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ON THE UNIQUE SOLVABILITY OF SEMILINEAR PROBLEMS WITH STRONGLY MONOTONE NONLINEARITY

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Abstract: *It is presented a method to solve semilinear equations in real Hilbert spaces. Some applications to differential equations are given.*

1. INTRODUCTION

In [1] it is studied semilinear equations of the form

(1) $$Au = F(u)$$

in a real Hilbert space H, where $A : D(A) \subset H \to H$ is a self-adjoint linear operator with the resolvent set $\rho(A)$ and $F : H \to H$ is a Gateaux differentiable gradient operator. In particular it is known that equation (1) possesses multiple solutions if the nonlinearity F interacts suitably with the spectrum of A. In [2] it is presented the following existence and uniqueness theorem, as a corollary to some general considerations on saddle points:

Theorem 1 (Amann). *Suppose that there exists real numbers $\nu < \mu$ such that $[\nu, \mu] \subset \rho(A)$ and*

(2) $$\nu \leq \frac{< F(u) - F(v), u - v >}{|u - v|^2} \leq \mu \ , \ \forall \ u, v \in H, u \neq v.$$

Then the equation $Au = F(u)$ possesses exactly one solution.

In this paper we consider the equation (1) of the form

(3) $$Au + F(u) = 0.$$

We establish an existence and uniqueness result for (3) asking a condition of type (2) for F and maximal monotony for A, but giving up from self-adjointness of A and Gateaux differentiability of F. The condition of maximal monotony for A is not very restrictive because the most known differential equations have this property.

2. THE MAIN RESULT

We give the following

Theorem 2. *Assume that* $A : D(A) \subset H \to H$ *is maximal monotone and there exist* $m, M > 0$ *such that*
(i) $< F(u) - F(v), u - v > \geq m \cdot |u - v|^2 \quad , \forall\, u, v \in H;$
(ii) $|F(u) - F(v)| \leq M \cdot |u - v| \quad , \forall\, u, v \in H.$
Then the equation (3) *has an unique solution.*

Proof: We shall use the following known result:

Lemma. *Suppose that* $F : H \to H$ *satisfy (i) and (ii). Then there exists* $\lambda > 0$ *such that* $S_\lambda : H \to H,\ \ S_\lambda(u) := u - \lambda F(u)$ *is a contraction.*

Indeed,

$$|S_\lambda(u) - S_\lambda(v)|^2 = |u - v|^2 - 2\lambda < F(u) - F(v), u - v > +\lambda^2 |F(u) - F(v)|^2 \leq$$

$$\leq (1 - 2\lambda m + \lambda^2 M)\, |u - v|^2 ,$$

thus

(4) $$|S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v| ,$$

with $c := \sqrt{1 - 2\lambda m + \lambda^2 M} < 1$, if $\lambda \in (0, \frac{2m}{M})$.

Now equation (3) can be written as

(5) $$(I + \lambda A)u - (u - \lambda F(u)) = 0,$$

or

(6) $$(I + \lambda A)u = S_\lambda(u),$$

where $\lambda > 0$ is taken from the lemma. Using the fact that $(I+\lambda A)$ is inversable and $|(I + \lambda A)^{-1}| \leq 1$ for each $\lambda > 0$ (because A is maximal monotone, e.g.[3],p.101) the equation (6) is equivalent with

(7) $$u = (I + \lambda A)^{-1} S_\lambda(u).$$

We have

$$|(I + \lambda A)^{-1} S_\lambda(u) - (I + \lambda A)^{-1} S_\lambda(v)| = |(I + \lambda A)^{-1}(S_\lambda(u) - S_\lambda(v))| \leq$$
$$\leq |(I + \lambda A)^{-1}| \cdot |S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v| \ , \ u, v \in H.$$

Therefore, $u \mapsto (I + \lambda A)^{-1} S_\lambda(u)$ is a contraction having an unique fixed point, thus (7) and consequently (3) has an unique solution.□

A similar result can be proved in the next case:

Theorem 3. *Suppose that F satisfy* (i)+(ii) *and $A : D(A) \subset H \to H$ is bounded, compact and monotone. Then the equation* (3) *has an unique solution.*

Proof: Equation (3) can be equivalently written as

(8) $$(\lambda I + A)u = T_\lambda(u),$$

where $T_\lambda(u) := \lambda u - F(u),\ \lambda > 0$. We have

$$|T_\lambda(u) - T_\lambda(v)|^2 = \lambda^2|u-v|^2 - 2\lambda < F(u) - F(v), u - v > +|F(u) - F(v)|^2 \leq$$
$$\leq (\lambda^2 - 2\lambda m + M^2)|u - v|^2,$$

therefore

(9) $$|T_\lambda(u) - T_\lambda(v)| \leq \sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|.$$

Let us choose $\lambda > \max\{\|A\|, \frac{M^2}{2m}\}$. In particular, $\lambda > \|A\|$ imply that $\lambda I + A$ is inversable because $\sigma(A) \subset [-\|A\|, \|A\|]$. Moreover,

(10) $$|(\lambda I + A)u|^2 = \lambda^2|u|^2 + 2\lambda(Au, u) + |Au|^2 \geq \lambda^2|u|^2,$$

