LIBERTAS MATHEMATICA, VOL. VII (1987)

SOME CONSIDERATIONS ABOUT A MATHEMATICAL MODEL FOR REPRESENTING DATA IMPERFECTION

E. Roventa and I. Popescu

ABSTRACT

In this paper we continue the study of some indefinite integrals on a class $\tilde{\mathcal{M}}(X,\mathcal{B})$ of fuzzy measurable subsets of a given space X .

These integrals can distinguish the fuzzy sets which have the same fuzzy degree.

The main result is a convergence theorem which still holds in the new context under suitable assumptions.

KEY WORDS: Fuzzy Set, Transom and Support Measures, Transom Indefinite Integral.

I. INTRODUCTION.

One knows that artificial intelligence studies the treatment of intellectual data that reflect three aspects: syntactic, semantic and pragmatic.

Knowledge treated by brain are not precise because the measure instruments are imperfect and/or the human is the only source of information.

One can consider that the imperfection of information is equal with uncertainty and/or imprecision of information.

Also, the imprecision of information is equal with inexactitude and/or fuzziness of information.

There are two classical approaches for representing the imperfection of information: calculus of errors and probability theory. The first approach is not shading and it works only for numerical parameters. The second is not supple (the fundamental axiom concern the probability of disjoints events).

With the progress of data processing and the development of artificial intelligence techniques it appeared the necessity of a new theory of imperfection of judgement: the fuzzy sets theory.

If X is a "universal" set, we denote by $\mathcal{P}(X)$ the set of all subsets of X . It is patent that the quadruple $(\mathcal{P}(X),\cup,\cap,C)$ is a Boolean algebra with respect to the union, intersection and complementary operations.

We can also define in the set of characteristic functions $\{0,1\}^X = \{\varphi_A | \varphi_A : X \to \{0,1\}\}$ a structure of Boolean algebra with the operations $\vee$-max , $\wedge$-min , C-complement, so these sets are isomorphic as Boolean algebras $(C\varphi_A(x) = 1 - \varphi_A(x))$.

Now, we consider the set $[0,1]^X = \{\tilde{F} \mid \tilde{F} : X \to [0,1]\}$. Then $[0,1]^X$ will be in correspondence one-to-one with the class of all fuzzy subsets of X (denoted by $\tilde{\mathcal{F}}(x)$) .

1.1 DEFINITION. We call fuzzy subset of X a part $\tilde{F} \subseteq X \times [0,1]$ such as:

(i) $pr_1(\tilde{F}) = X$;

(ii) $(x,y_1) \in \tilde{F}$ and $(x,y_2) \in \tilde{F} \Rightarrow y_1 = y_2$.

Thus, $\tilde{F} \in [0,1]^X$, $x \mapsto \tilde{F}(x) \in [0,1]$.

Examples 1) A precise sentence A = "John is between 20 and 25 years old".

$$\varphi_A(x) = \begin{cases} 1 , & x \in [20,25] \\ 0 , & x \in C[20,25] \end{cases} .$$

2) A fuzzy imprecise sentence $\tilde{F}$ = "John is young".

$$\tilde{F}(x) = \begin{array}{ll} [1 + (\frac{20-x}{5})^2]^{-1} & , x < 20 \\ 1 & , x \in [20,25] \\ [1 + (\frac{x-25}{5})^2]^{-1} & , x > 25 \end{array} .$$

1.2 DEFINITION. If $\{\tilde{F}_i\}_{i\in I} \subseteq \tilde{\mathcal{F}}(X)$, then:

(a) $\tilde{F}_i = \tilde{F}_j \Leftrightarrow \tilde{F}_i(x) = \tilde{F}_j(x)$, $i,j \in I$, $x \in X$

(b) $\tilde{F}_i \subseteq \tilde{F}_j \Leftrightarrow \tilde{F}_i(x) \leq \tilde{F}_j(x)$, $i,j \in I$, $x \in X$

