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DIRECT PRODUCTS OF MEASURES

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The purpose of this paper is to give detailed analysis of product lattices and their associated Wallman spaces, and to investigate how certain lattice properties carry over to the product lattices. In addition, we proceed from a measure theoretic point of view. We note that some of the material presented here has been developed from a filter approach by F.Kost [5], but the measure approach lends itself to generalizations to non-two valued measures, and, in addition, leads to an easier treatment of the carry over of topological style lattice properties to the product lattices, mainly by making systematic use of the support of a measure. This also enables us to get in a more direct and simpler manner results of M.Kerner [4], and to get extensions of his results (see Theorem 12). Finally, by selecting specific lattices in topological spaces, one gets a variety of applications in the two valued case to topological completeness matters, while in the non-two valued case to measure compactness questions.

We begin with some notations and background material which will be used throughout.

1. Background and notation

We follow the notation and terminology in [1]. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. It is assumed that $\emptyset, X \in \mathcal{L}$. We denote by $\mathcal{A}(\mathcal{L})$, the algebra generated by $\mathcal{L}$; $\sigma(\mathcal{L})$, the σ-algebra generated by $\mathcal{L}$; $\delta(\mathcal{L})$, the lattice of all countable intersections of sets from $\mathcal{L}$; $\tau(\mathcal{L})$, the lattice of arbitrary intersections of sets of $\mathcal{L}$; $\rho(\mathcal{L})$, the smallest class closed under countable intersections and unions which contains $\mathcal{L}$; $s(\mathcal{L})$, the lattice derived Souslin sets; $M(\mathcal{L})$, the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$. A measure $\mu \in M(\mathcal{L})$ is called: σ-smooth on $\mathcal{L}$ if $L_n \in \mathcal{L}$, $n=1,2,\dots$, and $L_n \downarrow \emptyset$ implies $\mu(L_n) \to 0$; σ-smooth on $\mathcal{A}(\mathcal{L})$ if $A_n \in \mathcal{A}(\mathcal{L})$, $n=1,2,\dots$, and $A_n \downarrow \emptyset$ implies $\mu(A_n) \to 0$; τ-smooth on $\mathcal{L}$ if for every net $\{L_\alpha\}$, $L_\alpha \in \mathcal{L}$, such that $L_\alpha \downarrow \emptyset$, we have $\mu(L_\alpha) \to 0$; $\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup\{\mu(L) / L \subset A, L \in \mathcal{L}\}$. We tacitly assume that all measures are non-negative. In addition we denote by $M_R(\mathcal{L})$, the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$; $M_\sigma(\mathcal{L})$, the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$; $M_\tau(\mathcal{L})$, the set of τ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$; $M_R^\sigma(\mathcal{L})$, the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$; $M_R^\tau(\mathcal{L})$, the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$ which are also τ-smooth on $\mathcal{L}$. $I(\mathcal{L})$, $I_R(\mathcal{L})$, $I_R^\sigma(\mathcal{L})$, and $I_R^\tau(\mathcal{L})$ are the subsets of the corresponding M's consisting of the non-trivial zero-one valued measures. For $\mu \in M(\mathcal{L})$, the support of μ, $S(\mu) = \bigcap\{L \in \mathcal{L} / \mu(L) = \mu(X)\}$. $\mathcal{L}$ is replete if for any $\mu \in I_R^\sigma(\mathcal{L})$, $\mu \neq 0$, $S(\mu) \neq \emptyset$.

We next recall some lattice terminology and some well-known lattice characterizations (see [1],[2],and[3]). $\mathcal{L}$ is called: delta lattice(δ-lattice) if $\mathcal{L}$ is closed under countable intersections; disjunctive if for $x \in X$ and $L_1 \in \mathcal{L}$ such that $x \notin L_1$, there exists $L_2 \in \mathcal{L}$ with $x \in L_2$ and $L_1 \cap L_2 = \emptyset$; separating if $x, y \in X$ and $x \neq y$ implies there exists $L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$. $\mathcal{L}$ is a compact lattice iff $S(\mu) \neq \emptyset$ for every $\mu \in I_R(\mathcal{L})$. $\mathcal{L}$ is a normal lattice iff for each $\mu \in I(\mathcal{L})$, there is a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$.

If $x \in X$, then μ_x is the measure concentrated at x:

$$\mu_x(A) = \begin{cases} 1 , & x \in A \\ 0 , & x \notin A \end{cases} \qquad \text{where } A \in \mathcal{A}(\mathcal{L})$$

and $\mu_x \in I_R(\mathcal{L})$ iff $\mathcal{L}$ is disjunctive.

$I_R(\mathcal{L})$ is the general Wallman space associated with X, a subset of $M_R(\mathcal{L})$. The Wallman topology on $I_R(\mathcal{L})$ is obtained by taking all $W(L) = \{\mu \in I_R(\mathcal{L}) / \mu(L) = 1\}$, where $L \in \mathcal{L}$ as a base for the closed sets.

2.The Finite Case

Let X,Y be abstract sets and let $\mathcal{L}_1$ be a lattice of subsets of X and $\mathcal{L}_2$ be a lattice of subsets of Y. We denote:

(1) $\mathcal{M} = \mathcal{L}_1 \times \mathcal{L}_2 = \{L_1 \times L_2 / L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$

(2) $\mathcal{L} = \mathcal{L}(\mathcal{M})$, the lattice generated by $\mathcal{M}$.

We have:

(3) $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathcal{L})$

(4) $S_{\mathcal{L}}(\mu) = S_{\mathcal{M}}(\mu)$

(5) $I_\sigma(\mathcal{M}) = I_\sigma(\mathcal{L})$

(6) $I_R(\mathcal{M})=I_R(\mathcal{L})$

The proofs of all these are quite direct and will be omitted.

<u>Theorem 1</u> Let X,Y be abstract sets and let $\mathcal{L}_1$, $\mathcal{L}_2$ be lattices of subsets of X and Y respectively. Then

$I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)=I_R(\mathcal{L})$

<u>Proof.</u> It is easy to see that $\mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2)=\mathcal{A}(\mathcal{L}_1 \times \mathcal{L}_2)$.

Note that since $\mathcal{A}(\mathcal{L}_1)$ and $\mathcal{A}(\mathcal{L}_2)$ are algebras, so is $\mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2)$ and each element of $\mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2)$ is a finite disjoint union of sets of the form:

$A_1 \times A_2$ with $A_1 \in \mathcal{A}(\mathcal{L}_1)$ and $A_2 \in \mathcal{A}(\mathcal{L}_2)$.

Therefore, for $B \in \mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2)=\mathcal{A}(\mathcal{L}_1 \times \mathcal{L}_2)$ we have:

$B=\bigcup_{i=1}^{n} A_1^i \times A_2^i$, disjoint union and $A_1^i \in \mathcal{A}(\mathcal{L}_1)$, $A_2^i \in \mathcal{A}(\mathcal{L}_2)$.

Now, let $\mu \in I_R(\mathcal{L}_1)$ and $\nu \in I_R(\mathcal{L}_2)$. Since μ defined on $\mathcal{A}(\mathcal{L}_1)$ and ν on $\mathcal{A}(\mathcal{L}_2)$, $\mu \times \nu$ defined on $\mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2)=\mathcal{A}(\mathcal{L}_1 \times \mathcal{L}_2)$.

Suppose that $\mu \times \nu (B)=1$; then $\mu \times \nu (A_1^i \times A_2^i)=1$ for some i. But, by the definition of a direct product of measures we get $\mu(A_1^i)\nu(A_2^i)=1$, and since μ and ν are zero-one valued measures,

$\mu(A_1^i)=1$ and $\nu(A_2^i)=1$.

