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ON MULTIVARIATE GENERALIZATIONS OF HARDY'S INEQUALITY

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Abstract. The object of this paper is to establish some new multivariate generalizations of the well known Hardy's integral inequality. Our results are obtained by using the Fubini's theorem and the Jensen's integral inequality.

1. Introduction

In a celebrated paper of 1920, G. H. Hardy [3] proved a remarkable integral inequality which can be stated as follows.

If $p > 1$, $f(x) \geq 0$, $0 < x < \infty$ and $F(x) = \int_0^x f(t)dt$, then

$$(1)\qquad \int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p\, dx ,$$

unless $f \equiv 0$.

In the past seventy five years many alternative proofs of Hardy's inequality (1) were found, and more general results were obtained, see [1-10] and the references given therein. Although stimulating research works have been undertaken in this direction, it appears that there is no natural multivariate version of (1) in the literature. Our objective here is to present some new multivariate generalizations of Hardy's inequality (1) which in turn are the natural multivariate versions of some of the results given by Levinson in [6]. The method we use to obtain our results is based on the applications of the well known Fubini's theorem and the Jensen's integral inequality.

2. Statement of results

In what follows, we let B be a subset of the n-dimensional Euclidean space R^n defined by $B = \{ x \in R^n : 0 < x < \infty \}$ where $0 = (o,\ldots,o) \in R^n$. We denote by $\int_B u(z)dz$ and $\int_{B_{x,y}} u(z)dz$ the n-fold integrals $\int_0^\infty \ldots \int_0^\infty u(z_1,\ldots,z_n)dz_n\ldots dz_1$ and $\int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} u(z_1,\ldots,z_n)dz_n\ldots dz_1$ respectively, where $x = (x_1,\ldots,x_n) \in B$, $y = (y_1,\ldots,y_n) \in B$ such that $x < y$ i.e. $x_i < y_i$. Throughout this paper without further mention, we assume that all inequalities between vectors are componentwise and all the integrals exists on the respective domains of their definations.

A generalization of Hardy's inequality (1) to the case of multivariate functions is contained in the following theorem.

Theorem 1. Let $p > 1$ be a constant. Let $f(x)$ be a nonnegative and integrable function on B and let $r_i(x_i)$, $i = 1,\ldots,n$ be positive and absolutely continuous functions on $(0,\infty)$ and let

$$(2) \qquad R_i(x_i) = \int_0^{x_i} r_i(y_i)dy_i ,$$

exists. Let

$$(3) \qquad 1 + \left(\frac{1}{p-1}\right) \frac{R_i(x_i)r_i'(x_i)}{r_i^2(x_i)} \geq \frac{1}{\alpha_i} ,$$

for all $x_i > 0$ and for some positive constants α_i for $i = 1,\ldots,n$. If $F(x)$ is defined by

$$(4) \qquad F(x) = \frac{1}{\prod_{i=1}^{n} R_i(x_i)} \int_{B_{0,x}} \left(\prod_{i=1}^{n} r_i(y_i) \right) f(y)dy,$$

for $x \in B$, then

$$(5) \qquad \int_B F^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1} \right)^p \int_B f^p(x)dx .$$

The equality holds in (5) if $f(x) \equiv 0$.

A slightly different version of Theorem 1 is established in the following theorem.

Theorem 2. Let p and f be as in Theorem 1. Let $r_i(x_i)$, $i = 1,\ldots,n$ be positive, continuous and monotone nondecreasing functions on $(0,\infty)$. If $R_i(x_i)$ and $F(x)$ be as defined in (2) and (4) respectively, where $r_i(x_i)$ are as defined above, then

$$\int_B F^p(x)dx \le \left(\frac{p}{p-1}\right)^{np} \int_B f^p(x)dx \quad (6)$$

The equality holds in (6) if $f(x) \equiv 0$.

Remark 1. We note that in the special cases when $r_i(x_i) = 1$ and $\alpha_i = 1$ in (3), the inequalities established in Theorems 1 and 2 reduces to the natural multivariate version of the Hardy's inequality given in (1) which in turn yields the inequality (1) when n=1.

We next establish the following multivariate version of the inequality given by Levinson in [6].

