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LIMIT PROPERTIES FOR TRANSOM $\tilde{M}$-MEASURES

Eugene Roventa

ABSTRACT: One of the aims of the present paper is to introduce the notion of convergence for sequences of fuzzy sets without using a topological structure on the reference set X. Also, some results concerning operations with convergent sequences are mentioned. The essential part of the paper is devoted to the study of some important limit properties and counter-examples connected to some functions called transom $\tilde{M}$-measures.

1. SEQUENCES OF FUZZY SETS

Let X be a reference set and $[0,1]$ the closed interval on the real line. Throughout the paper (X,β,μ) will denote a measure space, where β is a σ-algebra of parts of X, and μ a finite, positive measure on β. A fuzzy set $\tilde{F}$ is a map from X to $[0,1]$. The family of fuzzy sets is a lattice with respect to the following operations: $\tilde{A} \subseteq \tilde{B} \Leftrightarrow A(x) \leq \tilde{B}(x)$, $\tilde{A} \cup \tilde{B} \Leftrightarrow \tilde{A}(x) \vee \tilde{B}(x)$, $\tilde{A} \cap \tilde{B} \Leftrightarrow \tilde{A}(x) \wedge \tilde{B}(x)$, $C\tilde{A} \Leftrightarrow C\tilde{A}(x) = 1 - \tilde{A}(x)$. A fuzzy set $\tilde{F}$ is called $\beta-$measurable if for any $\alpha \in [0,1]$, the set $\{x \in X \mid \tilde{F}(x) \geq \alpha\} \in \beta$. The class of all measurable fuzzy subsets of X is denoted by $\tilde{M}(X,\beta)$. The terminology that will be used is the one given in [5]. We call sequence of fuzzy sets each application $s{:}\mathbb{N} \rightarrow [0,1]^X$. In many cases the application s will be identified with the set $\{s(k)\}_{k \in \mathbb{N}}$, or with the indexed notation $\{\tilde{F}_k\}_{k \in \mathbb{N}}$ or $\{\tilde{F}_k\}$.

Definition 1.1. Let $\{\tilde{F}_k\}$ be a sequence of fuzzy sets. Then the fuzzy set $\tilde{F}^*$ (resp. $\tilde{F}_*$) defined by

$$\tilde{F}^* = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} \tilde{F}_k(x) \qquad (\text{resp. } F_* = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} \tilde{F}_k(x))$$

is called the upper (resp. the lower) limit of $\{\tilde{F}_k\}$.

Remark 1.1. Obviously $\tilde{F}_* \subseteq \tilde{F}^*$.

Definition 1.2. The sequence $\{\tilde{F}_k\}$ is said to be convergent if $\tilde{F}_* = \tilde{F}^*$. In such a case, the two above mentioned equal sets will be denoted by $\tilde{F}$ and $\tilde{F}$ will be called the limit of $\{\tilde{F}_k\}$. If $\{\tilde{F}_k\}$ converges to $\tilde{F}$, the notation $\tilde{F}_k \to \tilde{F}$ will be used.

It can be easily shown that

Proposition 1.1.

(i) $\tilde{F}_k \to \tilde{F}$ implies $C\tilde{F} \to C\tilde{F}$.

(ii) If $\tilde{E}_k \to E$, $\tilde{F}_k \to \tilde{F}$ then $\tilde{E}_k \cup \tilde{F}_k \to \tilde{E} \cup \tilde{F}$ and $\tilde{E}_k \cap \tilde{F}_k \to \tilde{E} \cap \tilde{F}$.

2. THE MAIN RESULT

Let $m_T : \tilde{M}(X, \beta) \to \mathbb{R}_+$ the transom measure given in [5] (where $m_T(\tilde{F}) = \mu(T(\tilde{F}))$ and $T(\tilde{F}) = \{x \in X \mid 0 < \tilde{F}(x) < 1\}$).