(c) $\tilde{A} = \bigcup_{i\in I} \tilde{F}_i \Leftrightarrow \tilde{A}(x) = \sup_{i\in I}\{\tilde{F}_i(x)\}$, $x \in X$

(d) $\tilde{B} = \bigcap_{i\in I} \tilde{F}_i \Leftrightarrow \tilde{B}(x) = \inf_{i\in I}\{\tilde{F}_i(x)\}$, $x \in X$

(e) $\tilde{F}'_i = C\tilde{F}_i \Leftrightarrow \tilde{F}'_i = 1 - \tilde{F}(x)$, $i \in I$, $x \in X$.

The quadruple $(\tilde{F}(X),\vee,\wedge,C)$ is a Boolean lattice (is not a Boolean algebra) because it is possible to have $\tilde{F} \subseteq C\tilde{F}$, $\tilde{F} \cap C\tilde{F} \neq \tilde{0}$, $\tilde{F} \cup C\tilde{F} \neq \tilde{1}$. For example if $\tilde{F}(x) = 1/3$, $x \in X = [0,5]$, then $C\tilde{F} = 2/3$ and $\tilde{F} \subseteq C\tilde{F}$, $\tilde{F} \cap C\tilde{F} = 1/3 \neq \tilde{0}$, $\tilde{F} \cup C\tilde{F} = 2/3 \neq \tilde{1}$.

Generally speaking, the imprecise concern the content of information, while the uncertain is relative to its truth (confidence).

The uncertainty can be expressed by words like probable, possible, necessary, plausible and credible. Probable has two meanings: physical (frequency to turn up of an event) and epysthemical (subjective judgement). Possible has also two meanings: physical (measure of material difficulty in doing an action) and epysthemical (subjective judgement which engages little his author). Necessary is closed of certain, plausible is closed of possible and credible leads to necessary.

Example. "It is probable that John weighs at least 80 Kg" = (weight, John, 80 Kg, probable).

The imprecise can be expressed by words like fuzzy (vague), general and ambiguous (that is not considered in this context).

Examples 1) A fuzzy imprecise sentence: "x is approximatively equal to y" = (equality, (x,y) , approx., 1).

2) A non fuzzy imprecise sentence: "x is equal to y with ε prescribed degree of accurancy" = (equality, (x,y) , with ε pres. degree of acc., 1).

Concerning the measure of imperfection, one can summarize [1]:

measure of imperfection	[event]	=	value .
↑	↑		↑
· probability	precise		precise
· possibility	imprecise		imprecise
· necessity	fuzzy		fuzzy
· plausibility			
· credibility			
· transom and support			

Transom and Support Measures

Let $(X,\mathcal{B},\mu)$ a space with measure where $\mathcal{B}$ is σ-algebra and $\mu : \mathcal{B} \to \mathbb{R}_+$ is finite. It is said that a fuzzy set $\tilde{F}$ is measurable if for each $\alpha \in [0,1]$ it results $\{x \in X | \tilde{F}(x) > \alpha\} \in \mathcal{B}$ (we denote by $\tilde{\mathcal{M}}(X,\mathcal{B})$ the class of all measurable fuzzy subsets of X).

1.3 DEFINITION. We shall call transom of $\tilde{F} \in \tilde{F}(X)$ (resp. support) the usual set $T(\tilde{F}) = \{x \in X | 0 < \tilde{F}(x) < 1\}$ (resp. $S(\tilde{F}) = \{x \in X | \tilde{F}(x) > 0\}$) . Further, we shall also use the notations:

$$Z(\tilde{F}) = \{x \in X | \tilde{F}(x) = 0\} \quad \text{and} \quad H(\tilde{F}) = \{x \in X | \tilde{F}(x) = 1\} .$$

We notice that:

$$S(\tilde{F}) = T(\tilde{F}) \cup H(\tilde{F}) \ , \ T(\tilde{F}) \cap H(\tilde{F}) = \emptyset \ .$$

1.4 DEFINITION [3]. We call transom measure (resp. support measure) (associated to μ) the function $m_T\colon \tilde{\mathcal{M}}(X,\mathcal{B}) \to \mathbb{R}_+ = [0,\infty)$ (resp. $m_S\colon \tilde{\mathcal{M}}(X,\mathcal{B}) \to \mathbb{R}_+$) defined by:

$$m_T(\tilde{F}) = \mu(T(\tilde{F})) \quad (\text{resp.} \quad m_S(\tilde{F}) = \mu(S(\tilde{F}))\ .$$

REMARKS.

