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MINKOWSKI TYPES OF INEQUALITIES

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§1. INTRODUCTION

The module of a family of measures or surfaces is defined in [2] and [8] using an arbitrary increasing function

$$\Phi : x \to \Phi(x) \ , \ x \geq 0 \ .$$

However, for most of the applications, only

$$\Phi_r : x \to x^r \ , \ r > 0$$

are used.

It appeared in [3], [4] and [5] that Φ_r are too restrictive functions. A weaker condition on Φ, as Δ_2-Birnbaum and Orlicz condition ([5], ch. 1) under equality form, allows the development of a general and consistent enough extremal lengths theory. This condition is: there exists $M > 0$ such that:

$$\Phi(2x) = M\Phi(x) \ , \text{ for every } x \geq 0 \ .$$

Φ_r fulfills the (strong) Δ_2-condition with $M = 2^r$.

A whole family $\{\Phi_{r,M}\}$ of functions fulfilling the strong Δ_2-condition has been described in [1] and it has been shown in [1] and [5] that these functions verify some other well-known properties, like Hölder, Clarkson and Jensen-Cooper types of inequalities. It resulted from [3] that the fundamental theorem of existence of an almost extremal function with respect to a Φ-module uses only the strong Δ_2-condition and Clarkson and Minkowski types of inequalities for Φ.

While $\Phi_{r,M}$ verify the first two properties, the third is true only if $M = 2^r$, therefore only when $\Phi_{r,M}$ is in fact Φ_r.

The question arises whether there exist some other functions Φ different from Φ_r, which verify Minkowski types of inequalities.

In this note it will be shown that such functions can be obtained using $\Phi_{r,M}$.

§2. SOME GENERAL PROPERTIES OF THE FUNCTIONS $\Phi_{r,M}$

Let us denote for each $r \in \mathbb{R}$, and every $M > 0$, $M \neq 1$:

$$\Phi_{r,M}(x) = \sum_{n=-\infty}^{+\infty} M^n \left[\left(\frac{x}{2^n} \right)^r - \frac{M - 2^r}{M - 1} \right] \chi_{I_n}(x) \ , \ x > 0$$

$$\Phi_{r,M}(0) = 0 \ , \tag{1}$$

where χ_{I_n} is the characteristic function of the interval

$I_n = [2^n, 2^{n+1}[\ , \ n \in \mathbb{Z}$.

We mention the following immediate properties of the functions $\Phi_{r,M}$:

1) $\Phi_{r,M}(2^k x) = M^k \Phi_{r,M}(x)$, $k \in \mathbb{Z}$, $x \geq 0$; $\Phi_{r,M}(x) > 0$ if $x > 0$, except for $r > 0$, $0 < M < 1$ and $r < 0$, $M > 1$, when $\Phi_{r,M}(x) < 0$ if $x > 0$.

2) $\Phi_{r,M}$ is strictly increasing if $r > 0$ and strictly decreasing on $]0,\infty[$ if $r < 0$. $\Phi_{r,M}(x) \equiv 0$ if $r = 0$.

3) $\Phi_{r,M}$ is continuous on $]0,\infty[$ if $0 < M < 1$ and on $[0,\infty[$ if $M > 1$; $\Phi_{r,M}(0_+) = 0$ if $M > 1$, but if $0 < M < 1$, then $\Phi_{r,M}(0_+) = \begin{cases} +\infty , & \text{if } r < 0 \\ -\infty , & \text{if } r > 0 \end{cases}$.

4) $\Phi_{r,M}$ is indefinitely differentiable on each interval $]2^n, 2^{n+1}[$.

5) $\lim\limits_{x \uparrow 2^n} \Phi'_{r,M}(x) = r \cdot 2^{r-1}\left[\frac{M}{2}\right]$, $\lim\limits_{x \downarrow 2^n} \Phi'_{r,M}(x) = r\left[\frac{M}{2}\right]^n$.
It results particularly that $\Phi_{r,M}$ is continuously differentiable on $]0,\infty[$ if and only if $M = 2^r$. In this last case, $\Phi_{r,M}(x) = x^r$.

