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UNIQUENESS OF BOUNDED SOLUTIONS FOR SOME ABSTRACT DIFFERENTIAL EQUATIONS

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Introduction In this work we present some results about the "uniqueness of bounded solutions" for differential equations with unbounded operators in Hilbert spaces. As seen for instance in [4] - ch. 5, § 1, there are classes of interesting differential equations of the form: $u'(t) = Au(t)$ in Banach or Hilbert spaces which have $u(t) \equiv \theta$ as the only solution which exists and is norm bounded on the whole real line (norm boundedness can be replaced by "integral boundedness" of the form: $\sup_{t \in \mathbb{R}} \int_t^{t+1} \|u(s)\|^2 ds < +\infty$).

In this paper we shall present some more results of the same form which are found essentially in the papers [1], [2].

1. We start with an extension of a proposition appearing in Th. 2.1 of [5], from the case of bounded and constant operator A to the case of unbounded and time-dependent operator A(t). The precise statement is as follows:

Theorem 1.1.

Let be H - a hilbert space; A(t), a family of linear operators, defined on D(A(t) ⊂ H, $t \in \mathbb{R}$. Let be $D = \bigcap_{t \in \mathbb{R}} D(A(t))$, and assume that operator A(t) is symmetric for all t, on D.

Assume also that $\exists\ m < 0$ such that $(A(t)\,x, x) \le m\,\|x\|^2$, $\forall\, x \in D, \forall\, t \in \mathbb{R}$. Let be $v(t)$, $\mathbb{R} \to D$ a solution of the differential equation $v'(t) = A(t)\,v(t)$ on $\mathbb{R}$ such that

$$\|v(t)\| \le L\ , t \in \mathbb{R}\ .$$

Then $v(t) = \theta \quad \forall\, t \in \mathbb{R}$.

Proof. We have, taking scalar products, the simple relations:

$$(v'(t)\ , v(t)) = (A(t)\,v(t), v(t))$$
$$(v(t)\ , v'(t)) = (v(t)\ ,\ A(t)\,v(t)) = (A(t)\,v(t)\ , v(t))$$

It follows that:

$$\frac{d}{dt}\ \|v(t)\|^2 = 2(A(t)\,v(t)\ , v(t)) \le 2m\,\|v(t)\|^2\ , t \in \mathbb{R}\ , (m < 0)\ .$$

Let us integrate between $-n$ and 0 ($n \in \mathbb{N}$) . We get the estimate

$$\|v(0)\|^2 - \|v(-n)\|^2 \le 2\,m \int_{-n}^{0} \|v(s)\|^2\,ds$$

or

$$\|v(-n)\|^2 - \|v(0)\|^2 \ge 2\,|m| \int_{-n}^{0} \|v(s)\|^2\,ds\ ;\ \text{hence} \int_{-n}^{0} \|v(s)\|^2\,ds \le \frac{1}{2|m|}\,[\|v(-n)\|^2 - \|v(0)\|^2]$$

We assumed that v is bounded on $\mathbb{R}$; therefore we get

$$\int_{-n}^{0} \|v(s)\|^2\,ds \le c\ , \forall\ n = 1, 2\ldots\ , \text{hence} \int_{-\infty}^{0} \|v(s)\|^2\,ds < +\infty\ .$$

On the other hand, we see that $t \to \|v(t)\|^2$ is non-increasing. Hence $\|v(\sigma)\| \ge \|v(0)\|$ for $\sigma < 0$, and we get

$$\int_{-n}^{0} \|v(s)\|^2\,ds > \|v(0)\|^2\,.\,n \to +\infty \text{ as } n \to \infty\ , \text{ if } v(0) \ne \theta\ ,$$ a contradiction.

