PERIODIC SOLUTIONS OF CERTAIN INTEGRODIFFERENTIAL SYSTEMS

C. Corduneanu

We shall consider in this paper some integrodifferential systems of the form

$$\dot{x}(t) = \int_R [dA(S)]x(t-s) + f(t) , \quad t \in R , \tag{1}$$

where x and f take values in C^n (the Hilbert space of n-vectors with complex coordinates), while A stands for a matrix of the type n by n , whose entries are real valued functions with bounded variation on the real line R.As usual , we will assume that A(s) is left continuous.

The problem we shall discuss consists in finding periodic solutions , of period $\omega > 0$, for the system (1) , assuming that $f \in L^2([0,\omega], C^n)$ and satisfies the periodicity condition $f(t+\omega) = f(t)$, a.e. on R.

Since periodic functions are uniquely determined by the corresponding Fourier series , it is natural - especially if we keep in mind the linearity of the equation - to try to construct such series for the periodic solutions of (1), if such solutions do exist.

Suppose that $f \in L^2([0,\omega], C^n)$ is such that

$$f(t) \sim \sum_{k \in Z} f_k \exp(i\omega_k t) , \tag{2}$$

where $\omega_k = 2k\pi/\omega$, $k \in Z$. It is then known that the so-called Bessel-Parseval equation

$$\sum_{k \in Z} |f_k|^2 = \omega^{-1}\int_0^{\omega} |f(t)|^2 dt \tag{3}$$

holds true.

Let us now seek a periodic solution of the equation (1) in the form of a Fourier series , namely

$$x(t) = \sum_{k \in Z} x_k \exp(i\omega_k t). \tag{4}$$

We certainly expect the series in the right hand side of (4) to be convergent in some sense. And to be practical, we should look for the best available kind of convergence under our assumptions. Nevertheless, it should be pointed out that having the Fourier series of a periodic function, it is possible to determine that function by means of constructive procedures (see, for instance, [4]). Cesaro's summability procedure is an example of this kind.

Since the solution $x(t)$ of (1) must have a derivative that belongs to the space $L^2([0,\omega],C^n)$, there results that the Fourier series of the derivative is obtained by formal differentiation of the Fourier series of the function. If we keep this in mind and substitute the series for $x(t)$, $\dot{x}(t)$, and $f(t)$ in equation (1), then the following relationships are obtained by equating the coefficients appearing in both sides of the equation:

$$i\omega_k x_k = \left(\int_R [\,dA(s)]\exp(-i\omega_k s)\right) x_k + f_k \ , \ k \in Z \ . \tag{5}$$

The equations (5) can be rewritten in the form

$$[\,i\omega_k I - \tilde{A}(i\omega_k)]\ x_k = f_k \ , \ k \in Z \ , \tag{6}$$

where

$$\tilde{A}(is) = \int_R [\,dA(t)]\exp(-ist) \tag{7}$$

is the Fourier-Stieltjes transform of $A(t)$. It does exist for all $s \in R$ due to our assumptions on the matrix valued function $A(t)$.

If we make now the assumption

$$\det[\,isI - \tilde{A}(is)\,] \neq 0 \quad \text{for } s \in R \ , \tag{8}$$

then each linear system (6) has unique solution for any f_k, $k \in Z$, and any positive ω. But condition (8) is obviously too strong if we are concerned with a fixed $\omega > 0$. It is, indeed, sufficient to impose the much weaker condition

$$\det[\,i\omega_k I - \tilde{A}(i\omega_k)\,] \neq 0 \ , \ k \in Z \ , \tag{9}$$

in order to secure the existence of a unique x_k, $k \in Z$. Moreover, it appears that even condition (9) requires too much from $A(t)$ in order to secure the above mentioned property. This will be made clear if we prove the following

<u>Lemma</u>. If $s \in R$ satisfies the equation

$$\det[\, isI - \tilde{A}(is)\,] = 0 , \tag{10}$$

then necessarily

$$|s| \le \int_R |dA(t)| = \gamma < \infty . \tag{11}$$

Proof. If condition (10) holds true for some $s \in R$, then the linear system $is\xi = \tilde{A}(is)\xi$ has nontrivial solutions $\xi \in C^n$. This implies obviously

$$|s| \le |\tilde{A}(is)| , \tag{12}$$

where the norm in the right hand side could be the Euclidean norm for matrices. On the other hand , from (7) one derives the inequality

$$|\tilde{A}(is)| \le \int_R |dA(t)| , \tag{13}$$

which combined with (12) implies (11) . Let us notice that the right hand side of (13) means the total variation of the matrix-valued function $A(t)$,i.e. , the sum of total variations of its entries .

Remark. The main consequence we can derive from the Lemma is the fact that the system (6) may fail to produce a unique solution x_k only for those values of $k \in Z$ for which $|\omega_k| \le \gamma$, which means actually $|k| \le \omega\gamma/(2\pi)$. Therefore,there can be only finitely many values of k for which the system (6) does not produce a unique solution (for every f_k).

Of course , the loss of uniqueness can occur only for those $k \in Z$ for which ω_k is a solution of the equation (10). One can say that ω_k is a characteristic root of the equation (1) .

Before we state the main result on the existence and uniqueness of the periodic solution for the equation (1) , let us notice the fact there exists a unique nonnegative integer p , such that

$$\omega_p \le \gamma < \omega_{p+1} . \tag{14}$$

Theorem. Consider the system (1) in which $f \in L^2([0,\omega], C^n)$, $f(t + \omega) = f(t)$ a.e. on R , and $A(t)$ is a matrix of type n by n whose entries are functions with bounded variation on R , continuous from the left .