6) $\Phi_{r,M}$ is convex on $]0,\infty[$ if

   i) $r < 0$, $0 < M \leq 2^r$, or

   ii) $r > 1$, $M \geq 2^r$

   and it is concave on $]0,\infty[$ if $0 < r < 1$, $0 < M \leq 2^r$, $M \neq 1$.

7) If $r \neq 0$, then:

$$\Phi^{-1}_{r,M}(y) = \sum_{n=-\infty}^{+\infty} 2^n \left[\frac{y}{M^n} + \frac{M - 2^r}{M - 1}\right]^{1/r} \chi_{J_n}(y) , \qquad (2)$$

where χ_{J_n} is the characteristic function of the interval

$J_n = \left[M^n \frac{2^r - 1}{M - 1} , M^{n+1} \frac{2^r - 1}{M - 1}\right[$, when $r > 0$, and

$J_n = \left]M^{n+1} \frac{2^r - 1}{M - 1} , M^n \frac{2^r - 1}{M - 1}\right]$; when $r < 0$.

Proposition 2.1. For $x > 0$, $y > 0$ *the following are true:*

if   *i)* $r > 0$ *and* $M > 2^r$, *or*

ii) $r < 0$ *and* $M > 1$, *or* $0 < M < 2^r$, *then*

$$\Phi_{r,M}(x \cdot y) > \Phi_{r,M}(x) \cdot \Phi_{r,M}(y) ,$$

if iii) $r > 0$ and $1 < M < 2^r$, *or* *iv)* $r < 0$ *and* $2^r < M < 1$ *, then*

$$\Phi_{r,M}(x\cdot y) < \Phi_{r,M}(x) \cdot \Phi_{r,M}(y) .$$

In this last case $\Phi_{r,M}$ *verifies in particular the* (Δ')*-condition* ([7], page 29) *with* $c = 1$.

<u>Proof</u>: Let us denote

$$H(x,y) = \Phi_{r,M}(x) \cdot \Phi_{r,M}(y) - \Phi_{r,M}(x\cdot y) .$$

Since for $n , m \in \mathbb{Z}$, we have

$$H(2^n x, 2^m y) = M^{n+m} H(x,y) ,$$

it is sufficient to study $H(x,y)$ in $A = \{(x,y) : 1 \le x < 2 , 1 \le y < 2\}$.

For $(x,y) \in A$ we have

$$H(x,y) = \left[x^r - \frac{M - 2^r}{M - 1}\right] \left[y^r - \frac{M - 2^r}{M - 1}\right] - \left[\frac{M}{2^r}\right]^k x^r y^r + M^k \frac{M - 2^r}{M - 1} ,$$

where $k = 0$ when $1 \le xy < 2$, and $k = 1$ when $2 \le xy < 4$.

$H(x,1) = \frac{2^r - M}{M - 1} \Phi_{r,M}(x)$, $H(x,2) = MH(x,1)$ and H is continuous in y for each x . It is obvious that each one of the conditions i), ii) implies

$$H(x,1) < 0 \quad \text{and} \quad H(x,2) < 0$$

and each one of the conditions iii), iv) implies:

$$H(x,1) > 0 \quad \text{and} \quad H(x,2) > 0 .$$

Let us remark that for $1 < x < 2$, $1 < y < 2$, $xy \ne 2$ we have:

$$\frac{\partial H}{\partial y} = \begin{cases} ry^{r-1} \dfrac{2^r - M}{M - 1} & , \ 1 < y < \dfrac{2}{x} \\ ry^{r-1} \left[x^r\left[1 - \dfrac{M}{2^r}\right] + \dfrac{2^r - M}{M - 1}\right] & , \ \dfrac{2}{x} < y < 2 \end{cases}$$

which allows us to see that $H(x,y)$ doesn't change sign between $H(x,1)$ and $H(x,2)$ in the two cases analysed and the affirmation of the proposition is proved.