From $\mu \in I_R(\mathcal{L}_1)$ and $\nu \in I_R(\mathcal{L}_2)$ it follows that there exist $L_1 \subset A_1^i$ with $\mu(L_1)=1$ and $L_2 \subset A_2^i$ with $\nu(L_2)=1$, respectively and $L_1 \in \mathcal{L}_1$, $L_2 \in \mathcal{L}_2$.

Therefore $\mu \times \nu (L_1 \times L_2)=\mu(L_1)\nu(L_2)=1$ and $L_1 \times L_2 \in \mathcal{M}$.

If we let

$M=L_1 \times L_2 \subset A_1^i \times A_2^i \subset B$, then

$\mu \times \nu (B)= \sup\{\mu \times \nu (M)/M \subset B,\ M \in \mathcal{M}\}$, which shows that $\mu \times \nu \in I_R(\mathcal{M})$.

Conversely, let $\mu \in I_R(\mathcal{M})=I_R(\mathcal{L})$ and define μ_1 on $\mathcal{A}(\mathcal{L}_1)$ by

$\mu_1(A)=\mu(A \times Y)$, $A \in \mathcal{A}(\mathcal{L}_1)$.

Since μ is a zero-one measure on $\mathcal{A}(\mathcal{L}_1 \times \mathcal{L}_2)$, from the above definition it follows that μ_1 is a zero-one measure on $\mathcal{A}(\mathcal{L}_1)$, i.e. $\mu_1 \in I_R(\mathcal{L}_1)$. Suppose that $\mu_1(A)=1=\mu(A \times Y)$; since μ is $\mathcal{L}$-regular there exists $A \times Y \supset L_1 \times L_2 \in \mathcal{M}$ such that $\mu(L_1 \times L_2)=1$ and also $\mu(L_1 \times Y)=1$. Then

$\mu_1(L_1)=\mu(L_1 \times Y)=1$ and $L_1 \subset A$, which proves that $\mu_1 \in I_R(\mathcal{L}_1)$.

Similarily, let define μ_2 on $\mathcal{A}(\mathcal{L}_2)$ by $\mu_2(B)=\mu(X \times B)$, $B \in \mathcal{A}(\mathcal{L}_2)$. Then, as before $\mu_2 \in I_R(\mathcal{L}_2)$.

Now, for any $A \in \mathcal{A}(\mathcal{L}_1)$ and any $B \in \mathcal{A}(\mathcal{L}_2)$ we have:

$$\mu_1 \times \mu_2(A \times B)=\mu_1(A)\mu_2(B)=\mu(A \times Y)\mu(X \times B)=$$
$$=\mu[(A \times Y)\cap(X \times B)]=\mu[(A\cap X) \times (Y\cap B)]=\mu(A \times B)$$

which shows that $\mu=\mu_1 \times \mu_2$, and therefore $I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)=I_R(\mathcal{L})$.

<u>Theorem 2</u> Let X,Y be abstract sets and let $\mathcal{L}_1$, $\mathcal{L}_2$ be lattices of subsets of X and Y respectively.Then

$$I_R^{\sigma}(\mathcal{L}_1) \times I_R^{\sigma}(\mathcal{L}_2)=I_R^{\sigma}(\mathcal{L}_1 \times \mathcal{L}_2).$$

<u>Proof.</u> Let $\mu \in I_R^{\sigma}(\mathcal{L}_1)$ and $\nu \in I_R^{\sigma}(\mathcal{L}_2)$.
μ σ-smooth on $\mathcal{A}(\mathcal{L}_1)$ implies that for $A_{1n} \in \mathcal{A}(\mathcal{L}_1)$ with $A_{1n} \downarrow \emptyset$, we have $\mu(A_{1n}) \to 0$; ν σ-smooth on $\mathcal{A}(\mathcal{L}_2)$ implies that for $A_{2n} \in \mathcal{A}(\mathcal{L}_2)$ with $A_{2n} \downarrow \emptyset$, we have $\nu(A_{2n}) \to 0$, $n=1,2,\dots$.

Let consider now the sequence $\{B_n\}$ of sets from $\mathcal{A}(\mathcal{L}_1) \times \mathcal{A}(\mathcal{L}_2)$
Then, as in Theorem 1

$$B_n=\bigcup_{i=1}^{k} A_{1n}^i \times A_{2n}^i$$, disjoint union with $A_{1n}^i \in \mathcal{A}(\mathcal{L}_1)$; $A_{2n}^i \in \mathcal{A}(\mathcal{L}_2)$.

Suppose that $B_n \downarrow \emptyset$, i.e. $A_{1n}^i \times A_{2n}^i \downarrow \emptyset$ for all i.Therefore :

$A_{1n}^i \downarrow \emptyset$ or $A_{2n}^i \downarrow \emptyset$ or both.

$$\mu \times \nu(B_n)=\mu\times\nu[\bigcup_{i=1}^{k}(A_{1n}^i \times A_{2n}^i)]=\sum_{i=1}^{k}\mu\times\nu(A_{1n}^i \times A_{2n}^i)=$$
$$=\sum_{i=1}^{k}\mu(A_{1n}^i)\nu(A_{2n}^i) \longrightarrow 0$$, therefore $\mu \times \nu \in I_R^{\sigma}(\mathcal{L}_1 \times \mathcal{L}_2)$.

Conversely, let $\mu \in I_R^{\sigma}(\mathcal{L}_1 \times \mathcal{L}_2)$ and define μ_1 on $\mathcal{A}(\mathcal{L}_1)$ by:
$\mu_1(A)=\mu(A \times Y)$, $A \in \mathcal{A}(\mathcal{L}_1)$.

If $\{A_n\}$ is a sequence of sets with $A_n \in \mathcal{A}(\mathcal{L}_1)$ and $A_n \downarrow \emptyset$, then $A_n \times Y \downarrow \emptyset$ and since $\mu \in I^{\sigma}(\mathcal{L}_1 \times \mathcal{L}_2)$ it follows that $\mu(A_n \times Y) \longrightarrow 0$.
Therefore $\mu_1(A_n) \longrightarrow 0$ and then $\mu_1 \in I^{\sigma}(\mathcal{L}_1)$.
Defining μ_2 on $\mathcal{A}(\mathcal{L}_2)$ by $\mu_2(B)=\mu(X \times B)$, $B \in \mathcal{A}(\mathcal{L}_2)$ and taking

$\{B_n\}$ a sequence of sets $B_n \in \mathcal{A}(\mathcal{L}_2)$ with $B_n \downarrow \emptyset$ and since $\mu \in I^{\sigma}(\mathcal{L}_1 \times \mathcal{L}_2)$ it follows that $\mu(X \times B_n) \longrightarrow 0$. Then $\mu_2(B_n) \longrightarrow 0$ which shows that $\mu_2 \in I^{\sigma}(\mathcal{L}_2)$. Since $\mu = \mu_1 \times \mu_2$ we proved that

$I^{\sigma}(\mathcal{L}_1 \times \mathcal{L}_2) = I^{\sigma}(\mathcal{L}_1) \times I^{\sigma}(\mathcal{L}_2)$ and by the previous theorem, the statement is proved.

Lemma 1 Let $\mathcal{L}_1$ be a lattice of subsets of X and let $\mathcal{L}_2$ be a lattice of subsets of Y. Then:

a) If $\mu = \mu_1 \times \mu_2 \in I(\mathcal{L}_1 \times \mathcal{L}_2) = I(\mathcal{L}_1) \times I(\mathcal{L}_2)$ then $S(\mu) = S(\mu_1) \times S(\mu_2)$;

b) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are compact lattices, then $\mathcal{L}$ is compact.