Lemma 2.1 (i) If $\{\tilde{F}_k\} \subseteq \tilde{M}(X, \beta)$, $\tilde{F}_k \uparrow \tilde{F}$ and $H(\tilde{F}_k) = H(\tilde{F})$, $k \in \mathbb{N}$, then

$$m_T(\tilde{F}) = \lim_{k \to \infty} m_T(\tilde{F}_k).$$

(ii) If $\{\tilde{F}_k\} \subseteq \tilde{M}(X, \beta)$, $\tilde{F}_k \downarrow \tilde{F}$ and $Z(\tilde{F}_k) = Z(\tilde{F})$, $k \in \mathbb{N}$, then

$$m_T(\tilde{F}) = \lim_{k \to \infty} m_T(\tilde{F}_k).$$

Proof. (i) From the hypotheses we deduce that $T(\tilde{F}) = \bigcup_{k=1}^{\infty} T(\tilde{F}_k)$ and $T(\tilde{F}_k) \uparrow T(\tilde{F})$, hence $\mu(T(\tilde{F})) = \lim_{k \to \infty} \mu(T(\tilde{F}_k))$.

(ii) We follow an analogous argument as for part (i).

The following counter-example shows that the equality

$$m_T(\lim_{k \to \infty} \tilde{F}_k) = \lim_{k \to \infty} m_T(\tilde{F}_k).$$

does not hold for any monotonic sequence, without making some supplementary hypotheses.

Let

$$F_k(x) = \begin{cases} 0 & \text{if } x < k \\ x - k & \text{if } k \leq x < k+1 \\ 1 & \text{if } x \geq k+1 \end{cases}$$

It is easily seen that $m_T(\tilde{F}_k) = 1$, $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} m_T(\tilde{F}_k) = 1$ but $m_T(\lim\limits_{k\to\infty} \tilde{F}_k) = m_T(\tilde{0}) = 0$.

Also, the equality

$$m_S(\lim_{k\to\infty} \tilde{F}_k) = \lim_{k\to\infty} m_S(\tilde{F}_k)$$

where $m_S(\tilde{F}_k) = \mu(S(\tilde{F}_k))$ [5] does not hold.

Indeed, let

$$\tilde{F}_k(x) = \begin{cases} 0 & \text{if } x < k,\ x \geq k+1 \\ 1 & \text{if } k \leq x < k+1 \end{cases}$$

Then we have $m_S(\tilde{F}_k) = 1$, $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} m_S(\tilde{F}_k) = 1$. However $m_S(\lim\limits_{k\to\infty} \tilde{F}_k) = m_S(\tilde{0}) = 0$.

Theorem 2.1 For each sequence $\{\tilde{F}_k\} \subseteq \tilde{M}(X,\beta)$ with $H(\tilde{F}'_n) = H(\tilde{F}_*)$ and $H(\tilde{F}'_n) = H(\tilde{F}_k)$, $k \geq n$, $n \in N$ where $\tilde{F}'_n = \bigcap\limits_{k=n}^{\infty} \tilde{F}_k$, the following inequality holds:

$$m_T(\tilde{F}_*) \leq \liminf m_T(\tilde{F}_k).$$

Proof. From $\tilde{F}_*(x) = \bigvee\limits_{n=1}^{\infty} \bigwedge\limits_{k=n}^{\infty} \tilde{F}_k(x)$ we get $\tilde{F}'_n \uparrow \tilde{F}_*$. Using lemma 2.1 (i) we obtain

$$m_T(\tilde{F}_*) = \lim_{k\to\infty} m_T(\tilde{F}'_n) = \liminf m_T(\tilde{F}'_n).$$

However, $\tilde{F}'_n(x) \leq \tilde{F}_k(x)$, $k \geq n$ and from $H(\tilde{F}'_n) = H(\tilde{F}_k)$, $k \geq n$ we have $m_T(\tilde{F}'_n) \leq m_T(\tilde{F}_k)$ for $k \geq n$. Therefore, $\liminf m_T(\tilde{F}'_n) \leq \liminf m_T(\tilde{F}_k)$.

Remark 2.1. The Theorem 2.1 holds under the more general conditions:

$$H(\tilde{F}_*) \subseteq \bigcap_{n=1}^{\infty} H(\tilde{F}'_n) \text{ and } H(\tilde{F}'_n) = H(\tilde{F}_k),\ k \geq n,\ n \in \mathbb{N}.$$

Indeed, we have

$$m_T(\tilde{F}_*) = (\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \tilde{F}_k) = \mu(T(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \tilde{F}_k)).$$