Proposition 2.2. The function $\Phi_{r,M}$ *increases when* r *increases from* 0 *to* ∞ *if* $M > 1$ *and it decreases if* $0 < M < 1$.

<u>Proof</u>: Let $0 < r < s$ and define $f(x)$ by:

$M^n f(x) = \Phi_{s,M}(x) - \Phi_{r,M}(x)$ for $2^n \le x < 2^{n+1}$. Then

$$f(x) = \left[\frac{x}{2^n}\right]^s - \left[\frac{x}{2^n}\right]^r + \frac{2^s - 2^r}{M - 1} \quad \text{for} \quad 2^n \le x < 2^{n+1} \quad \text{and}$$

$$2^n \cdot f'(x) = s\left[\frac{x}{2^n}\right]^{s-1} - r\left[\frac{x}{2^n}\right]^{r-1} \quad \text{for} \quad 2^n < x < 2^{n+1} .$$

From $0 < r < s$ we have $0 < \frac{r}{s} < 1$ and $s - r > 0$. It follows that $\left(\frac{r}{s}\right)^{\frac{1}{s-r}} < 1$ and hence $x \geq 2^n > 2^n\left(\frac{r}{s}\right)^{\frac{1}{s-r}}$, which implies $\frac{x}{2^n} > \left(\frac{r}{s}\right)^{\frac{1}{s-r}}$, or $\left(\frac{x}{2^n}\right)^{s-r} > \frac{r}{s}$, or $\left(\frac{x}{2^n}\right)^{s-1} : \left(\frac{x}{2^n}\right)^{r-1} > \frac{r}{s}$ and finally $s\left(\frac{x}{2^n}\right)^{s-1} - r\left(\frac{x}{2^n}\right)^{r-1} > 0$ or $f'(x) > 0$.

Since f is a continuous function on $[2^n, 2^{n+1}[$ and $f(2^n) = \frac{2^s - 2^r}{M - 1}$, $\lim_{x \uparrow 2^{n+1}} f(x) = M\frac{2^s - 2^r}{M - 1}$ the affirmation of the proposition results.

§3. ON THE MINKOWSKI INEQUALITY

In this paragraph, it will be shown that for some suitable values of r and M, $\Phi_{r,M}$ can be changed in such a way that most of the properties mentioned in §2 still hold and a Minkowski type of inequality becomes true.

Lemma 3.1. There is a function $\varphi_{r,M}: \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

a) $\varphi_{r,M}(2^n) = 1$, $n \in \mathbb{Z}$

b) $\varphi_{r,M}(2^k x) = \varphi_{r,M}(x)$, $x \geq 0$, $k \in \mathbb{Z}$

c) $\Psi_{r,M} = \varphi_{r,M} \cdot \Phi_{r,M}$ *is continuously differentiable on* $]0,\infty[$

d) For M *close enough to* 2^r, $\Psi_{r,M}$ *is convex (resp. concave) when* $\Phi_{r,M}$ *is convex (resp. concave).*

Proof: We search for a function $\varphi_{r,M}$ of the form $\varphi_{r,M}(x) = e^{\theta(x)}$, where $\theta(x) = \frac{\alpha}{2^{2n}}(x - 2^n)(x - 2^{n+1})$, if $2^n \leq x < 2^{n+1}$, $n \in \mathbb{Z}$ and $\theta(0) = 0$. Here α is a constant depending on r and M, which will be determined later.

It is easily seen that $\theta(2^n) = 0$, $n \in \mathbb{Z}$ and $\theta(2x) = \theta(x)$ for $x \geq 0$. This implies a) and b).