Proof. a) $S(\mu_i) = \bigcap \{L_i \in \mathcal{L}_i / \mu_i(L_i) = \mu_i(X)\}$ $\quad i=1,2.$

$S(\mu) = S(\mu_1 \times \mu_2) = \bigcap \{L_1 \times L_2 \in \mathcal{L}_1 \times \mathcal{L}_2 / \mu(L_1 \times L_2) = \mu(X \times Y)\}$

$S(\mu_1) \times S(\mu_2) = \bigcap \{L_1 \times L_2 / L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2, \mu_1(L_1) = \mu_1(X), \mu_2(L_2) = \mu_2(Y)\}$

But $\mu(L_1 \times L_2) = \mu_1 \times \mu_2(L_1 \times L_2) = \mu_1(L_1)\mu_2(L_2)$ and $\mu(X \times Y) = \mu_1 \times \mu_2(X \times Y) = \mu_1(X)\mu_2(Y)$; then $S(\mu_1) \times S(\mu_2) = S(\mu_1 \times \mu_2)$.

b) $S(\mu) = S(\mu_1) \times S(\mu_2) \neq \emptyset$, since $S(\mu_i) \neq \emptyset$, $\mathcal{L}_i$ being compact.

Theorem 3 Consider the spaces $I_R(\mathcal{L}_i)$ with the Wallman topologies $tW_i(\mathcal{L}_i)$, $i=1,2$. It is known that the topological spaces $(I_R(\mathcal{L}_i), tW_i(\mathcal{L}_i))$ are compact and T_1. Then the topological space $(I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2), tW_1(\mathcal{L}_1) \times tW_2(\mathcal{L}_2))$ is also compact and T_1.

Proof. Since $I_R(\mathcal{L}_1)$ and $I_R(\mathcal{L}_2)$ compact topological spaces, $S_{\mathcal{L}_1}(\hat{\mu}) \neq \emptyset$ and $S_{\mathcal{L}_2}(\hat{\nu}) \neq \emptyset$ where

$S_{\mathcal{L}_1}(\hat{\mu}) = \bigcap \{W_1(L_1) \in W_1(\mathcal{L}_1) / \hat{\mu}(W_1(L_1)) = 1\}$

$S_{\mathcal{L}_2}(\hat{\nu}) = \bigcap \{W_2(L_2) \in W_2(\mathcal{L}_2) / \hat{\nu}(W_2(L_2)) = 1\}$. We have:

$\hat{\mu}(W_1(A)) = \mu(A)$, $\mu \in I_R(\mathcal{L}_1)$, $A \in \mathcal{A}(\mathcal{L}_1)$, $\hat{\mu} \in I_R(W_1(\mathcal{L}_1))$ and $\hat{\nu}(W_2(B)) = \nu(B)$, $\nu \in I_R(\mathcal{L}_2)$, $B \in \mathcal{A}(\mathcal{L}_2)$, $\hat{\nu} \in I_R(W_2(\mathcal{L}_2))$. Therefore $\widehat{\mu \times \nu}(W_1(A) \times W_2(B)) = \mu \times \nu(A \times B) = \mu(A)\nu(B)$; $\hat{\mu} \times \hat{\nu}(W_1(A) \times W_2(B)) = \hat{\mu}(W_1(A))\hat{\nu}(W_2(B)) = \mu(A)\nu(B)$ i.e. $\widehat{\mu \times \nu} = \hat{\mu} \times \hat{\nu} \in I_R(W_1(\mathcal{L}_1)) \times I_R(W_2(\mathcal{L}_2))$

$S_{\mathcal{L}_1 \times \mathcal{L}_2}(\widehat{\mu \times \nu}) = S_{\mathcal{L}_1 \times \mathcal{L}_2}(\hat{\mu} \times \hat{\nu}) = S_{\mathcal{L}_1}(\hat{\mu}) \times S_{\mathcal{L}_2}(\hat{\nu}) \neq \emptyset$ which proves that the product space $I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)$ is compact.

To show that $I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)$ is a T_1 space, let $\mu,\nu \in I_R(\mathcal{L})$ and suppose $\mu \neq \nu$.
Since $\mu=\mu_1 \times \mu_2$ with $\mu_1,\nu_1 \in I_R(\mathcal{L}_1)$ and $\nu=\nu_1 \times \nu_2$ with $\nu_1,\nu_2 \in I_R(\mathcal{L}_2)$ we get $\mu_1 \neq \nu_1$ and $\mu_2 \neq \nu_2$. There exist $L_1,\tilde{L}_1 \in \mathcal{L}_1$ and $L_2,\tilde{L}_2 \in \mathcal{L}_2$ with

$$\mu_1 \in W_1(L_1) \ , \ \nu_1 \in W_1(L_1)' \ ; \ \nu_1 \in W_1(\tilde{L}_1) \ , \ \mu_1 \in W_1(\tilde{L}_1)'$$

$$\mu_2 \in W_2(L_2) \ , \ \nu_2 \in W_2(L_2)' \ ; \ \nu_2 \in W_2(\tilde{L}_2) \ , \ \mu_2 \in W_2(\tilde{L}_2)'.$$ Therefore:

$\mu_1(L_1)=\mu_2(L_2)=1$, $\nu_1(L_1)=\nu_2(L_2)=0$, $\mu_1(\tilde{L}_1)=\mu_2(\tilde{L}_2)=0$, $\nu_1(\tilde{L}_1)=\nu_2(\tilde{L}_2)=1$ which implies $\mu \in W(L_1 \times L_2), \nu \in W(L_1 \times L_2)'; \nu \in W(\tilde{L}_1 \times \tilde{L}_2)$ $,\mu \in W(\tilde{L}_1 \times \tilde{L}_2)'$.

Theorem 4 Consider the space $I_R(\mathcal{L}_1 \times \mathcal{L}_2)$ with the Wallman topology $t(W_1(\mathcal{L}_1) \times W_2(\mathcal{L}_2))$ for the generated lattice $\mathcal{L}$. Then the mapping $T: I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2) \longrightarrow I_R(\mathcal{L}_1 \times \mathcal{L}_2)$ is a homeomorphism.

Proof. We define T by $T(\mu,\nu)=\mu \times \nu$ $,\mu \in I_R(\mathcal{L}_1), \nu \in I_R(\mathcal{L}_2)$.
Clearly, T is one-to-one and onto (by Theorem 1).

Next, we show that T is closed. For this consider

$$W_1(L_1) \times W_2(L_2)=\{\mu \in I_R(\mathcal{L}_1)/\mu(L_1)=1\} \times \{\nu \in I_R(\mathcal{L}_2)/\nu(L_2)=1\} =$$
$$= \{(\mu,\nu) \in I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)/\mu \times \nu (L_1 \times L_2)=1\}$$

The image of $W_1(L_1) \times W_2(L_2)$ under T is

$$\{\mu \times \nu \in I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)/ \mu \times \nu (L_1 \times L_2)=1\} = W(L_1 \times L_2).$$

T is also continuous, since

$$T^{-1}(W(L_1 \times L_2))=T^{-1}\{\rho \in I_R(\mathcal{L}_1 \times \mathcal{L}_2)/\rho(L_1 \times L_2)=1\}=T^{-1}\{\mu \times \nu \in I_R(\mathcal{L}_1 \times \mathcal{L}_2)/$$
$$\mu \times \nu(L_1 \times L_2)=1\} = \{(\mu,\nu) \in I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)/\mu(L_1)=\nu(L_2)=1\} =$$
$$=\{\mu \in I_R(\mathcal{L}_1)/\mu(L_1)=1\} \times \{\nu \in I_R(\mathcal{L}_2)/\nu(L_2)=1\}= W_1(L_1) \times W_2(L_2),$$

therefore T is a homeomorphism.