(because A is monotone), or

$$|(\lambda I + A)u| \geq \lambda|u|,$$

hence $|(\lambda I + A)^{-1}| \leq \frac{1}{\lambda}$. Equation (8) is equivalent with

(11) $$u = (\lambda I + A)^{-1}T_\lambda(u).$$

We have

$$|(\lambda I + A)^{-1}T_\lambda(u) - (\lambda I + A)^{-1}T_\lambda(v)| = |(\lambda I + A)^{-1}(T_\lambda(u) - T_\lambda(v))| \leq$$
$$\leq |(\lambda I + A)^{-1}| \cdot |T_\lambda(u) - T_\lambda(v)| \leq \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda m + M^2} \cdot |u - v|.$$

Because $\lambda > \frac{M^2}{2m}$, it results that $\gamma := \frac{1}{\lambda}\sqrt{\lambda^2 - 2\lambda m + M^2} < 1$, therefore $u \longmapsto (\lambda I + A)^{-1}T_\lambda(u)$ is a contraction. Now equation (11) and consequently (3) has an unique solution.

Remark. Compactness and boundedness of A was used to choose a number $\lambda > 0$ such that $\lambda I + A$ is inversable. This is possible in weaker hypotesis. Indeed, the condition "*A compact and bounded*" can be replaced with "*spectrum of A is bounded*". We can state the more general result:

Theorem 3. *Let $F : H \to H$ satisfy* (i)+(ii) *and $A : D(A) \subset H \to H$ be monotone and the spectrum $\sigma(A)$ is bounded from below. Then equation (3) has an unique solution.*

Indeed, it can be repeated the proof from theorem 2 taking $\lambda > \dfrac{M^2}{2m}$ such that $-\lambda \in \rho(A)$.

3. APPLICATIONS

(A1). SEMILINEAR ELLIPTIC BOUNDARY PROBLEMS

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $a_{ij} \in C^1(\overline{\Omega})$, $1 \le i,j \le N$ having the ellipticity property

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \alpha\,|\xi|^2\,,\ \forall\,\xi \in \mathbb{R}^N$$

for some $\alpha > 0$. Let us consider the following elliptic problem

$$\text{(12)}\qquad \begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + g(x,u) = f(x) & in\ \ \Omega \\ u = 0 & on\ \ \partial\Omega \end{cases}$$

where the nonlinearity is given by the real valued function $f \in L^2(\Omega)$.
The particular case when $g(x,u) = a_0(x)u$, with $a_0 \in C(\overline{\Omega})$, $a_0 > p > 0$ is studied in [3],p.177 using Lax-Milgram theorem and in [1],p.165 using the above theorem 1. Now we suppose that $g(x,u)$ has partial derivative in u of the first order and

$$\text{(13)}\qquad m \le \frac{\partial g}{\partial u} \le M \quad in\ \Omega,\ (m, M > 0).$$

Under these hypoteses, problem (12) has an unique solution in weak sense, for every $f \in L^2(\Omega)$. Indeed, we can apply theorem 2 for the following functional background:

$$H = L^2(\Omega),\quad Au := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right),\quad D(A) := H^2(\Omega) \cap H_0^1(\Omega),$$

$F(u) := g(\cdot,u) - f$. A is monotone:

$$(Au,u) = \int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} \ge 0$$

and $I + A$ is surjective ([3],p.177), thus A is maximal monotone. The conditions (i) and (ii) follows from (13).

(A2). In [5] is studied the perturbed Laplace problem

$$(14)\qquad \begin{cases} -\Delta u + Pu = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

using the variational theorem of Langenbach.We can apply theorem, asking that $P : L^2(\Omega) \to L^2(\Omega)$ satisfy (i) and (ii). In particular, if P is Gateaux differentiable with

$$m \cdot |h|^2 \leq < (DP)(u)h, h > \leq M \cdot |h|^2, \quad (m, M > 0)$$

then (14) has an unique solution, because $Au := -\Delta u$, $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ is maximal monotone.

(A3). PERIODIC SOLUTIONS OF SEMILINEAR WAVE EQUATION

Suppose that $A : D(A) \subset H \to H$ is maximal monotone and $F \in C(\mathbb{R} \times H, H)$ such that, for some $T > 0$,

$$F(t+T, \cdot) = F(t, \cdot), \quad \forall\, t \in \mathbb{R}.$$

Then we are interested in the existence of $T-$periodic solutions for the semilinear abstract equation:

$$(15)\qquad \begin{cases} -u'' + Au + F(t,u) = 0\,, & t \in \mathbb{R} \\ u(0) = u(T)\,,\ u'(0) = u'(T) \end{cases} .$$

Let now $H := L^2((0,T); H)$ and $Lu := -u'' + Au$, with $D(L) := \{u \in C^2([0,T]; H) \cap L^2((0,T), D(A)) \mid u(0) = u(T)\,,\ u'(0) = u'(T)\}$.

L is maximal monotone and if F satisfy (i) and (ii), in particular, a condition of type (13), then problem (15) has exactly one periodic solution.

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