From this and the property 1), §2, we deduce that

$$\Psi_{r,M}(2^k x) = M^k \, \Psi_{r,M}(x), \quad k \in \mathbb{Z}, \ x \geq 0. \tag{3}$$

If $x \neq 2^n$, $n \in \mathbb{Z}$, then

$$\Psi'_{r,M}(x) = e^{\theta(x)}[\theta'(x)\,\Phi_{r,M}(x) + \Phi'_{r,M}(x)] \tag{4}$$

which implies:

$$\lim_{x \downarrow 2^n} \Psi'_{r,M}(x) = \left(\frac{M}{2}\right)^n \left[-\alpha \frac{2^r - 1}{M - 1} + r\right] \tag{5}$$

$$\lim_{x \uparrow 2^n} \Psi'_{r,M}(x) = \left(\frac{M}{2}\right)^n \left[2\alpha \frac{2^r - 1}{M - 1} + r \frac{2^r}{M}\right] . \tag{6}$$

The right member in (5) and (6) are equal if and only if

$$\alpha = \frac{r}{3} \cdot \frac{M - 2^r}{M} \cdot \frac{M - 1}{2^r - 1} . \tag{7}$$

The continuity of $\Psi_{r,M}$ assures that for this value of α , $\Psi_{r,M}$ is continuously differentiable in $]0,\infty[$.

In the following, we'll assume that α is given by the relation (7).

Suppose $2^n < x < 2^{n+1}$, $n \in \mathbf{Z}$. Then:

$$\Psi''_{r,M}(x) = e^{\theta(x)}\{[\theta'^2(x) + \theta''(x)]\ \Phi_{r,M}(x) + \Phi''_{r,M}(x) + 2\theta'(x)\ \Phi'_{r,M}(x)\} . \tag{8}$$

From here, by computation, we find:

$$\Psi''_{r,M}(x) = e^{\theta(x)} \frac{M^n}{2^{2n}} H_{r,M}(x) \tag{9}$$

where

$$H_{r,M}(x) = 2\alpha\left[2\alpha\left(\frac{x}{2^n} - \frac{3}{2}\right)^2 + 1\right]\left[\left(\frac{x}{2^n}\right)^r - \frac{M - 2^r}{M - 1}\right] + r(r - 1)\left(\frac{x}{2^n}\right)^{r-2} + 4\alpha r\left(\frac{x}{2^n} - \frac{3}{2}\right)\left(\frac{x}{2^n}\right)^{r-1} . \tag{10}$$

Now, let $r < 0$ be fixed and M , $0 < M < 2^r(<1)$ be variable. Then $\alpha > 0$, and for $2^n < x < 2^{n+1}$,

$$\left(\frac{x}{2^n}\right)^r - \frac{M - 2^r}{M - 1} > M \frac{2^r - 1}{M - 1} ,$$

so

$$H_{r,M}(x) > 2\alpha M \frac{2^r - 1}{M - 1} + r(r - 1)2^{r-2} - 2\alpha r = A(M) .$$

A is a continuous function in $]0,2^r[$ and $\lim_{M \to 2^r} A(M) = r(r - 1)2^{r-2} > 0$.

Thus, there exists $\eta > 0$ such that $A(M) > 0$, hence $H_{r,M}(x) > 0$ and then $\Psi''_{r,M}(x) > 0$ for $2^n < x < 2^{n+1}$, provided $2^r - \eta < M < 2^r$.

Similarly, if we keep $r > 1$ fixed and let $M > 2^r$ vary, then $\alpha > 0$ and for $2^n < x < 2^{n+1}$,

$$\left[\frac{x}{2^n}\right]^r - \frac{M - 2^r}{M - 1} > \frac{2^r - 1}{M - 1} ,$$

so

$$H_{r,M}(x) > 2\alpha \frac{2^r - 1}{M - 1} + r(r - 1)2^{r-2} - 2\alpha r \cdot 2^{r-1} = B(M) .$$

Again, B being continuous for $M > 2^r$ and $\lim_{M \to 2^r} B(M) = r(r - 1)\, 2^{r-2} > 0$, there exists $\eta_1 > 0$ such that $\Psi''_{r,M}(x) > 0$ for $2^n < x < 2^{n+1}$, provided $2^r < M < 2^r + \eta_1$.

The property 6), §2, and the precedent property c) allow us to say that in the two cases analysed $\Phi_{r,M}$ and $\Psi_{r,M}$ are both convex.