(a) It is obvious that $m_S(\tilde{F}) = m_T(\tilde{F}) + \mu(H(\tilde{F}))$.

(b) If F is a usual set, we have $T(F) = \emptyset, H(F) = F$ and $m_S(F) = \mu(F)$.

(c) Denoting $\alpha(\tilde{F}) = \dfrac{m_T(\tilde{F})}{m_S(\tilde{F})}$ and $\beta(\tilde{F}) = \dfrac{\mu(H(\tilde{F}))}{m_S(\tilde{F})}$ we have:

$$\alpha(\tilde{F}) + \beta(\tilde{F}) = 1 \ , \ \forall\ \tilde{F} \in \tilde{\mathcal{M}}(X,\mathcal{B}) \ , \ m_S(\tilde{F}) \neq 0 \ .$$

If α is small, then $\tilde{F}$ is "almost usual set" and when β is small then $\tilde{F}$ is an "almost fuzzy set". Then, α characterizes the fuzziness of $\tilde{F}$.

(d) The transom measure is σ-additive, it is not monotonous and the equality $m_T(\lim\limits_{x\to\infty} \tilde{F}_x) = \lim\limits_{x\to\infty} m_T(\tilde{F}_x)$ does not hold without making supplimentary hypotheses [3].

We have introduced an indefinite integral on a class of fuzzy sets by:

1.5 DEFINITION [2]. We call transom indefinite integral of f, on $\tilde{F} \in \tilde{\mathcal{M}}(X,\mathcal{B})$ the function $I_f\colon \tilde{\mathcal{M}}(X,\mathcal{B}) \to \mathbb{R}$ defined by:

$$I_f(\tilde{F}) = \int f\tilde{F}T(\tilde{F})d\mu \ .$$

REMARK. If F is a usual set, then $T(F) = \emptyset$ hence $I_f(F) = 0$, and $\int fF\,d\mu$ is the usual indefinite integral.

If $f = 1$ we shall denote $I_1(\tilde{F}) = \int \tilde{F}T(\tilde{F})d\mu = \int_{T(\tilde{F})} \tilde{F}\,d\mu$.

1.6 DEFINITION. If $\tilde{F}_1, \tilde{F}_2 \in \tilde{\mathcal{M}}(X,\mathcal{B})$ we shall designate:

$$\tilde{F}_1 \sim \tilde{F}_2 \iff m_T(\tilde{F}_1) = m_T(\tilde{F}_2)$$

where m_T is the transom measure generated by μ . Obviously "~"

is an equivalence relation. We shall denote the corresponding quotient set with $\mathcal{C}_T$.

1.7 DEFINITION. We call <u>fuzzy degree</u> any equivalence class from $\mathcal{C}_T$.

REMARK. The transom measure characterizes the fuzzy degree of a fuzzy set being identical for all the fuzzy sets whose transom have the same measure.

Using the function I_1 , we can distinguish the fuzzy sets which have the same fuzzy degree.

Some convergence properties of these indefinite integrals were studied in [2].