Taking into account our hypotheses it follows that:

$$m_T(\tilde{F}_*) = \mu(\bigcup_{n=1}^{\infty} T(\bigcap_{k=n}^{\infty} \tilde{F}_k)) = \mu(\lim_{n\to\infty} T(\bigcap_{k=n}^{\infty} \tilde{F}_k)) = \lim_{n\to\infty} \mu(T(\bigcap_{k=n}^{\infty} \tilde{F}_k)).$$

Further, having in view the monotonicity of m_T it follows that

$$m_T(\tilde{F}_*) \leq \lim_{k\to\infty} \mu(T(\tilde{F}_k)) = \liminf m_T(\tilde{F}_k).$$

Theorem 2.2. For each sequence $\{\tilde{F}_k\} \subseteq \tilde{M}(X,\beta)$ with $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$ and $H(\tilde{F}''_n) = H(\tilde{F}_k)$, $k \geq n$, $n \in \mathbb{N}$ where $\tilde{F}''_n = \bigcup\limits_{k=n}^{\infty} \tilde{F}_k$, the following inequality holds

$$\limsup m_T(\tilde{F}_k) \leq m_T(\tilde{F}^*) .$$

Proof. From $F^*(x) = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} \tilde{F}_k(x) = \bigwedge_{n=1}^{\infty} \tilde{F}''_n(x)$ we have $\tilde{F}''_n \downarrow \tilde{F}^*$. Further, from lemma 2.1 (*ii*) we have

$$m_T(\tilde{F}^*) = \lim_{n\to\infty} m_T(\tilde{F}''_n) = \lim\sup m_T(\tilde{F}''_n).$$

On the other hand $\tilde{F}_k(x) \leq \tilde{F}''_n(x)$, $k \geq n$ and $H(\tilde{F}_k) = H(\tilde{F}''_n)$, $k \geq n$, hence

$$m_T(\tilde{F}_k) \leq m_T(\tilde{F}''_n) \text{ for } k \geq n.$$

Therefore

$$\lim\sup m_T(F_k) \leq \lim\sup m_T(F''_n) .$$

Remark 2.2. Theorem 2.2 holds under more general conditions:

$$Z(\tilde{F}^*) \subseteq \bigcup_{n=1}^{\infty} Z(\tilde{F}''_n), \ H(\tilde{F}''_n) = H(\tilde{F}_k), \ k \geq n, \ n \in \mathbb{N}.$$

Indeed, we have

$$m_T(F^*) = m_T(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} F_k) = \mu(T(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tilde{F}_k))$$

Because $H(\tilde{F}''_n) = H(\tilde{F}_k)$, $k \geq n$ implies $H(\bigcup_{k=i}^{\infty} \tilde{F}_k) = H(\bigcup_{k=j}^{\infty} \tilde{F}_k)$, $i \neq j$ our hypotheses allow us to infer that

$$m_T(\tilde{F}^*) = \mu(\bigcap_{n=1}^{\infty} T(\bigcup_{k=n}^{\infty} F_k)) = \mu(\lim_{n\to\infty} T(\bigcup_{k=n}^{\infty} \tilde{F}_k)) = \lim_{n\to\infty} \mu(T(\bigcup_{k=n}^{\infty} F_k)).$$

Since $H(\tilde{F}''_n) = H(\tilde{F}_k)$, $k \geq n$ it follows that

$$m_T(\tilde{F}^*) \geq \lim \mu(T(\tilde{F}_k)) = \lim\sup m_T(\tilde{F}_k).$$

Remark 2.3. From $H(\tilde{F}_k) = H(\tilde{F}_*)$, $k \in \mathbb{N}$ (resp. $Z(\tilde{F}_k) = Z(\tilde{F}^*)$, $k \in \mathbb{N}$) we obtain $H(\tilde{F}'_n) = H(\tilde{F}_*)$, $n \in \mathbb{N}$ (resp. $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$, $n \in \mathbb{N}$) because

$$H(\tilde{F}'_n) = \{x \in X \mid \inf_{k \geq n} F_k(x) = 1\} = \{x \in X \mid \tilde{F}_k(x) = 1, \ k \geq n\} = \bigcap_{k=n}^{\infty} H(\tilde{F}_k) = H(\tilde{F}_*)$$

and $Z(\tilde{F}''_n) = \bigcap_{k=n}^{\infty} Z(\tilde{F}_k) = Z(\tilde{F}^*)$.