Theorem 3. Let p, f, r_i, $i = 1,\dots,n$ be as defined in Theorem 1. Let

$$(7) \qquad 1 + \left(\frac{p}{p-1} \right) \frac{x_i r_i'(x_i)}{r_i(x_i)} \geq \frac{1}{\beta_i} ,$$

for all $x_i > 0$ and for some positive constants β_i for $i = 1,\dots,n$. If $G(x)$ is defined by

$$(8) \qquad G(x) = \frac{1}{\prod_{i=1}^{n} x_i r_i(x_i)} \int_{B_{o,x}} \left(\prod_{i=1}^{n} r_i(y_i) \right) f(y)dy,$$

for $x \in B$, then

$$(9) \qquad \int_B G^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\beta_i}{p-1} \right)^p \int_B f^p(x)dx .$$

The equality holds in (9) if $f(x) \equiv 0$.

Another interesting variant of Theorem 3 is embodied in the following theorem.

Theorem 4. Let p, f, r_i, $i = 1,\ldots,n$ be as defined in Theorem 1. Let

$$(10) \qquad 1 - \left(\frac{p}{p-1}\right) \frac{x_i r_i'(x_i)}{r_i(x_i)} \geq \frac{1}{\gamma_i} ,$$

for all $x_i > 0$ and for some positive constants γ_i for $i = 1,\ldots,n$. If $H(x)$ is defined by

$$(11) \qquad H(x) = \prod_{i=1}^{n} \left(\frac{r_i(x_i)}{x_i}\right) \int_{B_{o,x}} \left(\frac{1}{\prod_{i=1}^{n} r_i(y_i)}\right) f(y)dy,$$

for $x \in B$, then

$$(12) \qquad \int_B H^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\gamma_i}{p-1}\right)^p \int_B f^p(x)dx .$$

The equality holds in (12) if $f(x) \equiv 0$.

Remark 2. In the special case when $n = 1$, the inequality established in Theorem 3 reduces to the inequality established by Levinson in [6, Theorem 4]. We also note that the method employed in the proofs of our Theorems 1-4 can very easily be extended to obtain multivariate versions of the inequalities recently given by Lee and Yang in [5, Theorems 3.1 and 3.3] which in turn are the further extensions of some of the results given by Levinson in [6].

In the following theorems we establish the multivariate versions of the inequalities given by Levinson in [6, Theorems 3 and 2] .

Theorem 5. Let $\phi(u) \geq 0$ be defined on an open interval, finite or infinite and at the ends of the interval let ϕ take its limiting values, finite or infinite. For some $p > 1$, let $\phi^{1/p}(u)$ be convex. Let $r_i(x_i)$ and $R_i(x_i)$ be as defined in Theorem 1 satisfying the condition (3). If for $x \in B$, the range of values of $f(x)$ lie in the closed interval of definition of ϕ and $\phi(f(x))$ is integrable on B, and if $F(x)$ is defined by (4), then

$$(13) \qquad \int_B \phi(F(x))dx \leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1} \right)^p \int_B \phi(f(x))dx .$$

The equality holds in (13) if $\phi(f(x)) \equiv 0$.

Theorem 6. Let ϕ, p, $\phi^{1/p}$, f, $\phi(f)$ be as defined in Theorem 5. Let $r_i(x_i)$ and $R_i(x_i)$ be as defined in Theorem 2. If F(x) is defined as in Theorem 2, then

$$(14) \qquad \int_B \phi(F(x))dx \le \left(\frac{p}{p-1} \right)^{np} \int_B \phi(f(x))dx .$$

The equality holds in (14) if $\phi(f(x)) \equiv 0$.

Remark 3. The inequalities obtained in Theorems 5 and 6 can be considered as the multivariate versions of the inequalities established by Levinson in [6, Theorems 3 and 2]. However, our hypotheses on $\phi(u)$ differs slightly from those of used by Levinson in [6]. Here the condition $\phi''(u) \geq 0$ used in [6, p.389] is not needed and the condition

$$(15) \qquad \phi \phi'' \geq (1- \frac{1}{p})(\phi')^2 , \ p > 1 ,$$

used in [6,p.389] is replaced by $\phi^{1/p}$ is convex, since this is all that is required for the application of Jensen's inequality in the proofs of our results. We also note that the method employed in the proofs of our Theorems 5 and 6 can be very easily extended to obtain the multivariate versions of the results given by Lee and Yang in [5, Theorems 2.2 and 3.2] .