1.8 THEOREM [5]. Let $\{\tilde{F}_n\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$ and $\tilde{F} \in \tilde{\mathcal{M}}(X,\mathcal{B})$. If for any $\{\tilde{F}_{n_\ell}\} \subset \{\tilde{F}_n\}$, $H(\bigcup_{i=h}^{\infty} \tilde{D}_{n_i}) = H(\bigcup_{j=r}^{\infty} \tilde{D}_{n_j})$, $p \neq r$, $p,r \in N$, where $\tilde{D}_{n_i} = |\tilde{F}_{n_i} - \tilde{F}|$ and $\tilde{F}_n \underset{m_T}{\dashrightarrow} \tilde{F}$, there exists $\{\tilde{F}_{n_k}\} \subset \{\tilde{F}_n\}$ so that $\tilde{F}_{n_k} \to \tilde{F}$ a.e.m_T .

1.9 THEOREM [2].

(i) If the function f has an integral with respect to μ then for every $\{\tilde{F}_k\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$, with $\tilde{F}_k \nearrow \tilde{F}$ we have:

$$I_f(\tilde{F}) = \lim_{k\to\infty} I_f(\tilde{F}_k) \quad \text{if} \quad H(\tilde{F}_k) = H(\tilde{F}) \ , \ \forall k \in N \ .$$

(ii) If the function f has an integral with respect to μ then for every $\{\tilde{F}_k\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$, with $\tilde{F}_k \searrow \tilde{F}$ we have:

$$I_f(\tilde{F}) = \lim_{k\to\infty} I_f(\tilde{F}_k) \ , \ \text{if} \quad Z(\tilde{F}_k) = Z(\tilde{F}) \ , \ \forall k \in N \ .$$

II. CONVERGENCE PROPERTIES FOR INDEFINITE INTEGRALS ON A CLASS OF FUZZY SETS

In the following, we suppose that the measure μ is complete on $\mathcal{B}$ and $f : X \to \mathbb{R}_+$ is $\mathcal{B}$-measurable.

We show that the monotonic convergence theorem and Fatou's lemma are still true in this new setting by adding supplementary hypotheses.

2.1 THEOREM. Let $\{\tilde{F}_k\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$ and $\tilde{F}_* = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \tilde{F}_k$, then: $I_f(\tilde{F}_*) \leq \liminf I_f(\tilde{F}_k)$, if $H(\tilde{F}_k) = H(\tilde{F}_*)$, $k \in N$.

PROOF. Denoting by $\tilde{F}'_n = \bigcap_{k=n}^{\infty} \tilde{F}_k$, we obtain an increasing sequence $\{\tilde{F}'_n\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$. But $\tilde{F}_* = \lim_{n\to\infty} \tilde{F}'_n = \bigcup_{n=1}^{\infty} \tilde{F}'_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \tilde{F}_k$. According to Theorem 1.9 (i), we notice that $I_f(\lim_{n\to\infty} \tilde{F}'_n) = \lim_{n\to\infty} I_f(\tilde{F}'_n) = \liminf I_f(\tilde{F}'_n)$ if $H(\tilde{F}'_n) = H(\tilde{F}_*)$, $n \in N$. Since $\tilde{F}'_n \subset \tilde{F}_n$, $n \in N$ if $H(\tilde{F}'_n) = H(\tilde{F}_n)$, $n \in N$ we deduce (according to the property $\tilde{F}_1 \subset \tilde{F}_2$ and $H(\tilde{F}_1) = H(\tilde{F}_2) \Rightarrow T(\tilde{F}_1) \subset T(\tilde{F}_2)$ [3]) that $I_f(\tilde{F}'_n) \leq I_f(\tilde{F}_n)$, therefore $\liminf I_f(\tilde{F}'_n) \leq \liminf I_f(\tilde{F}_n)$, when $I_f(\tilde{F}_*) \leq \liminf I_f(\tilde{F}_n)$. Conditions $H(\tilde{F}'_n) = H(\tilde{F}_*)$, $H(\tilde{F}'_n) = H(\tilde{F}_n)$, $n \in N$ are equivalent with the condition $H(\tilde{F}_n) = H(\tilde{F}_*)$, $n \in N$ and the proof is therefore over.