Finally, let r , $0 < r < 1$ be fixed and M , $1 < M < 2^r$ be variable. Then $\alpha > 0$ and for $2^n < x < 2^{n+1}$,

$$H_{r,M}(x) < \alpha \frac{2^r - 1}{M - 1} (\alpha M + 2) + r(r - 1) \cdot 2^{r-2} - 2\alpha r = C(M) .$$

The function C is continuous for $1 < M < 2^r$ and $\lim_{M \to 2^r} C(M) = r(r - 1)2^{r-2} < 0$.

Thus, there exists $\eta_2 > 0$ such that $C(M) < 0$, hence $\Psi''_{r,M}(x) < 0$ for $2^n < x < 2^{n+1}$, provided $2^r - \eta_2 < M < 2^r$ and the property d) is completely proved.

Lemma 3.2. There is a function $\varepsilon_{r,M}: \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

a) $\varepsilon_{r,M}(2^n) = 0$, $n \in \mathbb{Z}$

b) $\varepsilon_{r,M}(2^k x) = M^k \varepsilon_{r,M}(x)$, $x \geq 0$, $k \in \mathbb{Z}$

c) $\Phi_{r,M} + \varepsilon_{r,M}$ *is continuously differentiable on* $]0,\infty[$

d) For M *close enough to* 2^r , $\Phi_{r,M} + \varepsilon_{r,M}$ *is convex (resp. concave) when* $\Phi_{r,M}$ *is convex (resp. concave).*

It is easily seen that $\varepsilon_{r,M} = \Psi_{r,M} - \Phi_{r,M}$, where $\Psi_{r,M}$ is the function from lemma 3.1, verifies all these properties.

It is said that a function Φ defined on $\mathbb{R}^+$ verifies a direct (resp. reversed) Minkowski type of inequality, if for any positive integer m and for any $p = (p_1,\ldots,p_m)$, $a = (a_1,\ldots,a_m)$, $b = (b_1,\ldots,b_m) \in \mathbb{R}^+_m$, the following inequality holds:

$$2\Phi^{-1}\left[\Sigma p_i\Phi\left[\frac{a_i + b_i}{2}\right]\right] \underset{(\text{resp. } \geq)}{\leq} \Phi^{-1}(\Sigma p_i\Phi(a_i)) + \Phi^{-1}(\Sigma p_i\Phi(b_i)) \ . \qquad (*)$$

This is equivalent to saying ([6], page 86) that

$$\Phi^{-1}\{\Sigma p_i\Phi(x_i\}$$

is, for given p , a convex (resp. concave) function of the m variables $x_1, x_2, \ldots, x_m$.

Proposition 3.1. There are $\delta' > 0$, $\delta'' > 0$, *such that* $\Psi_{r,M}$ *verifies a direct Minkowski type of inequality, if* $r > 1$, $2^r - \delta' \leq M \leq 2^r$, *and a reversed Minkowski type of inequality, if* $r < 1$, $2^r - \delta'' < M \leq 2^r$.

Since $\Psi_{r,M}(2x) = M\Psi_{r,M}(x)$, the relation $(*)$ for $\Psi_{r,M}$ becomes

$$\Psi^{-1}_{r,M}(\Sigma p_i \ \Psi_{r,M}(a_i + b_i)) \underset{(\text{resp. } \geq)}{\leq} \Psi^{-1}_{r,M}(\Sigma p_i \ \Psi_{r,M}(a_i)) + \Psi^{-1}_{r,M}(\Sigma p_i \ \Psi_{r,M}(b_i)) \ . \qquad (11)$$

<u>Proof</u>: Let $u = (u_1, \ldots, u_m)$, $x = (x_1, \ldots, x_m)$, $p = (p_1, \ldots, p_m)$ be arbitrary but fixed points in $\mathbb{R}^+_m$ and $t > 0$ be variable. Let us determine uniquely $n_i = n_i(t)$, $n = n(t)$ such that

$$2^{n_i} \leq x_i + u_i t < 2^{n_i + 1}$$
$$\Sigma p_i \ \Phi_{r,M}(x_i + u_i t) \in J_n \ , \qquad (12)$$

with J_n as defined in §2.