3. Proofs of Theorems 1-4

Let $a = (a_1,\dots,a_n) \in B$, $b = (b_1,\dots,b_n) \in B$, $0 = (o,\dots,o) \in R^n$ be such that $0 < a < b < \infty$ and define

$$(16) \qquad F_a(x) = \frac{1}{\prod_{i=1}^{n} R_i(x_i)} \int_{B_{a,x}} \left(\prod_{i=1}^{n} r_i(y_i) \right) f(y)dy,$$

for $x \in B$. From (16) and by Fubini's theorem (see, [1,p.18]) we have

$$(17)\quad \int_{B_{a,b}} F_a^p(x)dx$$

$$= \int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{\left(\prod_{i=1}^{n} R_i(x_i) \right)^p} \times$$

$$\times \left[\int_{a_n}^{b_n} R_n^{-p}(x_n) \left[\int_{a_n}^{x_n} r_n(y_n) \times \right.\right.$$

$$\times \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) \times \right.$$

$$\left.\left.\left. \times f(y_1,\ldots,y_n)dy_{n-1}\ldots dy_1 \right)dy_n \right]^p dx_n \right] \times$$

$$\times dx_{n-1}\ldots dx_1 .$$

By keeping $x_1,\ldots,x_{n-1}$ fixed and integrating by parts we have

$$(18)\quad \int_{a_n}^{b_n} R_n^{-p}(x_n)\left[\int_{a_n}^{x_n} r_n(y_n)\left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i)\right)\times \right.\right.$$

$$\left.\left. \times f(y_1,\ldots,y_n)dy_{n-1}\ldots dy_1 \right)dy_n \right]^p dx_n$$

$$= \int_{a_n}^{b_n} R_n^{-p}(x_n) r_n(x_n) \frac{1}{r_n(x_n)} \Big[\int_{a_n}^{x_n} r_n(y_n) \times$$

$$\times \Big(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \Big(\prod_{i=1}^{n-1} r_i(y_i)\Big) f(y_1,\ldots,y_n) \times$$

$$\times\, dy_{n-1} \cdots dy_1 \Big) dy_n \Big]^p dx_n$$

$$= \frac{R_n^{-p+1}(b_n)}{-p+1} \frac{1}{r_n(b_n)} \Big[\int_{a_n}^{b_n} r_n(y_n) \Big(\int_{a_1}^{x_1} \cdots \times$$

$$\times \int_{a_{n-1}}^{x_{n-1}} \Big(\prod_{i=1}^{n-1} r_i(y_i) f(y_1,\ldots,y_n) \times$$

$$\times\, dy_{n-1} \cdots dy_1 \Big) dy_n \Big]^p$$

$$+ \Big(\frac{p}{p-1} \Big) \int_{a_n}^{b_n} R_n^{-p+1}(x_n) \Big[\int_{a_n}^{x_n} r_n(y_n) \times$$

$$\times \Big(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \Big(\prod_{i=1}^{n-1} r_i(y_i) \Big) \times$$

$$\times\, f(y_1,\ldots,y_n) dy_{n-1} \cdots dy_1 \Big) dy_n \Big]^{p-1} \times$$

$$\times \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) f(y_1, \ldots, y_{n-1}, x_n) \times \right.$$

$$\left. \times \, dy_{n-1} \ldots dy_1 \right) dx_n$$

$$- \left(\frac{1}{p-1} \right) \int_{a_n}^{b_n} \frac{R_n^{-p+1}(x_n) r_n'(x_n)}{r_n^2(x_n)} \left[\int_{a_n}^{x_n} r_n(y_n) \times \right.$$

$$\times \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) f(y_1, \ldots, y_n) \times \right.$$

$$\left. \left. \times \, dy_{n-1} \ldots dy_1 \right) dy_n \right]^p dx_n .$$

Since $p > 1$, from (18) we observe that

$$(19) \quad \int_{a_n}^{b_n} \left[1 + \left(\frac{1}{p-1} \right) \frac{R_n(x_n) r_n'(x_n)}{r_n^2(x_n)} \right] R_n^{-p}(x_n) \times$$