Theorem 5 Suppose that $\mathcal{L}_1$ is a normal lattice of subsets of X and $\mathcal{L}_2$ is a normal lattice of subsets of Y. Then $\mathcal{L}$ is a normal lattice of subsets of X x Y.

Proof. Let $\mu \in I(\mathcal{L})$ and $\nu,\rho \in I_R(\mathcal{L})$ such that $\mu \leq \nu, \mu \leq \rho$ on $\mathcal{L}$.
Since $\mu=\mu_1 \times \mu_2 \in I(\mathcal{L}_1) \times I(\mathcal{L}_2)$, $\nu=\nu_1 \times \nu_2 \in I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)$ and $\rho=\rho_1 \times \rho_2 \in I_R(\mathcal{L}_1) \times I_R(\mathcal{L}_2)$ we get

$$\mu_1 \times \mu_2 \leq \nu_1 \times \nu_2 \quad \text{and} \quad \mu_1 \times \mu_2 \leq \rho_1 \times \rho_2 \quad \text{on } \mathcal{L}.$$

Then $\mu_i \leq \nu_i$ and $\mu_i \leq \rho_i$ on $\mathcal{L}_i$, i=1,2. But since $\mathcal{L}_i$ are normal lattices $\nu_1 = \rho_1$ and $\nu_2 = \rho_2$. Therefore $\nu_1 \times \nu_2 = \rho_1 \times \rho_2$ i.e. $\nu = \rho$ which shows that $\mathcal{L}$ is a normal lattice.

Lemma 2 Let $\mathcal{L}_1$ be a lattice of subsets of X and let $\mathcal{L}_2$ be a lattice of subsets of Y.

a) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are disjunctive, then $\mathcal{L}$ is disjunctive;

b) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are separating, then $\mathcal{L}$ is separating;

c) If we define

$pr_1 : X \times Y \longrightarrow X$ and $pr_2 : X \times Y \longrightarrow Y$, then pr_1 is $\mathcal{L}-\mathcal{L}_1$ continuous and pr_2 is $\mathcal{L}-\mathcal{L}_2$ continuous, i.e. $pr_1^{-1}(\mathcal{L}_1) \subset \mathcal{L}$ and $pr_2^{-1}(\mathcal{L}_2) \subset \mathcal{L}$.
(The proof is quite direct and will be omitted).

Theorem 6 Let $\mathcal{L}_1$ be a δ, replete lattice of subsets of X such that $\sigma(\mathcal{L}_1) = \rho(\mathcal{L}_1)$ and let $\mathcal{L}_2$ be a δ, replete lattice of subsets of Y such that $\sigma(\mathcal{L}_2) = \rho(\mathcal{L}_2)$. Let $\tilde{\mathcal{L}}$ be a lattice of subsets of $X \times Y$ such that $\mathcal{L} \subset \tilde{\mathcal{L}} \subset t\mathcal{L}$. Then $\tilde{\mathcal{L}}$ is replete.

Proof. We must show that for any $\mu \in I_R^{\sigma}(\tilde{\mathcal{L}})$, $S(\mu) \neq \emptyset$.
Consider the projection mappings:

$pr_1 : X \times Y \longrightarrow X$ and $pr_2 : X \times Y \longrightarrow Y$ which are $\mathcal{L}-\mathcal{L}_1$ and $\mathcal{L}-\mathcal{L}_2$ continuous, respectively; since $\mathcal{L} \subset \tilde{\mathcal{L}}$ pr_1 is $\tilde{\mathcal{L}}-\mathcal{L}_1$ continuous and pr_2 is $\tilde{\mathcal{L}}-\mathcal{L}_2$ continuous. If we define the mapping: $\widehat{pr}_1 : I_R^{\sigma}(\tilde{\mathcal{L}}) \longrightarrow I_R^{\sigma}(\mathcal{L}_1)$ by $\widehat{pr}_1\mu = \mu pr_1^{-1}$ where $\mu \in I_R^{\sigma}(\tilde{\mathcal{L}})$, then $\mu pr_1^{-1} \in I_R^{\sigma}(\mathcal{L}_1)$ and similarily $\mu pr_2^{-1} \in I_R^{\sigma}(\mathcal{L}_2)$. Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are replete $S(\mu pr_1^{-1}) \neq \emptyset$; $S(\mu pr_2^{-1}) \neq \emptyset$. Now, on $\mathcal{L}_1 \times \mathcal{L}_2$ we have:

$\mu pr_1^{-1} \times \mu pr_2^{-1}(L_1 \times L_2) = \mu pr_1^{-1}(L_1)\mu pr_2^{-1}(L_2) = \mu(L_1 \times Y)\mu(X \times L_2) =$
$= \mu[(L_1 \times Y) \cap (X \times L_2)] = \mu[(L_1 \times L_2) \cap (X \times Y)] = \mu(L_1 \times L_2)$.

Therefore

$\mu pr_1^{-1} \times \mu pr_2^{-1} = \mu$ on $\mathcal{L}$ and then $S_{\mathcal{L}}(\mu) = S(\mu pr_1^{-1}) \times S(\mu pr_2^{-1}) \neq \emptyset$ which implies $S_{\tilde{\mathcal{L}}}(\mu) \neq \emptyset$, since from $\tilde{\mathcal{L}} \subset t\mathcal{L}$ we get $S_{\tilde{\mathcal{L}}}(\mu) = S_{\mathcal{L}}(\mu)$.

Before presenting the next theorem, we introduce some of notations and background used in the proof. Let X be an abstract set and let $\mathcal{L}$ be a lattice of subsets of X.

(7) $\mathcal{L}$ is measure replete if for any $\mu \in M_R^{\sigma}(\mathcal{L})$, $\mu \neq 0$ it follows that $S(\mu) \neq \emptyset$.

(8) If $\mathcal{L}$ is a δ-lattice then $M_R^{\sigma}(\mathcal{L}) = M_R^{\tau}(\mathcal{L})$ if and only if $S(\mu) \neq \emptyset$ for all $\mu \in M_R^{\sigma}(\mathcal{L})$, $\mu > 0$.

(9) If $\mathcal{L}$ is a separating, disjoint lattice then $M_R^{\sigma}(\mathcal{L}) = M_R^{\tau}(\mathcal{L})$ if and only if $S(\mu) \neq \emptyset$ for all $\mu \in M_R^{\sigma}(\mathcal{L})$, $\mu > 0$.

(10) Let $\mu \in M_R^{\sigma}(\mathcal{L})$, $\mu \geqslant 0$. Then μ is called $\mathcal{L}$-tight if for every $\varepsilon > 0$ there exists a $K \in \mathcal{K}(\mathcal{L})$ = $\mathcal{L}$-compact sets, such that $\mu^*(K') \leq \varepsilon$

The collection of $\mathcal{L}$-tight measures is denoted by $M_R^t(\mathcal{L})$ and we have: $M_R^t(\mathcal{L}) \subset M_R^{\tau}(\mathcal{L}) \subset M_R^{\sigma}(\mathcal{L}) \subset M_R(\mathcal{L})$

<u>Theorem 7</u> Let $\mathcal{L}_1$ be a lattice of subsets of X and let $\mathcal{L}_2$ be a lattice of subsets of Y such that each $\mathcal{L}_i (i=1,2)$ is δ-lattice separating, disjunctive, normal (or T_2), tight and $\sigma(\mathcal{L}_i) = \rho(\mathcal{L}_i)$. Let $\tilde{\mathcal{L}}$ be a δ, separating and disjunctive lattice such that $\mathcal{L}(\mathcal{L}_1 \times \mathcal{L}_2) \subset \tilde{\mathcal{L}} \subset t\mathcal{L} = t\tilde{\mathcal{L}}$ and let $\mu \in M_R^{\tau}(\tilde{\mathcal{L}})$. Then $\mu \in M_R^t(\tilde{\mathcal{L}})$.