With the notations:

$$a = \left[\frac{2^r}{M}\right]^n \ , \qquad b = \frac{M - 2^r}{M - 1} \cdot 2^{nr}$$
$$\alpha_i = \left[\frac{M}{2^r}\right]^{n_i} \ , \qquad \beta_i = M^{n_i} \ \frac{2^r - 1}{M - 1} \ , \qquad (13)$$

let's put:

$$x_{r,M}(t) = \Phi^{-1}_{r,M}\{\Sigma p_i \ \Phi_{r,M}(x_i + u_i t\}$$
$$= \{a\Sigma p_i[\alpha_i(x_i + u_i t)^r + \beta_i)] + b\}^{1/r} \qquad (14)$$

The functions $x_{r,M}$ are piecewise indefinitely differentiable and where $x''_{r,M}(t)$ exists, it is of the form

$$\begin{aligned} x''_{r,M}(t) = (r-1)\, x^{1-2r}_{r,M}(t)\{&[a\,\Sigma p_i\,\alpha_i\,u_i^2(x_i+u_it)^{r-2}]\\ &[a\,\Sigma p_i\,\alpha_i(x_i+u_it)^r + a\,\Sigma p_i\,\beta_i + b]\\ &-[a\,\Sigma p_i\,u_i\,\alpha_i(x_i+u_it)^{r-1}]^2\}\ . \end{aligned} \tag{15}$$

Let us examine the expression

$$a\,\Sigma p_i\,\beta_i + b = \left(\frac{2^r}{M}\right)^n \frac{2^r-1}{M-1}\,[\Sigma\, p_i\, M^{n_i} - M^n]\ . \tag{16}$$

If there is a $t > 0$ such that

$$x_i + u_it = 2^{n_i}\ ,\quad i = 1,2,\ldots,m$$

then

$$\begin{aligned} &M^n \le \Sigma p_i\, M^{n_i} = \frac{M-1}{2^r-1}\,\Sigma p_i\,\Phi_{r,M}(x_i+u_it) < M^{n+1}\ ,\quad \text{for } M > 1\\ &M^{n+1} < \frac{M-1}{2^r-1}\,\Sigma p_i\,\Phi_{r,M}(x_i+u_it) = \Sigma p_i\,M^{n_i} \le M^n\ ,\quad \text{for } 0 < M < 1\ . \end{aligned} \tag{17}$$

If for at least one index i, $x_i + u_it > 2^{n_i}$, then „ = " is replaced by „ < " in (17). The first and the last inequality becomes equalities when $\Sigma p_i\, M^{n_i}$ becomes a power of M.

When t increases such that $2^{n_i} \le x_i + u_it < 2^{n_i+1}$, the expression $\Sigma p_i\, M^{n_i}$ does not change and the relations (17) take place with $n+1$ instead of n when $\frac{2^r-1}{M-1}\,\Sigma p_i\,\Phi_{r,M}(x_i+u_it)$ reaches M^{n+1}. Therefore, we always have:

$$\begin{aligned} &\Sigma p_i\, M^{n_i} - M^n \ge 0\ , &&\text{if } M > 1\\ &\Sigma p_i\, M^{n_i} - M^n \le 0\ , &&\text{if } 0 < M < 1\ . \end{aligned} \tag{18}$$