$$\times \left[\int_{a_n}^{x_n} r_n(y_n) \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) \times \right. \right.$$

$$\left. \left. \times \, f(y_1, \ldots, y_n) dy_{n-1} \ldots dy_1 \right) dy_n \right]^p dx_n$$

$$\leq \left(\frac{p}{p-1}\right) \int_{a_n}^{b_n} R_n^{-p+1}(x_n) \left[\int_{a_n}^{x_n} r_n(y_n) \left(\int_{a_1}^{x_1} \cdots \times \right.\right.$$

$$\times \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) f(y_1,\ldots,y_n) \times$$

$$\left.\times\, dy_{n-1}\cdots dy_1 \right) dy_n \Big]^{p-1} \times$$

$$\times \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) \times \right.$$

$$\left. \times\, f(y_1,\ldots,y_{n-1},x_n)\,dy_{n-1}\cdots dy_1 \right) dx_n \,.$$

From (3) and applying Hölder's inequality with indices $p/(p-1)$, p on the right side of (19) we obtain

$$(20) \qquad \int_{a_n}^{b_n} R_n^{-p}(x_n) \left[\int_{a_n}^{x_n} r_n(y_n) \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i)\right) \times \right.\right.$$

$$\left.\times\, f(y_1,\ldots,y_n)\,dy_{n-1}\cdots dy_1\right) dy_n \Big]^{p} dx_n$$

$$\leq \left(\frac{p\alpha_n}{p-1}\right)\left[\int_{a_n}^{b_n} R_n^{-p}(x_n)\left[\int_{a_n}^{x_n} r_n(y_n)\left(\int_{a_1}^{x_1}\cdots\times\right.\right.\right.$$

$$\times\int_{a_{n-1}}^{x_{n-1}}\left(\prod_{i=1}^{n-1} r_i(y_i)\right)f(y_1,\ldots,y_n)\times$$

$$\left.\left.\left.\times\, dy_{n-1}\ldots dy_1\right)dy_n\right]^p dx_n\right]^{(p-1)/p}\times$$

$$\times\left[\int_{a_n}^{b_n}\left[\int_{a_1}^{x_1}\cdots\int_{a_{n-1}}^{x_{n-1}}\left(\prod_{i=1}^{n-1} r_i(y_i)\right)\times\right.\right.$$

$$\left.\left.\times\, f(y_1,\ldots,y_n)dy_{n-1}\ldots dy_1\right]^p dx_n\right]^{1/p} .$$

Dividing both sides of (20) by the first integral factor on the right side of (20) and then raising both sides to the p th power we obtain

$$(21)\qquad \int_{a_n}^{b_n} R_n^{-p}(x_n)\left[\int_{a_n}^{x_n} r_n(y_n)\left(\int_{a_1}^{x_1}\cdots\int_{a_{n-1}}^{x_{n-1}}\times\right.\right.$$

$$\times\left(\prod_{i=1}^{n-1} r_i(y_i)\right)f(y_1,\ldots,y_n)\times$$

$$\left.\left.\times\, dy_{n-1}\ldots dy_1\right)dy_n\right]^p dx_n$$

$$\leq \left(\frac{p\alpha_n}{p-1} \right)^p \int_{a_n}^{b_n} \left[\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) \times \right.$$

$$\left. \times\, f(y_1,\ldots,y_{n-1},x_n)dy_{n-1}\ldots dy_1 \right]^p dx_n \,.$$

Substituting (21) in (17) and using Fubini's theorem we have

$$(22) \quad \int_{B_{a,b}} F_a^p(x)dx$$

$$\leq \left(\frac{p\alpha_n}{p-1} \right)^p \int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} \frac{1}{\left(\prod_{i=1}^{n-1} R_i(x_i) \right)^p} \times$$

$$\times \left[\int_{a_n}^{b_n} \left[\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) \times \right.\right.$$

$$\left. \times\, f(y_1,\ldots,y_{n-1},x_n)dy_{n-1}\ldots dy_1 \right]^p \times$$