On the other hand, if $v(0) = \theta \Rightarrow \|v(t)\| \leq \theta$ for all $t \geq 0$. If however $v(\bar{t}) \neq \theta$ for some $\bar{t} < 0$, we integrate again (now betwen $-n$ and $\bar{t}$), and obtain immediately $\int_{-n}^{\bar{t}} \|v(s)\|^2 \, ds \geq \|v(\bar{t})\|^2 (\bar{t} + n)$; this again converges to $+\infty$ as $n \to \infty$, which contradicts finiteness of the integral $\int_{-\infty}^{0} \|v(s)\|^2 \, ds$. □

2. In this section we present "Theorem 1" in the paper [2] (in [4] - Th. 1.3, p. 74 we have given a particular case of it; now we explain the complete result).

Let us first remember some elementary facts about time-dependent strongly differentiable linear unbounded operators.

Thus, let B a Banach space, and $A(t)$, $a < t < b$, a family of linear operators in B with common definition domain D. Assume also that $\frac{d}{dt}(A(t)\,x)$ exists, strongly in B, for all $x \in D$. Define, $\forall\, t \in (a, b)$, an operator

$$\mathring{A}(t) \text{ on } D \text{, by the relation : } \mathring{A}(t)\,x = \frac{d}{dt}(A(t)x) \;,\; \forall\, x \in D \,.$$

We see that it is a linear operator on D.

We shall have to use a certain (known) result, precisely:

Lemma 1. In the Hilbert space H consider a family of linear operators $A(t)$, defined for $t \in (a, b)$ and on $D \subset H$; we assume strong differentiability (as above) and symmetry : $(A(t)\,h, k) = (h, A(t)k)$, $\forall\, h, k \in D$, $t \in (a, b)$. Consider also a function $\varphi(t) \in C^1(a, b; H)$, such that $\varphi(t) \in D$ $\forall\, t \in (a, b)$ and $A(t)\varphi(t) \in C^0(a, b; H)$. Then the derivative

$$\frac{d}{dt}(\varphi(t), A(t)\varphi(t)) \text{ exists and } = 2\,\mathbb{R}e\,(\varphi'(t), A(t)\varphi(t)) + (\mathring{A}(t)\varphi(t), \varphi(t)).$$

<u>The proof</u> is quite straightforward.

We put $\phi(t) = (\varphi(t), A(t)\,\varphi(t))$ and note that (for small h)

$$\phi(t+h) - \phi(t) = (\varphi(t+h), A(t+h)\varphi(t+h)) - (\varphi(t), A(t)\varphi(t)) =$$

$$(\varphi(t+h) - \varphi(t),\ A(t+h)\,\varphi(t+h)) + (\varphi(t), A(t+h)\varphi(t+h) - A(t)\varphi(t)) =$$

$$(\varphi(t+h) - \varphi(t),\ A(t+h)\,\varphi(t+h)) + (\varphi(t), A(t+h)\varphi(t+h) - A(t+h)\,\varphi(t)) +$$

$$(\varphi(t), A(t+h)\,\varphi(t) - A(t)\,\varphi(t)).$$

Then

$$\frac{1}{h}(\phi(t+h) - \phi(t)) = (\frac{1}{h}(\varphi(t+h) - \varphi(t)),\ A(t+h)\,\varphi(t+h)) + (A(t+h)\,\varphi(t),\ \frac{\varphi(t+h) - \varphi(t)}{h})$$
$$+ (\varphi(t), \frac{1}{h}(A(t+h) - A(t)) \,.\, \varphi(t)).$$

Note also that $A(t)\,x$ is strongly continuous, $\forall\, x \in D$. From all the above we shall obtain: $\phi'(t)$ exists and $= (\varphi'(t), A(t)\,\varphi(t)) + (A(t)\,\varphi(t), \varphi'(t)) + (\varphi(t), \dot{A}(t) \,.\, \varphi(t)) =$ $(\varphi'(t), A(t)\,\varphi(t)) + \overline{(\varphi'(t), A(t)\,\varphi(t))} + (\varphi(t), \dot{A}(t)\,\varphi(t))$ □

We are thus ready to present

Theorem 2.1. In the Hilbert space H, consider a family of linear symmetric operators $A(t)$, defined for for $t \in (-\infty, \infty)$ and for $x \in D$; assume also strongly continuous differentiability, and that $0 \notin \sigma_p(A(t))$, $\forall\, t \in \mathbb{R}$; also $(\dot{A}(t)\,x, x) \geq -\gamma(t)\,\|A(t)\,x\|^2$, $x \in D, t \in \mathbb{R}$ where $\gamma(t)$ is a continuous function on $\mathbb{R}$ with $\gamma(t) < 2$, $\forall\, t \in \mathbb{R}$

Let $u(t): \mathbb{R} \to D$ be a $C^1(\mathbb{R}; H)$ solution of $u'(t) = A(t)\,u(t)$.