<u>Proof.</u> By hypothesis $M_R^{\sigma}(\mathcal{L}_1) = M_R^t(\mathcal{L}_1)$; $M_R^{\sigma}(\mathcal{L}_2) = M_R^t(\mathcal{L}_2)$ i.e. $\mathcal{L}_1$ and $\mathcal{L}_2$ are strongly measure replete. We define the mappings: $\widehat{pr}_1: M_R^{\sigma}(\tilde{\mathcal{L}}) \longrightarrow M_R^{\sigma}(\mathcal{L}_1)$ and $\widehat{pr}_2: M_R^{\sigma}(\tilde{\mathcal{L}}) \longrightarrow M_R^{\sigma}(\mathcal{L}_2)$ by:

$\widehat{pr}_1\mu = \mu pr_1^{-1} = \mu_1 \in M_R^{\sigma}(\mathcal{L}_1) = M_R^t(\mathcal{L}_1)$; $\widehat{pr}_2\mu = \mu pr_2^{-1} = \mu_2 \in M_R^{\sigma}(\mathcal{L}_2) = M_R^t(\mathcal{L}_2)$.

Since $\mu_1 \in M_R^t(\mathcal{L}_1)$ and $\mu_2 \in M_R^t(\mathcal{L}_2)$, there exist $K_1 \in \mathcal{K}(\mathcal{L}_1)$, $K_2 \in \mathcal{K}(\mathcal{L}_2)$ such that

$$\mu_1^*(K_1) \geqslant \mu_1(X) - \varepsilon \quad \text{and} \quad \mu_2^*(K_2) \geqslant \mu_2(Y) - \varepsilon$$

By Theorem 2.5[1] μ can be extended uniquely to $\nu \in M_R^{\tau}(t\tilde{\mathcal{L}}) = M_R^{\tau}(t\mathcal{L})$ and similarily μ_1 can be extended to $\nu_1 \in M_R^{\tau}(t\mathcal{L}_1)$ and μ_2 to $\nu_2 \in M_R^{\tau}(t\mathcal{L}_2)$

Now, to show that $\mu \in M_R^t(\tilde{\mathcal{L}})$ we must show that for $K_1 \times K_2 \in \mathcal{K}(\mathcal{L}_1) \times \mathcal{K}(\mathcal{L}_2)$ we have $\mu^*(K_1 \times K_2) \geqslant \mu(X \times Y) - 2\varepsilon$

But: $\mu^*(K_1 \times K_2) + 2\varepsilon = \mu^*[(K_1 \times Y) \cap (X \times K_2)] + 2\varepsilon = \nu[(K_1 \times Y) \cap (X \times K_2)] + 2\varepsilon = \nu(K_1 \times Y) + \varepsilon + \nu(X \times K_2) + \varepsilon - \nu[(K_1 \times Y) \cup (X \times K_2)] \geqslant \nu_1(K_1) + \varepsilon + \nu_2(K_2) + \varepsilon - \nu(X \times Y) = \mu_1^*(K_1) + \varepsilon + \mu_2^*(K_2) + \varepsilon - \nu(X \times Y) \geqslant \mu_1(X) - \varepsilon + \varepsilon + \mu_2(Y) - \varepsilon + \varepsilon - \mu(X \times Y) = \mu(X \times Y) + \mu(X \times Y) - \mu(X \times Y) = \mu(X \times Y)$. Therefore $\mu \in M_R^t(\tilde{\mathcal{L}})$.

Note that $\mu^* = \nu$ on $t\mathcal{L}$ and similarily $\mu_1^* = \nu_1$ on $t\mathcal{L}_1$ and $\mu_2^* = \nu_2$ on $t\mathcal{L}_2$.

Example 1 Let X,Y be topological spaces and let $\mathcal{O}_1$ be the lattice of open sets of X and $\mathcal{O}_2$ be the lattice of open sets of Y. Consider the product space X x Y with a base of open sets given by $\{O_1 \times O_2 / O_1 \in \mathcal{O}_1, O_2 \in \mathcal{O}_2\}$. We have $(O_1 \times O_2)' = \{(x,y) \in X \times Y / (x,y) \notin$ $\notin (O_1 \times O_2)\} = \{(x,y)/(x,y) \in (X \times O_2') \text{ or } (x,y)\ (O_1' \times Y)\} = (X \times O_2') \cup$ $\cup (O_1' \times Y) = (X \times F_2) \cup (F_1 \times Y)$, therefore $\mathcal{F} \subset t(\mathcal{L}(\mathcal{F}_1 \times \mathcal{F}_2)) \subset \mathcal{F}$ where $\mathcal{F}_1$ is the lattice of closed sets of X and $\mathcal{F}_2$ is the lattice of closed sets of Y. Thus we have $\mathcal{F} = t(\mathcal{L}(\mathcal{F}_1 \times \mathcal{F}_2))$.

Example 2 Let X and Y be topological $T_{3\frac{1}{2}}$ spaces and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the corresponding lattices of zero sets. Then for the product space X x Y we consider a base of open sets $\{Z_1' \times Z_2' / Z_1 \in \mathcal{Z}_1,\ Z_2 \in \mathcal{Z}_2\}$ such that any open set from X x Y is of the form:
$O = \bigcup_\alpha Z_{1\alpha}' \times Z_{2\alpha}'$ and any closed set is $F = O' = \bigcap_\alpha (Z_{1\alpha} \times Y) \cup (X \times Z_{2\alpha}) \in$ $\in t(\mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_2))$ and therefore $\mathcal{F} \subset t(\mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_2)) \subset \mathcal{F}$ which implies that $\mathcal{F} = t(\mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_2))$.

Lemma 3 Consider any topological spaces X and Y such that X and Y are $T_{3\frac{1}{2}}$ and let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be replete lattices of subsets of X and Y, respectively. Consider a lattice of subsets of X x Y, call it $\mathcal{Z}$ such that $\mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_2) \subset \mathcal{Z} \subset t(\mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_2)) = \mathcal{F}$ Then by Theorem 6 $\mathcal{Z}$ is replete.

Just for comparison with an example in Kost[5] we add:

Lemma 4 Let X be a discrete and infinite set.Then

$$\beta(X \times X) \neq \beta(X) \times \beta(X)$$

Proof. We use the theory of direct product of two spaces with X=Y and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{Z}_X = \mathcal{P}(X)$.By Theorem 1 we have in this case with $\mathcal{L}_1 \times \mathcal{L}_2 = \mathcal{P}(X) \times \mathcal{P}(X)$; $\mathcal{L} = \mathcal{L}(\mathcal{Z}_X \times \mathcal{Z}_X) \subset \mathcal{P}(X \times X)$ and $\mathcal{Z}_{X \times X} = \mathcal{P}(X \times X)$ that $I_R(\mathcal{Z}_X) \times I_R(\mathcal{Z}_X) = I_R(\mathcal{Z}_X \times \mathcal{Z}_X) = I_R(\mathcal{L})$. $\mathcal{L}$ is an algebra, since as in Example 2 for $Z_1 \in \mathcal{Z}_X, Z_2 \in \mathcal{Z}_X$ we have $(Z_1 \times Z_2)' \in \mathcal{L}$.But then $\mathcal{L}$ semi-separates $\mathcal{P}(X \times X)$, since any algebra semi-separates any lattice containing it.