It results that $a\,\Sigma p_i\,\beta_i + b$ is positive if

$$0 < M \le 2^r\ ,\ (M \ne 1)\ . \tag{19}$$

In this case, if $x''_{r,M}(0)$ exists, we have:

$$\begin{aligned} \frac{x^{2r-1}_{R,M}(0)}{(r-1)a}\, x''_{r,M}(0) &= [a\,\Sigma p_i(\alpha_i\, x_i^r + \beta_i) + b]\,\Sigma p_i\,\alpha_i\,u_i^2\,x_i^{r-2}\\ &\quad - a(\Sigma p_i\,\alpha_i\,u_i\,x_i^{r-1})^2\\ &\ge a[(\Sigma p_i\,\alpha_i\,x_i^r)(\Sigma p_i\,\alpha_i\,u_i\,x_i^{r-2}) \end{aligned}$$

$$- (\Sigma p_i\ \alpha_i\ u_i\ x_i^{r-1})^2]$$
$$= a \sum_{i<j} p_i p_j\ \alpha_i \alpha_j\ x_i^{r-2}\ x_j^{r-2}(u_i\ x_j - u_j\ x_i)^2 \geq 0 .$$

Therefore:

$$x''_{r,M}(0) \geq 0 \ , \ \text{if} \ \ r > 1 \ , \ 0 < M \leq 2^r \ , \ M \neq 1 \tag{20}$$

$$x''_{r,M}(0) \leq 0 \ , \ \text{if} \ \ r < 1 \ , \ 0 < M \leq 2^r \ , \ M \neq 1 \tag{21}$$

In fact, equality can appear in (20) and (21) for $p_i \neq 0$, $i = 1,2,\ldots,m$ only if $u_i\ x_j - u_j\ x_i = 0$, $i,j = 1,2,..,m$.

Let us return to the function $\Psi_{r,M} = \Phi_{r,M} + \varepsilon_{r,M}$.

It is easy to see that for M close enough to 2^r, say $|M - 2^r| < \eta'$, $\Psi_{r,M}$ is monotonic in the same direction as $\Phi_{r,M}$. The mentioned properties of these functions and the fact that $\lim_{M\to 2^r} \varepsilon_{r,M}(x) = 0$ allows us to write:

$$\Phi^{-1}_{r,M}(y) = \Psi^{-1}_{r,M}(y) + w_{r,M}(y) \tag{22}$$

where $\Psi^{-1}_{r,M}$ is continuously differentiable and

$$\lim_{M\to 2^r} w_{r,M}(y) = 0 . \tag{23}$$

Then we can apply the mean value theorem for $\Psi^{-1}_{r,M}(t)$ and write:

$$\begin{aligned} x_{r,M}(t) &= \Phi^{-1}_{r,M}(\Sigma p_i\ \Phi_{r,M}(x_i + u_i t)) \\ &= \Psi^{-1}_{r,M}(\Sigma p_i\ \Psi_{r,M}(x_i + u_i t) - \Sigma p_i\ \varepsilon_{r,M}(x_i + u_i t)) \\ &+ w_{r,M}(\Sigma p_i\ \Phi_{r,M}(x_i + u_i t)) = \Psi^{-1}_{r,M}(\Sigma p_i\ \Psi_{r,M}(x_i + u_i t)) \\ &- \Sigma p_i\ \varepsilon_{r,M}(x_i + u_i t)\ \frac{d}{dy}\ \Psi^{-1}_{r,M}(y') \\ &+ w_{r,M}(\Sigma p_i\ \Phi_{r,M}(x_i + u_i t)) \ , \end{aligned}$$

where y' is a point between $\Sigma p_i\ \Psi_{r,M}(x_i + u_i t) - \Sigma p_i\ \varepsilon_{r,M}(x_i + u_i t)$ and $\Sigma p_i\ \Psi_{r,M}(x_i + u_i t)$.