Then

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|u(\eta)\|^2\, d\eta < +\infty \Rightarrow u \equiv \theta.$$

(Note that density of D is not assumed!).

<u>Proof</u>. The method used here is based on some properties of convex functions.

We consider the function

$$F(t) = \int_t^{t+1} \| u(\eta) \|^2 \, d\eta$$

We show that is posseses a second derivative $F''(t)$ which is non-negative on the real line. It is obvious that the first derivative $F'(t)$ exists and =

$$\|u(t+1)\|^2 - \|u(t)\|^2 = \int_t^{t+1} \frac{d}{d\eta} \|u(\eta)\|^2 \, d\eta = 2 \int_t^{t+1} \mathbb{R}e\,(u'(\eta), u(\eta))d\eta =$$

$$2 \int_t^{t+1} \mathbb{R}e\,(A(\eta)\,u(\eta), u(\eta))\,d\eta = 2 \int_t^{t+1} (u(\eta), A(\eta)\,u(\eta))\,d\eta .$$

Then: $F''(t) = 2(u(t+1), A(t+1)\,u(t+1)) - 2(u(t), A(t)u(t)) =$

$$2 \int_t^{t+1} \frac{d}{d\eta} (u(\eta), A(\eta)u(\eta))d\eta .$$

In the expression under integral sign we apply Lemma 1 (note that $A(\eta)u(\eta) = u'(\eta) \in C(\mathbb{R}; H)$); we obtain

$$\frac{d}{d\eta}(u(\eta), A(\eta)u(\eta)) = 2\,\mathbb{R}e\,(u'(\eta), A(\eta)\,u(\eta)) + (\dot{A}(\eta)\,u(\eta), u(\eta)) =$$

$$2\,\| A(\eta)\,u(\eta) \|^2 + (\dot{A}(\eta)\,u(\eta), u(\eta)) .$$

Therefore it follows:

$$F''(t) = 4 \int_t^{t+1} \| A(\eta)\, u(\eta) \|^2 \, d\eta + 2 \int_t^{t+1} (\dot{A}(\eta)\, u(\eta), u(\eta))\, d\eta \geq$$

$$4 \int_t^{t+1} \| A(\eta)\, u(\eta) \|^2 \, d\eta - 2 \int_t^{t+1} \gamma(\eta) \| A(\eta)\, u(\eta) \|^2 \, d\eta =$$

$$2 \int_t^{t+1} (2 - \gamma(\eta)) \| A(\eta)\, u(\eta) \|^2 \, d\eta \geq 0 \,, \forall\, t \in \mathbb{R} \;.$$

This shows convexity of $F(t)$. We also assumed $F(t)$ to be bounded on $\mathbb{R}$. It follows that $F(t)$ is a constant function on $\mathbb{R}$, hence $F''(t)$ must be 0 for all $t \in \mathbb{R}$.
We finally see that this implies $u(t) = \theta \;\; \forall\, t \in \mathbb{R}$.
Otherwise, $\exists\; t_0 \in \mathbb{R}$, such that $u(t_0) \neq \theta$. As $\lambda = 0$ is not an eigen-value of $A(t_0)$, it follows $A(t_0)\, u(t_0) \neq \theta$. By continuity of $A(t)\, u(t)$ we find $\| A(t)\, u(t) \|^2 > 0$ in some interval $(t_0 - \delta \,, t_0 + \delta)$.