2.2 THEOREM. Let $\{\tilde{F}_k\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$, and $\tilde{F}^* = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tilde{F}_k$. Then $\limsup I_f(\tilde{F}_k) \leq I_f(\tilde{F}^*)$ if $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$ and $H(\tilde{F}''_n) = H(\tilde{F}_n)$, $n \in N$ where $\tilde{F}''_n = \bigcup_{k=n}^{\infty} \tilde{F}_k$.

PROOF. Denoting by $\tilde{F}''_n = \bigcup_{k=n}^{\infty} \tilde{F}_k$ we obtain a decreasing sequence $\{\tilde{F}''_n\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$. But $\tilde{F}^* = \lim_{n\to\infty} \tilde{F}''_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tilde{F}_k$. According to Theorem 1.9 (ii) we get $I_f(\tilde{F}^*) = I_f(\lim_{n\to\infty} \tilde{F}''_n) = \lim_{n\to\infty} I_f(\tilde{F}''_n) = \limsup I_f(\tilde{F}''_n)$ if $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$, $n \in N$. Since $\tilde{F}''_n \supset \tilde{F}$, $n \in N$ if $H(\tilde{F}''_n) = H(\tilde{F}_n)$ it results that $I_f(\tilde{F}_n) \leq I_f(\tilde{F}''_n)$, therefore

$\limsup I_f(\tilde{F}_n) \le \limsup I_f(\tilde{F}''_n) = I_f(\tilde{F}^*)$.

2.3 DEFINITION. We shall say that a property P (which makes sense for each point of the $\tilde{\mathcal{M}}$-measure space $(X, \tilde{\mathcal{M}}(X,\mathcal{B}), m_T)$) *holds almost everywhere with respect to $\tilde{\mathcal{M}}$-measure m_T* (briefly a.e.m_T.) if there exists $\tilde{A} \in \tilde{\mathcal{F}}(X)$ with:

$S(\tilde{A}) = \{x \in X \mid \tilde{A}(x) > 0\} = \{x \in X \mid P \text{ is false}\}$ such that:

$S(\tilde{A}) \in \tilde{\mathcal{M}}(X,\mathcal{B})$ and $m_T(\tilde{A}) = 0$.

Denoting by $\tilde{\mathcal{F}}_X(\mu)$ or simply by $\tilde{\mathcal{F}}(\mu)$ the class of fuzzy integrable sets with respect to the measure μ on X, we have:

2.4 THEOREM. If $\{\tilde{F}_k\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$, and if there exists $G \in \tilde{\mathcal{F}}(\mu)$ with $\tilde{F}_k \subset G$ a.e.m_T., $k \in N$ and:

$H(\tilde{F}_k) = H(\tilde{F}_*) = H(\tilde{G})$, $k \in N$, $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$, $n \in N$ and

$H(\tilde{F}''_n) = H(\tilde{F}_n)$, $n \in N$ where $\tilde{F}''_n = \bigcup_{k=n}^{\infty} \tilde{F}_k$,

then for any $k \in N$, $\tilde{F}_k \in \tilde{\mathcal{F}}(\mu)$, $\tilde{F}_*, \tilde{F}^* \in \tilde{\mathcal{F}}(\mu)$ and:

$I_f(\tilde{F}_*) = \liminf I_f(\tilde{F}_k) \le \limsup I_f(\tilde{F}_k) \le I_f(\tilde{F}^*)$.

PROOF. Because $\tilde{F}_k \subset \tilde{G}$ a.e.m_T. , $k \in N$ it results that $\tilde{A} \in \tilde{\mathcal{F}}(X)$ exists so that $S(\tilde{A}) = \{x \in X \mid \tilde{F}_k \supseteq \tilde{G}\}$ and $m_T(\tilde{A}) = 0$. In view of the hypotheses, $\tilde{F}_k \subset \tilde{G}$ implies $I_f(\tilde{F}_k) \le I_f(\tilde{G})$, therefore $\tilde{F}_k \in \tilde{\mathcal{F}}(\mu)$, $k \in N$. We observe that $\tilde{F}_* \subset \tilde{G}$, $\tilde{F}^* \subset \tilde{G}$ and the same hypotheses implies $H(\tilde{G}) = H(\tilde{F}^*) = H(\tilde{F}_*)$ hence $\tilde{F}_*, \tilde{F}^* \in \tilde{\mathcal{F}}(\mu)$.