Finally

$$x_{r,M}(t) = \lambda_{r,M}(t) + \mu_{r,M}(t) \ , \tag{24}$$

where

$$\lambda_{r,M}(t) = \lambda_{r,M}(t\ ;\ x_1,\ldots,x_m) = \Psi^{-1}_{r,M}(\Sigma p_i\ \Psi_{r,M}(x_i + u_i t) \tag{25}$$

and

$$\lim_{M\to 2^r} \mu_{r,M}(t) = 0 . \tag{26}$$

The functions from the relation (25) are piecewise indefinitely differentiable and, where the second derivative exists, we have:

$$x''_{r,M}(t) = \lambda''_{r,M}(t) + \mu''_{r,M}(t) \tag{27}$$

and

$$\lim_{M\to 2^r} \mu''_{r,M}(t) = 0 . \tag{28}$$

The relations (27) and (28) allow us to say that if $x''_{r,M}(0) = x''_{r,M}(0 ; x_1,\dots,x_m) > 0$, then there is an $\eta_0 > 0$, $\eta_0 < \eta'$ such that

$$|2^r - M| < \eta_0 \tag{29}$$

implies

$$\lambda''_{r,M}(0) = \lambda''_{r,M}(0 ; x_1,\dots,x_m) > 0 . \tag{30}$$

Let us denote

$$I = \underbrace{[1,2] \times [1,2] \times \dots \times [1,2]}_{m \text{ times}}$$

and suppose $p , u \in R^+_m$ fixed and $r > 1$, $0 < M \le 2^r$, $M \ne 1$.

Then there is an open dense subset $S \subset I$ such that

$$x''_{r,M}(0 ; x_1,\dots,x_m) > 0 , \ (x_1,\dots,x_m) \in S .$$

Each point $(x^0_1,\dots,x^0_m) \in I$ has a neighborhood V such that for $\eta > 0$ suitably chosen,

$$\lambda''_{r,M}(0 ; x_1,\dots,x_m) > 0$$

if $2^r - \eta < M \le 2^r$, $(x_1,\dots,x_m) \in V$.

Let $V_1,\dots,V_p$ be a finite covering of I and let $\eta_1,\dots,\eta_p$ be the corresponding values of η .

Then for $2^r - \delta' < M \le 2^r$, where $\delta' = \min\{\eta_1,\dots,\eta_p\}$, we have

$$\lambda''_{r,M}(0 ; x_1,\dots,x_m) > 0 \tag{31}$$

on an open dense subset of I . The fact that for $k \in \mathbb{Z}$,

$$\Psi^{-1}_{r,M}(\Sigma p_i \ \Psi_{r,M}(2^k x_i)) = 2^k \Psi^{-1}_{r,M}(\Sigma p_i \ \Psi_{r,M}(x_i))$$

assures the validity of the relation (31) on an open dense subset of R^+_m .

Since $\lambda_{r,M}(t)$ is continuously differentiable, this implies that $\lambda_{r,M}$ is a convex function, if $r > 1$ and $2^r - \delta' < M \le 2^r$.

By ([6], page 86), $\Psi_{r,M}$ verifies a direct Minkowski type of inequality.

Similar arguments allow us to find a $\delta'' > 0$ such that, if $r < 1$, $2^r - \delta'' < M \le 2^r$, the function $\Psi_{r,M}$ verifies a reversed Minkowski type of inequality and the proposition is thus completely proved.

REFERENCES

[1] Allard, J. and Ghisa, D., *Mean Values Depending on Some General Functions*, Libertas Mathematica, t. IV (1984), pp. 1-9.

[2] Fuglede, B., Extremal Lengths and Functional Completion, Acta Math. 98 (1957), pp. 111-219.

[3] Ghisa, D., *Almost Extremal Functions by Means of a General Module*, Proc. of Atlantic Math. Days (1983), pp. 5-22.

[4] Ghisa, D. and Moore, M., *Module d'une famille de mesures et application aux ensembles aleatoires*, Ann. sci. math. Québec, vol. 8, no. 1 (1984), pp. 61-79.

[5] Ghisa, D., *On Clarkson Inequalities*, Libertas Mathematica, t. IV (1984), pp. 11-14.

[6] Hardy, G.H., Littlewood, J.E. and Polya, G., *Inequalities*, Cambridge Univ. Press (1973).

[7] Krasnosel'skii, M.A. and Rutickii, Ya.B., *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen (1961).

[8] Ohtsuka, M., *Extremal Lengths and Precise Functions in Three-Space*, Hiroshima Univ. (1973).