Then we get

$$F''\left(t_0 - \frac{1}{2}\right) = 2 \int_{t_0 - \frac{1}{2}}^{t_0 + \frac{1}{2}} (2 - \gamma(\eta)) \| A(\eta)\, u(\eta) \|^2 \, d\eta \quad \text{is}$$

stricly > 0

which contradicts $F''(t) = 0 \;\; \forall\, t \in \mathbb{R}$ □

Levine in [2] - Remark 1 points out to the following slightly different version of the above result:

Theorem 2.2. In the Hilbert space H consider a family of linear operators $A(t),\ t \in R$ with definition domain $D(t) \subset H$. Assume $(A(t)\, h \,, k) = (h \,, A(t)\, k) \;\; \forall\, h \,, k \in D(t) \,, t \in \mathbb{R}$, and existence of the inverse $A(t)^{-1}$, for all $t \in \mathbb{R}$. Let $u(t) \,, \mathbb{R} \to D(t)$ be a $C^1(\mathbb{R}\, ; H)$ solution of the equation $u'(t) = A(t)\, u(t),\ t \in \mathbb{R}$, and assume furthermore:

the function $f(t) = \frac{d}{dt} (u(t) \,, A(t)\, u(t)) - 2\, \mathbb{R}\mathrm{e}\, (u'(t) \,, A(t)\, u(t))$

exists, ($t \in \mathbb{R}$), and $f(t) > -\gamma(t) \| A(t) u(t) \|^2$ where $\gamma(t) < 2$, $t \in \mathbb{R}$ and $\gamma(t)$ is continuous.

Then, if $\sup_{t \in \mathbb{R}} \int_t^{t+1} \| u(\eta) \|^2 d\eta < +\infty$ it follows $u(t) \equiv \theta$.

The proof is very much the same as in Th. 2.1. With same notation for $F(t)$ we again obtain

$$F''(t) = 2 \int_t^{t+1} \frac{d}{d\eta} (u(\eta), A(\eta) u(\eta)) d\eta$$

But $\frac{d}{d\eta} (u(\eta), A(\eta) u(\eta)) = f(\eta) + 2 \mathbb{R}e (u'(\eta), A(\eta) u(\eta)) =$

$$f(\eta) + 2 \| A(\eta) u(\eta) \|^2 > (2 - \gamma(\eta)) \| A(\eta) u(\eta) \|^2$$

Thus $F''(t) > 2 \int_t^{t+1} (2 - \gamma(\eta)) \| A(\eta) u(\eta) \|^2 d\eta$

The rest of the proof is exactly as in Theorem 2.1.

3. In previous results, the invertibility of the operators $A(t)$ (equivalent to : "$\lambda = 0 \notin \sigma_p (A(t))$") was essentially used in order to end the proof of unicity of (integrally) bounded solutions of $u' = A(t) u$ on the real line.

Dropping this assumption seems to imply "uniqueness-modulo constants" that is a result similar to classical Liouville's theorem in analytic function theory: "bounded entire analytic functions are constant"

In this context we present (compare with Remark 3, p. 252 in [2]) .

Theorem 3.1. Let H be a Hilbert space, and $D \subseteq H$ a linear subspace. Let A be a symmetric operator, $D \to H$ and $u(t)$, $\mathbb{R} \to D$ be a $C^1(\mathbb{R}\,;H)$ solution of the equation $u'(t) = A\,u(t)\,,\, t \in \mathbb{R}$. Then, if $\sup_{t\in\mathbb{R}} \int_t^{t+1} \| u(\eta)\|^2 \, d\eta < +\infty$ it follows that $u(t) = u_0\,,\ \forall\, t \in \mathbb{R}$.

Proof. One can follow the proof in [4] p. 74 .

With

$$F(t) = \int_t^{t+1} \| u(\eta) \|^2 \, d\eta \quad \text{we get} \quad F''(t) = 4 \int_t^{t+1} \| Au(\eta) \|^2 \, d\eta = 4 \int_t^{t+1} \|u'(\eta) \|^2 \, d\eta$$

Thus, $F''(t) \geq 0$ on $\mathbb{R}$, hence $F(t)$ is convex and bounded on $\mathbb{R}$ and is therefore a constant function, and this in turn implies that $F'(t) = 0 \;\forall\, t \in \mathbb{R}$.