Now, suppose that $\beta(X \times X) = \beta(X) \times \beta(X)$ where $I_R(\mathcal{Z}_X) = \beta(X)$ and $I_R(\mathcal{Z}_{X \times X}) = \beta(X \times X)$.Then since $I_R(\mathcal{P}(X)) \times I_R(\mathcal{P}(X)) = I_R(\mathcal{L})$ we get $I_R(\mathcal{P}(X \times X)) = I_R(\mathcal{L})$ and therefore the mapping:

$\psi: I_R(\mathcal{P}(X \times X)) \longrightarrow I_R(\mathcal{L})$ is a homeomorphism. Since $\mathcal{L}$ separates $\mathcal{P}(X \times X)$ and since $\mathcal{P}(X \times X)$ is an algebra, it follows that $\mathcal{L} = \mathcal{P}(X \times X)$. But X infinite implies X x X infinite and $\{(x,x)/x \in X\} \neq \{\bigcup_{i=1}^{n} (A_i \times B_i)/A_i, B_i \in \mathcal{P}(X)\}$. Therefore $\beta(X \times X) \neq \beta(X) \times \beta(X)$.

3. The General Case

Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a collection of abstract sets (Λ an arbitrary index set) and $\mathcal{L}_\alpha$ be the lattice of subsets of X_α for all α. We denote

(11) $\mathcal{M} = \prod_{\alpha \in \Lambda} \mathcal{L}_\alpha = \{\prod_\alpha L_\alpha / L_\alpha \in \mathcal{L}_\alpha$ and $L_\alpha = X_\alpha$ for almost all $\alpha\}$.

Theorem 8 $\prod_\alpha I_R(\mathcal{L}_\alpha) = I_R(\mathcal{L}) = I_R(\prod \mathcal{L}_\alpha)$

Proof. First, we note that $\prod_\alpha \mathcal{a}(\mathcal{L}_\alpha) = \mathcal{a}(\prod_\alpha \mathcal{L}_\alpha) = \mathcal{a}(\mathcal{L})$ and that $\prod_\alpha \mathcal{a}(\mathcal{L}_\alpha)$ is the collection of all finite cylinder sets i.e. if $A \in \prod_\alpha \mathcal{a}(\mathcal{L}_\alpha)$ then A is a cylinder set for which there exists a non-empty finite subset $F = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of Λ and a subset $E_F \in \prod_{\alpha \in F} \mathcal{a}(\mathcal{L}_\alpha)$ such that $A = P_F^{-1}(E_F)$, where

$P_F: \prod_\alpha X_\alpha \longrightarrow \prod_\alpha X_\alpha = X_{\alpha_1} \times X_{\alpha_2} \times \ldots \times X_{\alpha_n}$ and $P_\alpha : \prod_\alpha X_\alpha \longrightarrow X_\alpha$

Now, let $\mu_\alpha \in I_R(\mathcal{L}_\alpha)$ for all $\alpha \in \Lambda$ with $\mu_\alpha : \mathcal{a}(\mathcal{L}_\alpha)$, $\alpha \in \Lambda$ and define:

$$\mu = \prod_\alpha \mu_\alpha \in \prod_\alpha I_R(\mathcal{L}_\alpha) \quad , \quad \mu : \prod_\alpha \mathcal{a}(\mathcal{L}_\alpha).$$

Let $A \in \prod_\alpha \mathcal{a}(\mathcal{L}_\alpha)$ with $\mu(A) = 1$. Then $(\prod_\alpha \mu_\alpha)(A) = (\prod_\alpha \mu_\alpha)(P_F^{-1}(E_F)) = 1$, i.e.

$$\prod_{\alpha \in F} \mathcal{a}(\mathcal{L}_\alpha) \xrightarrow{P_F^{-1}} \prod_\alpha \mathcal{a}(\mathcal{L}_\alpha) \xrightarrow{\prod_\alpha \mu_\alpha} \{0,1\}$$

Thus for $E_F \in \prod_{\alpha \in F} \mathcal{a}(\mathcal{L}_\alpha)$ we get

$$(\prod_\alpha \mu_\alpha \cdot P_F^{-1})(E_F) = (\prod_\alpha \mu_\alpha)(P_F^{-1}(E_F)) = (\prod_{\alpha \in F})(E_F) = (\mu_{\alpha_1} \times \mu_{\alpha_2} \times \ldots\ldots \mu_{\alpha_n})(E_F) = 1.$$

As in the finite case we get $E_F \supset L_{\alpha_1} \times L_{\alpha_2} \times \ldots \times L_{\alpha_n}$ where $L_{\alpha_i} \in \mathcal{L}_{\alpha_i}$ and $\mu_{\alpha_i}(L_{\alpha_i}) = 1$ for all $i = 1, 2, \ldots. n$.

Then $A = P_F^{-1}(E_F) \supset P_F^{-1}(L_{\alpha_1} \times L_{\alpha_2} \times \ldots. L_{\alpha_n})$ and $(P_F^{-1}(L_{\alpha_1} \times \ldots. \times L_{\alpha_n})) = 1$, which shows that

$$\mu(A) = \sup\{\mu(P_F^{-1}(L_{\alpha_1} \times \ldots \times L_{\alpha_n})) / P_F^{-1}(L_{\alpha_1} \times \ldots \times L_{\alpha_n}) \subset A \text{ and } P_F^{-1}(L_{\alpha_1} \times \ldots\ldots L_{\alpha_n}) \in \prod_\alpha \mathcal{L}_\alpha = \mathcal{M}\}$$

Conversely, let $\mu \in I_R(\mathcal{L}) = I_R(\prod \mathcal{L}_\alpha)$ and define μ_α on $\mathcal{a}(\mathcal{L}_\alpha)$ by: $\mu_\alpha(A) = \mu(A \times \prod_{\beta \in \Lambda - \{\alpha\}} X_\beta)$, $A \in \mathcal{a}(\mathcal{L}_\alpha)$ i.e. $\mu_\alpha(A) = \mu(P_\alpha^{-1}(A))$

Since μ is a zero-one valued measure on $\mathcal{A}(\prod_\alpha \mathcal{L}_\alpha)$ it follows from the above definition that $\mu_\alpha \in I(\mathcal{L}_\alpha)$. If $\mu_\alpha(A)=1$, then $\mu(P_\alpha^{-1}(A))=1$ and since μ is $\mathcal{L}$-regular, there exists $\prod_\beta L_\beta$ such that $P_\alpha^{-1}(A) \supset \prod_\beta L_\beta \in \mathcal{M}$ and $\mu(\prod_\beta L_\beta)=1$. then $P_\alpha^{-1}(L_\alpha) \subset P_\alpha^{-1}(A)$ and $\mu_\alpha(L_\alpha)= =\mu(P_\alpha^{-1}(L_\alpha))=1$. Therefore $\mu_\alpha(A)=\sup\{\mu_\alpha(L)/L_\alpha \subset A, L_\alpha \in \mathcal{L}_\alpha\}$ i.e. $\mu_\alpha \in I_R(\mathcal{L}_\alpha)$

Now, if $B \in \mathcal{M}$, we may consider $B=P_F^{-1}(L_{\alpha_1} x \dots x L_{\alpha_n})$ and then $\prod_\alpha \mu_\alpha(B)=(\prod_\alpha \mu_\alpha)(P_F^{-1}(L_{\alpha_1} x \dots x L_{\alpha_n}))=(\prod_{\alpha\in F}\mu_\alpha)(L_{\alpha_1} x \dots x L_{\alpha_n})= =(\mu_{\alpha_1} x \dots x \mu_{\alpha_n})(L_{\alpha_1} x \dots x L_{\alpha_n})=\mu_{\alpha_1}(L_{\alpha_1})\mu_{\alpha_2}(L_{\alpha_2})\dots\mu_{\alpha_n}(L_{\alpha_n})=\mu(P_{\alpha_1}^{-1}(L_{\alpha_1}))\dots \mu(P_{\alpha_n}^{-1}(L_{\alpha_n}))$. If $\prod_\alpha \mu_\alpha(B)=1$, then $\mu(P_{\alpha_i}^{-1}(L_{\alpha_i}))=1$ for all i. Therefore $\mu(\bigcap_{i=1}^{n} P_{\alpha_i}^{-1}(L_{\alpha_i}))=1$ and $\mu(P_F^{-1}(L_{\alpha_1} x \dots x L_{\alpha_n}))=\mu(\prod_\alpha L_\alpha)=1$, i.e. $\mu(B)=1$

Thus $I_R(\mathcal{M}) \ni \mu=\prod_\alpha \mu_\alpha$ on $\mathcal{M}$ and then $\mu=\prod_\alpha \mu_\alpha$ on $\prod_\alpha \mathcal{A}(\mathcal{L}_\alpha)$.