$$\left. \times\, dx_n \right] dx_{n-1}\ldots dx_1$$

$$= \left(\frac{p\alpha_n}{p-1}\right)^p \int_{a_1}^{b_1} \cdots \int_{a_{n-2}}^{b_{n-2}} \int_{a_n}^{b_n} \frac{1}{\left(\prod_{i=1}^{n-2} R_i(x_i)\right)^p} \times$$

$$\times \left[\int_{a_{n-1}}^{b_{n-1}} R_{n-1}^{-p}(x_{n-1}) \left[\int_{a_{n-1}}^{x_{n-1}} r_{n-1}(y_{n-1}) \times \right.\right.$$

$$\times \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-2}}^{x_{n-2}} \left(\prod_{i=1}^{n-2} r_i(y_i) \right) \times \right.$$

$$\left.\left. \times f(y_1,\dots,y_{n-1},x_n) dy_{n-2}\cdots dy_1 \right) dy_{n-1} \right]^p \times$$

$$\left. \times dx_{n-1} \right] dx_n dx_{n-2}\cdots dx_1 .$$

Now by following exactly the same arguments as above we obtain

$$(23) \quad \int_{a_{n-1}}^{b_{n-1}} R_{n-1}^{-p}(x_{n-1}) \left[\int_{a_{n-1}}^{x_{n-1}} r_{n-1}(y_{n-1}) \left(\int_{a_1}^{x_1} \cdots \times \right.\right.$$

$$\times \int_{a_{n-2}}^{x_{n-2}} \left(\prod_{i=1}^{n-2} r_i(y_i) \right) f(y_1,\dots,y_{n-1},x_n) \times$$

$$\left.\left. \times dy_{n-2}\cdots dy_1 \right) dy_{n-1} \right]^p dx_{n-1}$$

$$\leq \left(\frac{p\alpha_{n-1}}{p-1} \right)^p \int_{a_{n-1}}^{b_{n-1}} \left[\int_{a_1}^{x_1} \dots \int_{a_{n-2}}^{x_{n-2}} \left(\prod_{i=1}^{n-2} r_i(y_i)\right) \times \right.$$

$$\left. \times f(y_1,\dots,y_{n-2},x_{n-1},x_n) dy_{n-2}\dots dy_1 \right]^p dx_{n-1}.$$

Substituting (23) in (22) we have

$$(24) \quad \int_{B_{a,b}} F_a^p(x)dx$$

$$\leq \left(\frac{p\alpha_n}{p-1} \right)^p \left(\frac{p\alpha_{n-1}}{p-1} \right)^p \int_{a_1}^{b_1} \dots \int_{a_{n-2}}^{b_{n-2}} \int_{a_n}^{b_n} \times$$

$$\times \frac{1}{\left(\prod_{i=1}^{n-2} R_i(x_i) \right)^p} \left[\int_{a_{n-1}}^{b_{n-1}} \left[\int_{a_1}^{x_1} \dots \times \right.\right.$$

$$\times \int_{a_{n-2}}^{x_{n-2}} \left(\prod_{i=1}^{n-2} r_i(y_i) \right) \times$$

$$\times f(y_1,\dots,y_{n-2},x_{n-1},x_n) \times$$

$$\left.\left. \times dy_{n-2}\dots dy_1 \right]^p dx_{n-1} \right] \times$$

$$\times dx_n dx_{n-2}\dots dx_1 .$$

Continuing in this way, we finally get

$$(25)\quad \int_{B_{a,b}} F_a^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1}\right)^p \int_{B_{a,b}} f^p(x)dx .$$

Let $c = (c_1,\ldots,c_n) \in B$ and $a < c < b$. Then from (25) we have

$$(26)\quad \int_{B_{c,b}} F_a^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1}\right)^p \int_B f^p(x)dx .$$

Letting $a \to 0$ i.e. $a_i \to o$ on the left side of (26) we have

$$(27)\quad \int_{B_{c,b}} F^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1}\right)^p \int_B f^p(x)dx .$$

Since this holds for arbitrary $0 < c < b$, it follows that

$$(28)\quad \int_B F^p(x)dx \leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1}\right)^p \int_B f^p(x)dx .$$

The proof of the Theorem 1 is complete.