On the other hand, if $u(t)$ is not constant, it follows that $u'(t)$ is not identically null on $\mathbb{R}$. There is some $t_0 \in \mathbb{R}$ such that $\| u'(t_0) \| > 0$. As $u \in C^1(\mathbb{R}\,;H)$ we find an interval $(t_0 - \delta\,, t_0 + \delta)$ where $\|u'(t)\| > 0$. Then $F''(t_0 - \frac{1}{2}) = 4 \int_{t_0-\frac{1}{2}}^{t_0+\frac{1}{2}} \|u'(\eta\|^2 \, d\eta > 0$ strictly, a contradiction with $F''(t) \equiv 0$ on $\mathbb{R}$. □

Next, following Remark 4 in [2] , we shall see that for $A(t) = A$ (time-independent operator A) one can get the result in Th. 2.1 without assuming symmetry of the operator A . We state

Theorem 3.2. In the hilbert space H consider a linear operator A, defined on $D \subset H$, such that $A = A_+ + A_-$ where A_+ is symmetric and A_- is skew-symmetric on D, and

<u>furthermore</u> A_+ <u>is invertible</u>. <u>Assume also that</u> $\mathbb{R}e\,(A_- x, A_+ x) = 0, \forall x \in D$. <u>Then, if</u> $u(t) \in C^1(\mathbb{R}; H)$, $u'(t) = Au(t)$ <u>on</u> $\mathbb{R}$, $A_+ u \in C(\mathbb{R}; H)$ <u>and</u> $\sup_{t\in\mathbb{R}} \int_t^{t+1} \| u(\eta) \|^2 d\eta < +\infty$, <u>it follows that</u> $u(t) = \theta\ \forall t \in \mathbb{R}$.

Proof With usual notation: $F(t) = \int_t^{t+1} \| u(\eta) \|^2 d\eta$ we obtain

$$F'(t) = 2\int_t^{t+1} \mathbb{R}e\,(u'(\eta), u(\eta))\, d\eta = 2\int_t^{t+1} \mathbb{R}e\,(A_+ u(\eta) + A_- u(\eta), u(\eta))\, d\eta$$

$$= 2\int_t^{t+1} \mathbb{R}e\,(A_+ u(\eta), u(\eta))\, d\eta = 2\int_t^{t+1} (u(\eta), A+u(\eta))\, d\eta$$

Hence $F''(t) = 2\int_t^{t+1} \frac{d}{d\eta}(u(\eta), A_+ u(\eta))\, d\eta$. Use now Lemma 1 and get $F''(t) = 4 \int_t^{t+1} \mathbb{R}e\,(A+u(\eta), u'(\eta))\, d\eta =$

$$4\int_t^{t+1} \mathbb{R}e\,(A+u(\eta), A+u(\eta) + A-u(\eta))\, d\eta = 4\int_t^{t+1} \| A+u(\eta) \|^2 d\eta$$

Then we may finish the argument as in Theorem 2.1 above.

<u>Remark 1</u>. In the above result we can replace the condition that $\mathbb{R}e\,(A-x, A+x) = 0$ $\forall x \in D$ by the slightly more general one:

$\mathbb{R}e\,(A-x, A+x) \geq -c \| A+x \|^2, \forall x \in D$, where $c < 1$.

We obtain that $F''(t) = 4\int_t^{t+1} \| A+u(\eta) \|^2 d\eta + 4\int_t^{t+1} \mathrm{Re}\,(A+u(\eta), A-u(\eta))\, d\eta$

$$\geq 4(1-c)\int_t^{t+1} \| A+u(\eta) \|^2 d\eta$$

and we may finish in the same way.