<u>Theorem 9</u> The mapping $T: \prod_\alpha I_R(\mathcal{L}_\alpha) \dashrightarrow I_R(\prod_\alpha \mathcal{L}_\alpha)=I_R(\mathcal{L})$ defined by $T((\mu_\alpha)_\alpha)=\prod_\alpha \mu_\alpha$, is a homeomorphism.

<u>Proof.</u> T is one-to-one and onto.

Consider now the topological space $(I_R(\mathcal{L}_\alpha), \mathcal{O}_{W_\alpha})$ with the Wallman topology obtained by taking all

$W_\alpha(L_\alpha)=\{\mu_\alpha \in I_R(\mathcal{L}_\alpha)/\mu_\alpha(L_\alpha)=1\}$, $L_\alpha \in \mathcal{L}_\alpha$ as a base for closed sets and the topological space $(I_R(\mathcal{L}), \mathcal{O}_W)$. First, we show that

$T(\prod_\alpha W(L_\alpha))=W(\prod_\alpha L_\alpha)$, where $W(L_\alpha)=I_R(\mathcal{L}_\alpha)$ for almost all $\alpha\in\Lambda$, since $L_\alpha=X_\alpha$ for almost all α.

Let $(\mu_\alpha)_\alpha \in \prod_\alpha W(L_\alpha)=\prod_\alpha\{\mu_\alpha \in I_R(\mathcal{L}_\alpha)/\mu_\alpha(L_\alpha)=1, L_\alpha\in\mathcal{L}_\alpha$ and $L_\alpha=X_\alpha$ for almost all $\alpha\}=\{(\mu_\alpha)_\alpha \in I_R(\prod_\alpha\mathcal{L}_\alpha)/\prod_\alpha\mu_\alpha(\prod_\alpha L_\alpha)=1, L_\alpha\in\mathcal{L}_\alpha$, for almost all α, $L_\alpha=X_\alpha\}$ $=\{\prod_\alpha\mu_\alpha \in I_R(\prod_\alpha\mathcal{L}_\alpha)/\prod_\alpha\mu_\alpha(\prod_\alpha L_\alpha)=1\}=W(\prod_\alpha L_\alpha)$.

Since T is one-to-one and onto we have:

$T^{-1}(W(\prod_\alpha L_\alpha))=T^{-1}\{\prod_\alpha\mu_\alpha \in I_R(\prod_\alpha\mathcal{L}_\alpha)/\prod_\alpha\mu_\alpha(\prod_\alpha L_\alpha)=1\}=\{(\mu_\alpha)_\alpha \in \prod_\alpha I_R(\mathcal{L}_\alpha)/\mu_\alpha(L_\alpha)=1$ where $L_\alpha=X_\alpha$ for almost all $\alpha\}=\prod_\alpha W(L_\alpha)$.

Therefore T is continuous and hence a homeomorphism.

<u>Lemma 5</u> Let $\mathcal{L}_\alpha$ be a lattice of subsets of X_α. Then if

a) $\mu=\prod_\alpha\mu_\alpha \in I(\prod_\alpha\mathcal{L}_\alpha)=\prod_\alpha I(\mathcal{L}_\alpha)$, we have $S(\mu)=\prod_\alpha S(\mu_\alpha)$.

b) $\mathcal{L}_\alpha$ disjunctive for all $\alpha\in\Lambda$, then $\mathcal{L}=\mathcal{L}(\prod_\alpha\mathcal{L}_\alpha)$ is a disjunctive lattice of subsets of $\prod_\alpha X_\alpha$.

(Proof omitted).

<u>Theorem 10</u> Suppose that $\mathcal{L}_\alpha$ is a normal lattice of subsets of X_α for all $\alpha\in\Lambda$. Then $\mathcal{L} = \mathcal{L}(\prod_\alpha \mathcal{L}_\alpha)$ is a normal lattice of subsets of $\prod_\alpha X_\alpha$.
(Proof omitted).

<u>Theorem 11</u> Let $\mathcal{L}_\alpha$ be a δ, replete lattice of subsets of X_α such that $\sigma(\mathcal{L}_\alpha) = \rho(\mathcal{L}_\alpha)$ for all α. Let $\tilde{\mathcal{L}}$ be a lattice of subsets of $\prod_\alpha X_\alpha$ such that $\mathcal{L}\subset\tilde{\mathcal{L}}\subset t\mathcal{L}$. Then $\tilde{\mathcal{L}}$ is replete.

<u>Proof.</u> We must show that for any $\mu\in I_R^\sigma(\tilde{\mathcal{L}})$, $S(\mu)\neq\emptyset$. Consider first the projection mapping $P_\alpha : \prod_\alpha X_\alpha \dashrightarrow X_\alpha$ which is $\mathcal{L}$-$\mathcal{L}_\alpha$ continuous for all $\alpha\in\Lambda$. Since $\mathcal{L}\subset\tilde{\mathcal{L}}$, P_α is $\tilde{\mathcal{L}}$-$\mathcal{L}_\alpha$ continuous for all α. Now, let $\mu\in I_R^\sigma(\tilde{\mathcal{L}})$ and define the mapping

$$\hat{P}_\alpha : I_R^\sigma(\tilde{\mathcal{L}}) \dashrightarrow I_R^\sigma(\mathcal{L}_\alpha) \quad \text{by} \quad \hat{P}_\alpha\mu = \mu P_\alpha^{-1}, \ \mu\in I_R^\sigma(\tilde{\mathcal{L}}).$$

Then $\mu P_\alpha^{-1} = \mu_\alpha \in I_R^\sigma(\mathcal{L}_\alpha)$ and on $\prod_\alpha \mathcal{L}_\alpha$ we have $\prod_\alpha \mu P_\alpha^{-1} = \prod_\alpha \mu_\alpha = \bar{\mu}$, where $\bar{\mu}$ is the restriction of μ to $I_R^\sigma(\prod_\alpha \mathcal{L}_\alpha)$. Since by Lemma 8 $S(\bar{\mu}) = \prod_\alpha S(\mu_\alpha)$ and since each $\mathcal{L}_\alpha$ is a replete lattice, $S(\bar{\mu})\neq\emptyset$. Since $\mathcal{L}\subset\tilde{\mathcal{L}}\subset t\mathcal{L}$, $S(\mu) = S(\bar{\mu})$ and therefore $S(\mu)\neq\emptyset$, which shows that $\tilde{\mathcal{L}}$ is replete.

We next consider a case which has application to Borel-completness and in which the assumption of Theorem 11 that $\tilde{\mathcal{L}}\subset t\mathcal{L}$ is not assumed.