If condition (9) holds true only for $|k| \le p$, where p is determined by (14) , then there exists a unique periodic solution $x(t)$ of the system (1) which is absolutely continuos and satisfies the system a.e. The Fourier series of the solution is

$$\sum_{k \in Z} [i\omega_k I - \tilde{A}(i\omega_k)]^{-1} f_k \exp(i\omega_k t) , \tag{15}$$

and it converges absolutely and uniformly to $x(t)$.

Proof. It is obvious that the series (15) can be constructed unde the hypotheses of the Theorem , with condition (9) for $|k| \leq p$ playing the central

The absolute and uniform (on R) convergence of the series (15) follows immediately from the estimate

$$|[i\omega_k I - \tilde{A}(i\omega_k)]^{-1} f_k| \leq A_o(\omega_k^{-2} + |f_k|^2) , \tag{16}$$

which holds true for $|k| > p+1$, with $A_o > 0$ conveniently chosen .This is always possible , because the Fourier-Stieltjes transform of any function with bounded variation is bounded on R .

If we denote by $x(t)$ the sum of the series (15) , it only has to be shown that $x(t)$ is indeed a solution of (1) . We shall achieve this conclusion by comparing the Fourier series of both sides of equation (1) , after substituting the series (15) for $x(t)$.

The series obtained by formal differentiation of series (15) has the coefficients

$$i\omega_k[i\omega_k I - \tilde{A}(i\omega_k)]^{-1} f_k = b_k , \; k \in Z , \tag{17}$$

and it can be easily seen that

$$\sum_{k \in Z} |b_k|^2 < \infty .$$

Therefore , we can assert the existence of a unique function in $L^2([0,\omega] , C^n)$ whose Fourier coefficients are exactly b_k , $k \in Z$. Due to the uniqueness of the Fourier series , this function cannot be but $\dot{x}(t)$.

On the other hand , the integral in the right side of (1) does exist for $x(t)$ constructed above , and we can easily calculate (because of the uniform convergence of (15)) the Fourier coefficients of the convolution product . We obtain

$$\int_R [dA(s)] x(t-s) = \int_R [dA(s)] \sum_{k \in Z} x_k \exp(i\omega_k(t-s))$$

$$= \sum_{k \in Z} \left(\int_R [dA(s)] \exp(-i\omega_k s)\right) x_k \exp(i\omega_k t) =$$

$$= \sum_{k \in Z} \tilde{A}(i\omega_k)x_k \exp(i\omega_k t) ,$$

where x_k is given by (6) . Therefore , the Fourier coefficients of the right hand side of (1) are

$$\tilde{A}(i\omega_k)[i\omega_k I - \tilde{A}(i\omega_k)]^{-1} f_k + f_k =$$

$$[\tilde{A}(i\omega_k) + i\omega_k I - \tilde{A}(i\omega_k)][i\omega_k I - \tilde{A}(i\omega_k)]^{-1} f_k =$$

$$i\omega_k[i\omega_k I - \tilde{A}(i\omega_k)]^{-1} f_k = b_k .$$

Since b_k are the Fourier coefficients of $\dot{x}(t)$, the Theorem is thereby proven .

Remark 1. The estimate given in the Lemma for the solutions of the equation (10) is not certainly the best possible in all cases . A good example to think about is that of ordinary differential systems $\dot{x} = Ax + f(t)$, which are special cases of the system (1) . Consequently , the condition (9) , for $|k| \leq p$, may contain some superfluous assumptions (i.e. , the inequality is automatically verified when $i\omega_k$ does not belong to the smallest interval $[-iT,iT]$ which contains all the roots of equation (10)).

Remark 2. A very interesting situation in regard to the existence of periodic solutions to the equation (1) occurs when condition (9) is violated for certain k's , with $|k| \leq p$. In such a case , the system (6) is either deprived of solution , or it has infinitely many solutions . This situation is known as the second case of Fredholm's alternative , and the existence of solutions is guaranteed by an orthogonality condition which involves the term f(t) , as well as the periodic solutions of the homogeneous adjoint equation . We do not get into details in this regard , but we will notice that this problem is largely discussed in a series of publications on this subject (see , for instance , [2] , [3], [5 - 8] .

Remark 3. Under the assumptions of the Theorem , we easily derive from (6) an inequality of the form

$$|x_k| \leq \bar{A}|f_k|, \ k \in Z .$$

The positive constant $\bar{A}$ depends only of A(t) . If we take into account Bessel-Parseval's equation (3) , then we obtain the following inequality in the L^2-norm:

$$|x|_{L^2} \leq |f|_{L^2} ,$$

where x stands for the unique periodic solution , of period ω, of the system (1).

The estimate found above for the L^2 - norm of the unique periodic solution of (1) is useful in regard to the existence of periodic solutions to the nonlinear system obtained by perturbing the system (1) , namely

$$\dot{x}(t) = \int_R [dA(s)]x(t-s) + f(t;x) , \quad (18)$$

where $f(t;x) = (fx)(t)$ is an operator on the space $L^2([0,\omega],C^n)$, satisfying a Lipschitz condition of the form

$$|fx - fy|_{L^2} \leq \lambda |x - y|_{L^2} .$$

It can be easily shown by means of contraction mapping principle that the system (18) has also a unique periodic solution , of period $\omega > 0$, provided $\lambda\bar{A} < 1$.

More problems related to the topic discussed in this paper can be found in the references listed below .

R E F E R E N C E S

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