Remark 2. We mention here a quite related result:

Theorem 3.3. Let H be a hilbert space and A a linear operator, defined on the linear subspace $D \subset H$, such that $A = A_+ + A_-$, where $A_+ : D \to H$ is symmetric, $A_- : D \to H$ is skew symmetric, and the inequality $\mathbb{R}e\,(A_+ x, A_- x) \geq -c \| A_+ x \|^2$, $x \in D$ holds, with a constant $c \leq 1$. Let $u(t) \in C^1(\mathbb{R}\,; H)$ such that $u'(t) = Au(t)$ on $\mathbb{R}$ and $A_+ u \in C(\mathbb{R}\,; H)$. Then, if $\| u(t) \| \leq M$, $t \in \mathbb{R}$, it follows that $\| u(t) \| = \| u(0) \|\ \forall\, t \in \mathbb{R}$. (compare with [3]).

The proof of this result goes on similar lines to those previously given. We have $\frac{d}{dt} \| u(t) \|^2 = 2\mathbb{R}e\,(u'(t), u(t)) = 2\,\mathbb{R}e\,(A_+ u, u) + 2\,\mathbb{R}e\,(A_- u, u) = 2\,\mathbb{R}e\,(A_+ u, u) = 2\,(A_+ u, u)$.

Using again Lemma 1 we find that derivative $\frac{d}{dt}(A_+ u, u)$ exists and $= 2\,\mathbb{R}e\,(A_+ u, A_+ u + A_- u) = 2\,\mathbb{R}e\,\| A_+ u \|^2 + 2\,\mathbb{R}e\,(A_+ u, A_- u)$

Therefore

$$\frac{d^2}{dt^2} \|u(t)\|^2 = 4 \| A_+ u(t) \|^2 + 4\,\mathbb{R}e\,(A_+ u, A_- u) \geq 4\,(1 - c) \| A_+ u(t) \|^2 \geq 0$$

This shows that the function $t \to \| u(t) \|^2$, $t \in \mathbb{R}$, is convex. The result follows.

4. In this section, we note that second order equations can also be studied from the viewpoint of uniqueness of bounded solutions (see our papers [6], [7]) and [2], Th. 2. The equation studied here will be: $Pu''(t) = A(t)u(t)$, on the whole real line. The first result appears now as following.

Theorem 4.1. Let H be a hilbert space, and P, $D(P) \subseteq H \to H$ be a symmetric operator such that $(Ph, h) \geq 0$ $\forall h \in D(P)$. Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators with domain $D(t) \subseteq H$, such that $A(t) = A_1(t) + A_2(t)$, where $A_1(t)$ is symmetric for all t, $A_2(t)$ is skew symmetric $\forall t$, $(x, A_1(t)x) > 0$ $\forall x \in D(t)$, $x \neq \theta$. Let $u(t) \in C^2(\mathbb{R}; H)$ be a solution of $Pu''(t) = A(t)u(t)$, $t \in \mathbb{R}$ such that $u(t), u'(t), u''(t)$ belong to $D(P)$, $u(t) \in D(t)$, $t \in \mathbb{R}$, and $Pu \in C(\mathbb{R}; H)$.

Then, if $(u(t), Pu(t)) \leq c$ $\forall t \in \mathbb{R}$ it follows that $u(t) = \theta$, $t \in \mathbb{R}$.

Proof. We define $F(t) = (u(t), Pu(t))$; Using Lemma 1 we get $F'(t) = 2\mathbb{R}e\,(u'(t), Pu(t)) = 2\,\mathbb{R}e(Pu, u')$. Using Lemma 2 in [8] we get $F''(t) = 2\mathbb{R}e\,\{(Pu, u'') + (u', Pu')\} = 2\,\mathbb{R}e\,(u, Pu'') + 2(u', Pu') = 2\,\mathbb{R}e\,(u, A_1(t)u + A_2(t)u) + 2(u', Pu') = 2(u, A_1(t)u) + 2(u', Pu') \geq 0$.

Thus $F(t)$ being convex and bounded on $\mathbb{R}$ is constant. Then $F''(t)$ must be zero. Then $0 \leq (u, A_1(t)u) + (u', Pu') = 0$ implies that $(u(t), A_1(t)u(t)) = 0$, $t \in \mathbb{R}$; hence $u(t) = \theta$, $\forall t \in \mathbb{R}$.

