x,0) = y_0(x) \quad \text{in } \Omega$$
$$y = 0 \quad \text{in } \partial\Omega \times (0,\infty)$$

where β is a maximal monotone graph in $R \times R$ such that $D(\beta) = R$, $0 \in \beta(0)$ and the input function u belongs to $\mathcal{U}$. If denote by $S_1(t)$ the contraction semigroup in $L^1(\Omega)$ defined by the solution to homogeneous problem (4.5) then it is well known that

$$\| S_1(t)y_0\|_{L^\infty(\Omega)} \le Ct^{-\frac{N}{2}} \| y_0\|_{L^1(\Omega)} \qquad \forall t>0,\ y_0 \in L^1(\Omega).$$

On the other hand, as seen in [3], $\varphi(y_0) < \infty$ for all $y_0 \in L^1(\Omega)$. Then arguing as in the proof of Theorem 3 we get

$$\psi(y_0) \le \varepsilon + \psi(z_0) + C\,\varepsilon^{-N/2}\|y_0 - z_0\|_{L^1(\Omega)} \qquad \forall \varepsilon > 0$$

and therefore

$$|\psi(y_0) - \psi(z_0)| \le C\|y_0 - z_0\|_{L^1(\Omega)}^{\frac{1}{2+N}} \qquad \forall y_0, z_0 \in L^1(\Omega).$$

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