The proof of Theorem 2 proceeds in very much the same way as in the proof of Theorem 1. By following the same arguments as in the proof of Theorem 1 we obtain (18). Since $r_i(x_i)$ are monotone nondecreasing, from (18) we observe that (see [6,p.391])

$$\int_{a_n}^{b_n} R_n^{-p}(x_n) \Big[\int_{a_n}^{x_n} r_n(y_n) \Big(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \times$$

$$\times \Big(\prod_{i=1}^{n-1} r_i(y_i) \Big) f(y_1,\ldots,y_n) dy_{n-1} \ldots dy_1 \Big) \times$$

$$\times\, dy_n \Big]^p dx_n$$

$$\leq \Big(\frac{p}{p-1} \Big) \int_{a_n}^{b_n} R_n^{-p+1}(x_n) \Big[\int_{a_n}^{x_n} r_n(y_n) \times$$

$$\times \Big(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \Big(\prod_{i=1}^{n-1} r_i(y_i) \Big) \times$$

$$\times\, f(y_1,\ldots,y_n) dy_{n-1} \ldots dy_1 \Big) dy_n \Big]^{p-1} \times$$

$$\times \left(\int_{a_1}^{x_1} \cdots \int_{a_{n-1}}^{x_{n-1}} \left(\prod_{i=1}^{n-1} r_i(y_i) \right) \times \right.$$

$$\left. \times f(y_1,\ldots,y_{n-1},x_n)dy_{n-1}\ldots dy_1 \right)dx_n .$$

The rest of the proof of Theorem 2 follows exactly the same steps as in the proof of Theorem 1 below the inequality (19) with suitable changes and hence we omit further details.

The proofs of Theorems 3 and 4 follow by the similar arguments as in the proof of Theorem 1 (see, also [6,p.393]) with suitable modifications. We omit the details.

4. Proofs of Theorems 5 and 6

Let $\psi(u) = (\phi(u))^{1/p} \geq 0$. Then $\psi(u)$ is convex. By repeated application of Jensen's inequality (see, [4,p.133]) we have

$$(29)\quad \psi(F(x)) \leq \frac{1}{\prod_{i=1}^{n} R_i(x_i)} \int_{B_{o,x}} \prod_{i=1}^{n} r_i(y_i)\psi(f(y))dy,$$

for $x \in B$. Applying (5) to $\psi(f(x))$ instead of $f(x)$ we have

$$(30)\quad \int_B \left[\frac{1}{\prod_{i=1}^{n} R_i(x_i)} \int_{B_{o,x}} \left(\prod_{i=1}^{n} r_i(y_i)\right)\psi(f(y))dy \right]^p dx$$

$$\leq \prod_{i=1}^{n} \left(\frac{p\alpha_i}{p-1} \right)^p \int_B \left(\psi(f(x)) \right)^p dx .$$

Using $\phi(u) = \psi^p(u)$ and (29) we have

$$(31)\quad \phi(F(x)) = \left(\psi(F(x)) \right)^p$$

$$\leq \left[\frac{1}{\prod_{i=1}^{n} R_i(x_i)} \int_{B_{o,x}} \left(\prod_{i=1}^{n} r_i(y_i) \right) \times \psi(f(y))dy \right]^p .$$

From (31) and (30) we observe that

$$\int_B \Phi(F(x))dx \le \int_B \Big[\frac{1}{\prod_{i=1}^{n} R_i(x_i)} \times$$

$$\times \int_{B_{o,x}} \Big(\prod_{i=1}^{n} r_i(y_i) \Big) \psi(f(y))dy \Big]^p dx$$

$$\le \prod_{i=1}^{n} \Big(\frac{p\alpha_i}{p-1} \Big)^p \int_B \big(\psi(f(x)) \big)^p dx$$

$$= \prod_{i=1}^{n} \Big(\frac{p\alpha_i}{p-1} \Big)^p \int_B \Phi(f(x))dx \ .$$

This completes the proof of Theorem 5.

The proof of Theorem 6 proceeds in very much the same way as in the proof of Theorem 5 with suitable changes and hence we omit the details.

Remark 4. The multidimensional variants of the Hardy's inequality (1) are recently given by Wannebo [10] by using different technique. Here we note that our results are established by using elementary analysis and we believe that the inequalities obtained in Theorems 1-6 are of independent interest.

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