A similar result, under slightly different hypotheses is now given as

Theorem 4.2. In the Hilbert space H, let P, $D(P) \subseteq H \to H$ be a symmetric operator such that $(Px, x) > 0$ $\forall x \in D(P)$, $x \neq \theta$. Let $A(t)$, $t \in \mathbb{R}$, be a family of linear operators, $D(t) \subseteq H \to H$, where $A(t) = A_1(t) + A_2(t)$, $A_1(t)$ is symmetric $\forall t \in \mathbb{R}$, $A_2(t)$ is skew symmetric $\forall t \in \mathbb{R}$, $(x, A_1(t)x) \geq 0$ $\forall x \in D(t)$. Let $u(t) \in C^2(\mathbb{R}; H)$ be a solution of $Pu''(t) = A(t)u(t)$, $t \in \mathbb{R}$, such that $u, u', u'' \in D(P)$, $Pu \in C(\mathbb{R}; H)$, $u(t) \in D(t)$, $t \in \mathbb{R}$.

Then, if $(u(t), Pu(t)) \leq C$ $\forall t \in \mathbb{R}$ it follows that $u(t) = u(0)$, $\forall t \in \mathbb{R}$.

The proof is similar to the previous one. With same $F(t)$ we get again $F''(t) = 2\,(u(t), A_1(t)u(t)) + 2\,(u'(t), Pu'(t)) \geq 0$.

Hence F is constant and F" = 0 which implies (u'(t) , Pu'(t)) = 0 $\forall$ t and consequently u'(t) = θ $\forall$ t , □ .

5. In this (last section) we present an idea in [1] which permits further extension of some of previous considerations. It consists essentially in the remark that for some classes of vector-valued functions (not necessarily solutions of differential equations) a certain elementary inequality has as a consequence that S^2 boundedness $\left(\sup_{\mathbb{R}} \int_t^{t+1} \| u(t)\|^2 \, dt < \infty\right)$ implies $u(t) = u_0$, $t \in \mathbb{R}$.

Let therefore state the following

Theorem 5.1. Let $\mathcal{H}$ be a Hilbert space, and $D \subseteq \mathcal{H}$ a linear subspace. Let u(t), $\mathbb{R} \to D$ be a $C^1(\mathbb{R}\,;H)$ function, such that the continuous function $\frac{d}{ds}(u'(s), h)$ exists $\forall$ h $\in D$, $s \in \mathbb{R}$. Assume furthermore that $\mathbb{R}e\left(\frac{d}{ds}(u'(s), u(t))\,|_{s=t}\right) \geq -\alpha(t)\, \| u'(t)\|^2$, $t \in \mathbb{R}$, where $\alpha(t)$ is a continuous function, with $\alpha(t) < 1$, $t \in \mathbb{R}$. Then, if $\int_t^{t+1} \| u(\xi)\|^2 \, d\xi \leq c$, $\forall\, t \in \mathbb{R}$ it follows that $u(t) = u(0)$, $t \in \mathbb{R}$.

(Note that $\frac{d}{ds}(u'(s), u(t))\,|_{s=t}$ means $\frac{d}{ds}(u'(s) , h)$ for $h = u(t)$ and $s = t$.)

Proof. With the usual notation: $F(t) = \int_t^{t+1} \| u(\xi)\|^2 \, d\xi$ we obtain

$$F'(t) = \| u(t+1)\|^2 - \|u(t)\|^2 = \int_t^{t+1} \frac{d}{d\xi}\|u(\xi)\|^2 \, d\xi = 2\int_t^{t+1} \frac{d}{d\xi}\,\mathbb{R}e\,(u'(\xi) , u(\xi))\, d\xi$$