In view of our hypotheses we can apply Theorem 2.1, therefore $I_f(\tilde{F}_*) \le \liminf I_f(\tilde{F}_k)$.

The inequality $\liminf I_f(\tilde{F}_k) \le \limsup I_f(\tilde{F}_k)$ is obvious.

According to Theorem 2.2 we get $\limsup I_f(\tilde{F}_k) \le I_f(\tilde{F}^*)$.

THEOREM 2.5. If $\{\tilde{F}_k\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$, and if there exists $\tilde{G} \in \tilde{\mathcal{F}}(\mu)$

with $\tilde{F}_k \subset \tilde{G}$ a.e.m_T. , $k \in N$ and:

(i) $\tilde{F}_k \to \tilde{F}$ a.e.m_T. , $\tilde{F} \in \tilde{\mathcal{M}}(X,\mathcal{B})$.

(ii) $H(\tilde{F}_k) = H(\tilde{F}_*) = H(\tilde{G})$, $k \in N$, $Z(\tilde{F}''_n) = Z(\tilde{F}^*)$ and

$H(\tilde{F}''_n) = H(\tilde{F}_n)$, $n \in N$ where $\tilde{F}''_n = \bigcup_{k=n}^{\infty} \tilde{F}_k$

then $\tilde{F} \in \tilde{\mathcal{F}}(\mu)$, $\{\tilde{F}_k\} \subset \tilde{\mathcal{F}}(\mu)$ and $I_f(\tilde{F}) = \lim_{k\to\infty} I_f(\tilde{F}_k)$.

PROOF. The classical proof "almost" works. Since $\tilde{F}_k \to \tilde{F}$ a.e.m_T ., it results that $\tilde{D}^* = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \tilde{D}_k \in \tilde{\mathcal{M}}(X,\mathcal{B})$, where $\tilde{D}_k = |\tilde{F}_k - \tilde{F}|$ and $m_T(\tilde{D}^*) = 0$. We can apply Theorem 2.4 on the $C_{T(\tilde{D}^*)}$, therefore $\tilde{F} \in \tilde{\mathcal{F}}(\mu)$. Because $\mu(T(\tilde{D}^*)) = 0$ we have $I_f\tilde{F} = \int_X fT(\tilde{F})d\mu = \int_A fT(\tilde{F})d\mu < \infty$. Therefore $\tilde{F} \in \tilde{\mathcal{F}}(\mu)$. According to Theorem 2.4 for $\tilde{F}^* = \tilde{F}_* = \tilde{F}$ we get:

$$I_f(\tilde{F}) = \int_X f\tilde{F}T(\tilde{F})d\mu = \int_A f\tilde{F}T(\tilde{F})d\mu \le \liminf \int_A f\tilde{F}_kT(\tilde{F}_k)d\mu \le$$

$$\le \limsup \int_A f\tilde{F}_kT(\tilde{F}_k)d\mu \le \int_A f\tilde{F}T(\tilde{F})d\mu = \int_X f\tilde{F}T(\tilde{F})d\mu = I_f(\tilde{F}) .$$

Therefore the sequence $\int_A f\tilde{F}_kT(\tilde{F}_k)d\mu$ is convergent to $I_f(\tilde{F}) = \int_X f\tilde{F}T(\tilde{F})d\mu$.

Analogously we obtain:

$$\int_A f\tilde{F}_kT(\tilde{F}_k)d\mu = \int_X f\tilde{F}_kT(\tilde{F}_k)d\mu = I_f(\tilde{F}_k) , \quad k \in N .$$

Therefore, $I_f(\tilde{F}) = I_f(\lim_{k\to\infty} \tilde{F}_k) = \lim_{k\to\infty} I_f(\tilde{F}_k)$.