The conditions $H(\tilde{F}'_n) = H(\tilde{F}_*)$ and $H(\tilde{F}'_n)$, $n \in \mathbb{N}$, $k \geq n$ from Theorem 2.1 are equivalent to $H(\tilde{F}_k) = H(\tilde{F}_*)$, $k \in \mathbb{N}$ and therefore we can replace them with the latter.

The following counter-example shows that an analogous equivalence does not hold between the conditions $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$, $H(\tilde{F}''_n) = H(\tilde{F}_k)$, $n \in \mathbb{N}$, $k \geq n$ from Theorem 2.2 and the condition $Z(\tilde{F}_k) = Z(\tilde{F}^*)$. Let $\{r, r, \cdots r_k, \cdots\}$ be the rationals from $(0,1)$. For each r_k we define:

$$F_k(x) = \begin{cases} 0 & \text{if} \quad x \in C(0,\frac{3}{2}) \\ \frac{1}{2} & \text{if} \quad x \in (0,\frac{r_k}{2}) \cup (\frac{3r_k}{2},\frac{3}{2}) \\ 1 & \text{if} \quad x \in (\frac{r_k}{2},\frac{3r_k}{2}) \end{cases}$$

One sees that $m_T(\tilde{F}_k) = \frac{3}{2} - r_k$, therefore we get lim sup $m_T(\tilde{F}_k) = \frac{3}{2}$. If $\tilde{F}''_n = \bigcup_{k=n}^{\infty} \tilde{F}_k$, then

$$\tilde{F}''_n(x) = \begin{cases} 0 & \text{if} \quad x \in C(0,\frac{3}{2}) \\ 1 & \text{if} \quad x \in (0,\frac{3}{2}) \end{cases} \quad \text{and} \quad \tilde{F}^*(x) = \begin{cases} 0 & \text{if} \quad x \in C(0,\frac{3}{2}) \\ 1 & \text{if} \quad x \in (0,\frac{3}{2}) \end{cases}$$

hence $T(\tilde{F}^*) = \phi$. It follows that

$$m_T(\tilde{F}^*) = 0 < \text{lim sup } m_T(\tilde{F}_k) = \frac{3}{2} .$$

Corollary 2.1. For each sequence $\{\tilde{F}_k\} \subseteq \tilde{M}(X,\beta)$ with $H(\tilde{F}_k) = H(\tilde{F}_*)$, $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$, $H(\tilde{F}''_n) = H(\tilde{F}_k)$, $k \geq n$, $n \in \mathbb{N}$ where $\tilde{F}''_n = \bigcup_{n=k}^{\infty} \tilde{F}_k$ the following inequalities hold:

$$m_T(\tilde{F}_*) \leq \text{lim inf } m_T(\tilde{F}_k) \leq \text{lim sup } m_T(\tilde{F}_k) \leq m_T(\tilde{F}^*) .$$

Theorem 2.3. For each convergent sequence $\{\tilde{F}_k\} \subseteq \tilde{M}(X,\beta)$, $\tilde{F}_k \to \tilde{F}$, with $H(\tilde{F}_k) = H(\tilde{F}_*)$, $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$, $H(\tilde{F}''_n) = H(\tilde{F}_k)$, $k \geq n$, $n \in \mathbb{N}$, where $\tilde{F}''_n = \bigcup_{k=n}^{\infty} \tilde{F}_k$, the equality

$$m_T(\lim_{k\to\infty} \tilde{F}_k) = \lim_{k\to\infty} m_T(\tilde{F}_k)$$

holds.

Proof. Theorem 2.3 results directly from Corollary 2.1.

CONCLUSIONS

As was stated in the introduction the main purpose of the present work is to discuss the limit properties of transom $\tilde{M}$-measures. It is of interest to observe that, there are limit properties that hold for measures on crisp sets, which are not true in this new context. This fact appears natural if we take into account that for measures on crisp sets these properties concern the characteristic functions of crisp sets while $\tilde{M}$-measures are related to the membership functions of fuzzy sets and the structure in β is quite different from that in $\tilde{M}(X,\beta)$. On the other hand these differences are due to the special properties of transom and support functions. To establish the supplementary conditions which will assure that these limit properties are satisfied, the usual four sets attached to a fuzzy set, the transom (T), the support (S), the zero part (Z) and the height (H) are used. Throughout the paper the differences with respect to measure theory on crisp sets are highlighted.

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