Now we note that $\frac{d}{d\xi} \mathbb{R}e\,(u'(\xi), u(\xi)) = \lim_{\delta\to 0} \mathbb{R}e\,[(u'(\xi+\delta), u(\xi+\delta)) - (u'(\xi), u(\xi)]\,\frac{1}{\delta}$

$$= \lim_{\delta\to 0} \mathbb{R}e\left[\left(u'(\xi+\delta)\,,\ \frac{u(\xi+\delta) - u(\xi)}{\delta}\right) + \left(\frac{u'(\xi+\delta) - u'(\xi)}{\delta}\,,\ u(\xi)\right)\right] =$$

$\mathbb{R}e\,[\| u'(\xi) \|^2 + \frac{d}{dr}(u'(r)\,, u(\xi))\,|_{r=\xi}] = \| u'(\xi)\|^2 + \mathbb{R}e\,\frac{d}{dr}\,(u'(r)\,, u(\xi))|_{r=\xi}$.

Therefore we find

$$F''(t) = 2\int_t^{t+1} \left(\| u'(\xi) \|^2 + \mathbb{R}e\,\frac{d}{dr}(u'(r)\,, u(\xi))\,|_{r=\xi}\right) d\xi \geq$$

$$2\int_t^{t+1} [\| u'(\xi)\|^2 - \alpha(\xi)\,\| u'(\xi) \|^2]\,d\xi = 2\int_t^{t+1} (1 - \alpha(\xi))\,\| u'(\xi) \|^2\,d\xi \geq 0$$

Hence $F(t)$ is bounded convex, a constant function on $\mathbb{R}$.

If $u(t)$ is not constant on $\mathbb{R}$, there is some $t_0 \in \mathbb{R}$ where $\| u'(t_0) \| > 0$ and consequently $F''(t_0) > 0$ too, a contradiction.

We shall next deduce from the above result the following

Theorem 5.2. Let H be a Hilbert space, D a linear subset of H, and $A(t), t \in \mathbb{R}$ be a family of linear symmetric operators defined on D, which is strongly differentiable in D, with strongly continuous derivative $\dot{A}(t)\,, t \in \mathbb{R}$. Assume also the estimate: $(\dot{A}(t)x, x) \geq -\gamma(t)\,\| A(t)\,x \|^2, x \in D\,, t \in \mathbb{R}$, where $\gamma(t) < 2\ (t \in \mathbb{R})$ is a continuous function.

Let $u(t), \mathbb{R} \to D$ be a $C^1(\mathbb{R}\,; H)$ solution of $u'(t) = A(t)u(t)$ and suppose $\int_t^{t+1} \|$

$u(s)\|^2$ ds $\le$ c, $t \in \mathbb{R}$. Then $u(t) = u(0), t \in \mathbb{R}$.

Proof. First we note that $(u'(s), h) = (A(s)u(s), h) = (u(s), A(s)h)$ for $s \in \mathbb{R}$, $h \in D$. The derivative here exists and is equal to $(u'(s), A(s)h) + (u(s), \dot{A}(s)h)$, a continuous function.

Next we compute $\frac{d}{ds}$ $(u'(s) , u(t)) |_{s=t} = \frac{d}{ds}$ $(A(s)u(s) , u(t)) |_{s=t} = \frac{d}{ds}$ $(u(s) , A(s)u(t))_{s=t} = (u'(t), A(t)u(t)) + (u(t), \dot{A}(t)u(t)) = \| u'(t) \|^2 + (u(t), \dot{A}(t)u(t))$.

Note also that the operators $\dot{A}(t)$ are symmetric on $D(\forall\, t \in \mathbb{R})$, so that the expression right hand side is real.

We have then estimate

$$\mathbb{R}e \frac{d}{ds} (u'(s), u(t)) |_{s=t} = \| u'(t)\|^2 + (u(t), \dot{A}(t)u(t)) \ge \| u'(t) \|^2$$

$$- \gamma(t) \| A(t) u(t) \|^2 = (1 - \gamma(t)) \| u'(t) \|^2$$

If $\alpha(t) = \gamma(t) - 1$, we have $\alpha(t) < 2 - 1 = 1$ and previous theorem is applicable.

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