2.6 THEOREM. If $\{\tilde{F}_n\} \subset \tilde{\mathcal{M}}(X,\mathcal{B})$, and if there exists $\tilde{G} \in \tilde{\mathcal{F}}(\mu)$ with $\tilde{F}_n \subset \tilde{G}$ a.e.m_T. , $n \in N$ and:

(i) $\tilde{F}_n \xrightarrow[m_T]{} \tilde{F}$, $(\tilde{F} \in \tilde{\mathcal{M}}(X,\mathcal{B}))$

(ii) $H(\tilde{F}_n) = H(\tilde{F}_*) = H(\tilde{G})$, $n \in N$, $Z(\tilde{F}''_m) = Z(\tilde{F}^*)$ and $H(\tilde{F}''_m) = H(\tilde{F}_m)$, $m \in N$ where $\tilde{F}''_m = \bigcup_{n=m}^{\infty} \tilde{F}_n$, whatever would be $\{\tilde{F}_{n_\ell}\} \subset \{\tilde{F}_n\}$, $H(\bigcup_{i=p}^{\infty} \tilde{D}_{n_i}) = H(\bigcup_{j=r}^{\infty} \tilde{D}_{n_j})$ $p \neq r$, $p,r \in N$, then $\tilde{F}_n \in \tilde{\mathcal{F}}(\mu)$, $n \in N$, $\tilde{F} \in \tilde{\mathcal{F}}(\mu)$ and:

$$I_f(\tilde{F}) = \lim_{n\to\infty} I_f(\tilde{F}_n) .$$

PROOF. From $\tilde{F}_n \subset \tilde{G}$ a.e.m_T. , $\tilde{G} \in \tilde{\mathcal{F}}(\mu)$ we get $\tilde{F}_n \in \tilde{\mathcal{F}}(\mu)$, $n \in N$. According to Theorem 1.8, there exists $\{\tilde{F}_{n_k}\} \subset \{\tilde{F}_n\}$ such that $\tilde{F}_{n_k} \to \tilde{F}$ a.e.m_T. Since $\tilde{F}_{n_k} \subset \tilde{G}$ it results that $\tilde{F} \subset \tilde{G}$, therefore $\tilde{F} \in \tilde{\mathcal{F}}(\mu)$. For the sequence $\{\tilde{F}_{n_k}\}$ the Theorem 2.5 holds, therefore $I_f(\tilde{F}) = \lim_{k\to\infty} I_f(\tilde{F}_{n_k})$. Supposing that $I_f(\tilde{F}_n)$ is not convergent to $I_f(\tilde{F})$ (similarly to the classical proof) a contradiction with the last equality results, therefore $I_f(\tilde{F}) = \lim_{n\to\infty} I_f(\tilde{F}_n)$.

REMARK. $\lim_{n\to\infty} \tilde{F}_n$ should be considered as a transom $\tilde{\mathcal{M}}$-measure convergence, and $\lim_{n\to\infty} I_f(\tilde{F})$ related to the natural topology on $\mathbb{R}$.

In order to conclude this paper, we show that the supplimentary conditions cannot be dropped.

Indeed, consider:

$$\tilde{F}_k(x) = \begin{cases} 0 & , x < k \\ x - k & , k \le x < k+1 \\ 1 & , x > k+1 \end{cases} .$$

Clearly, $m_T(\tilde{F}_k) = 1$, $k \in N$, therefore $\lim_{k\to\infty} m_T(\tilde{F}_k) = 1$ but $m_T(\lim_{k\to\infty} \tilde{F}_k) = m_T(\tilde{0}) = 0$. For $f = 1$ we get:

$I_1(\tilde{F}_k) = \int \tilde{F}_k T(\tilde{F}_k) d\mu = 1/2$, while $I_1(\tilde{F}) = \int \tilde{F} T(\tilde{F}) d\mu = 0$.

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