<u>Theorem 12</u> Let $\{X_i\}_{i=1}^\infty$ be a collection of abstract sets and let $\mathcal{L}_i$ be the lattice of subsest of X_i, $i=1,2,\dots,n,\dots$ Suppose that $\mathcal{L}_i$ contains all singletons, that $\mathcal{L}_i$ is a δ-lattice, replete and that $\sigma(\mathcal{L}_i) = \rho(\mathcal{L}_i)$ for all i. Let $\tilde{\mathcal{L}}$ be a lattice of subsets of $\prod_{i=1}^\infty X_i$ such that $\mathcal{L} = \mathcal{L}(\prod_{i=1}^\infty \mathcal{L}_i)\subset\tilde{\mathcal{L}}$. Then $\tilde{\mathcal{L}}$ is replete.

<u>Proof.</u> Let $\mu\in I_R^\sigma(\tilde{\mathcal{L}})$ and consider the projection mapping: $P_i : \prod_{i=1}^\infty X_i \dashrightarrow X_i$ which is $\mathcal{L}$-$\mathcal{L}_i$ continuous for all i. Since $\mathcal{L}\subset\tilde{\mathcal{L}}$ it follows that it is $\tilde{\mathcal{L}}$-$\mathcal{L}_i$ continuous. Now we define the mapping $\hat{P}_i : I_R^\sigma(\tilde{\mathcal{L}}) \dashrightarrow I_R^\sigma(\mathcal{L}_i)$ by $\hat{P}_i\mu = \mu P_i^{-1}$ where $\mu\in I_R^\sigma(\tilde{\mathcal{L}})$. Then $\mu P_i^{-1} = \mu_i = \mu_{x_i}$ where $x = (x_i)_{i=1}^\infty \in \prod_{i=1}^\infty X_i$ and

$$\mu_{x_i}(A_i) = \begin{cases} 1, & x\in A_i \\ 0, & x\notin A_i \end{cases} \quad \text{where } A_i\in\mathcal{A}(\mathcal{L}_i)$$

$\{x_i\}\in\mathcal{L}_i$ for all i, implies that

$\{x_1\}\times\{x_2\}\times\ldots\ldots\times\{x_n\}\times\ldots.=\prod_{i=1}^{\infty}\{x_i\}\in\prod_{i=1}^{\infty}\sigma(\mathcal{L}_i)=\sigma(\prod_{i=1}^{\infty}\mathcal{L}_i)\subset\sigma(\tilde{\mathcal{L}})$.

We have $\mu P_i^{-1}(\{x_i\})=\mu_{x_i}(\{x_i\})=1$. Since $\mu P_i^{-1}=\mu_i$ and since for $\mu\in I_R^{\sigma}(\tilde{\mathcal{L}})$ we define $\mu_i(A_i)=\mu(A_i\times\prod_{j=1}^{\infty}X_j)$, $i\neq j$, $A_i\in\mathcal{a}(\mathcal{L}_i)$, we get:

$$\mu P_i^{-1}(\{x_i\})=\mu(\{x_i\}\times\prod_{j=1}^{\infty}X_j)=1,\quad i\neq j.$$

For $\{x_1\}\times\{x_2\}\times\ldots..\times\{x_n\}\times\ldots\in\sigma(\tilde{\mathcal{L}})$ we have

$\mu(\{x_1\}\times\{x_2\}\times\ldots.\times\{x_n\}\times\ldots)=(\prod_{i=1}^{\infty}\mu_i)(\prod_{i=1}^{\infty}\{x_i\})=\prod_{i=1}^{\infty}\mu_i(\{x_i\})=$

$=\prod_{i=1}^{\infty}[\mu(\{x_i\}\times\prod_{\substack{j=1\\ i\neq j}}^{\infty}X_j)]=1$;therefore

$\{x_1\}\times\{x_2\}\times\ldots\ldots..\times\{x_n\}\times..\subset\bigcap\{\prod_{i=1}^{\infty}A_i\in\prod_{i=1}^{\infty}\sigma(\mathcal{L}_i)/x_i\in A_i$ all i and $\mu(\prod_{i=1}^{\infty}A_i)=1\}$

Consider now $\tilde{L}\in\tilde{\mathcal{L}}$ with $\mu(\tilde{L})=1$.Then since $\mu(\{x_1\}\times\ldots\{x_n\}\times\ldots)=1$ we must have $\{x_1\}\times\ldots.\times\{x_n\}\times..\subset\tilde{L}$.

Therefore $S(\mu)=\bigcap\{\tilde{L}/\mu(\tilde{L})=1,\tilde{L}\in\tilde{\mathcal{L}}\}\neq\emptyset$.

We give next some examples of products of topological lattices

Example 3 Let X_α be a topological $T_{3\frac{1}{2}}$ space and let $\mathcal{L}_\alpha=\mathcal{Z}_\alpha$ the replete lattice of zero sets for all $\alpha\in\Lambda$.Then each X_α is said to be realcompact.Consider now a lattice $\mathcal{Z}$ of subsets of $\prod_\alpha X_\alpha$ such that $\prod_\alpha\mathcal{Z}_\alpha\subset\mathcal{Z}\subset t(\prod_\alpha\mathcal{Z}_\alpha)$.Then, by Theorem 11 $\mathcal{Z}$ is replete and therefore $\prod_\alpha X_\alpha$ is realcompact.

Example 4 Let X_α be a T_2 and 0-dimensional space and let $\mathcal{L}_\alpha=\mathcal{C}_\alpha$ the replete lattice of clopen sets for all $\alpha\in\Lambda$.Then each X_α is N-compact.Consider any lattice $\mathcal{C}$ of subsets of $\prod_\alpha X_\alpha$ such that $\prod_\alpha\mathcal{C}_\alpha\subset\mathcal{C}\subset t(\prod_\alpha\mathcal{C}_\alpha)\subset t(\prod_\alpha\mathcal{Z}_\alpha)=\mathcal{F}$,as in Example 2. Since all $\mathcal{C}_\alpha$ are algebras, we have $I_R^{\sigma}(\mathcal{C}_\alpha)=I^{\sigma}(\mathcal{C}_\alpha)$.Each $\mathcal{C}_\alpha$ replete, implies that $S(\mu_\alpha)\neq\emptyset$ for all $\alpha\in\Lambda$, where $\mu_\alpha\in I_R^{\sigma}(\mathcal{C}_\alpha)$.

Let $\mu\in I_R^{\sigma}(\mathcal{C})=I^{\sigma}(\mathcal{C})$.Then we have

$\mu P_\alpha^{-1}=\mu_\alpha\in I_R^{\sigma}(\mathcal{C}_\alpha)=I^{\sigma}(\mathcal{C}_\alpha)$ and $S(\mu)=\prod_\alpha S(\mu_\alpha)\neq\emptyset$. Therefore $\mathcal{C}$ is replete.

REFERENCES

[1] G.Bachman & M.Szeto, On Strongly Measure replete Lattices and the General Wallman Remainder ,Fundamenta Mathematicae CXXII(1984),pp 199-217.

[2] Z.Frolik, Real Compactness in a Baire-Measurable Property , Bull.Acad.Polon.Sci.Ser.Sci.Math.Astronom Phys.,19,1971, pp 617-621.

[3] H.Herrlich, E-Kompacte Raume ,Math.Zeit.96(1967),pp 228-255.

[4] M.Kerner, Lattice Repletions and Products , " Measure theory and its applications",Proc. 1980 Conf.Northern Ill.Univ.(1980)

[5] F.Kost, Wallman-Type Compactifications and Products , Proceedings of the American Mathematical Society, Vol.29, No.3,August 1971,pp 607-612.