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HAMILTON-JACOBI EQUATION RELATED TO A CONTROL PROBLEM

CĂTĂLINA DAVIDEANU

Abstract. In this paper a control problem governed by a multivalued differential equation is studied.

The basic feature of the approach consists in approximating the corresponding Hamilton-Jacobi equation by means of the fractional steps scheme.

Two particular examples are entirely solved using the proposed method.

1. Introduction

We present a control problem with nonconvex and nondiferentiable cost functional and governed by a multivalued differential equation (nonlinear state equation), which have been studied in [6].

The characterization of the optimal arcs (maximum principle) is presented in Theorem 2.2. The maximum principle which was established is compared to that obtained in [4].

The difficulties appear from the maximal monoton operator $\beta : D(\beta) \subset R^n \to R^n$ since $\beta \neq \partial\varphi$ where $\partial\varphi$ is the subdifferential of a proper convex and lower semicontinuous function. In this case we cannot apply the result: if $y \in W^{1,2}(0,T;H)$ so that $y(t) \in D(\partial\varphi)$ a.e. $t \in]0,T[$ and there is $g \in L^2(0,T;H)$ with $g(t) \in \partial\varphi(y(t))$ a.e. $t \in]0,T[$, then the function $t \to \varphi(y(t))$ is absolutely continuous on $[0,T]$ and a.e. $]0,T[$ we get the equality

$$\frac{d}{dt}\varphi(y(t)) = (h, \frac{dy}{dt}(t)) \quad \text{for every} \quad h \in \partial\varphi(y(t)),$$

where H is a Hilbert space with finite dimension. For the proof see Lemma 2.2 from [8], pp.57.

The particular case $\beta = \partial I_k$, where ∂I_k is the subdifferential of the indicator function of the convex, closed set $K = \{z \in R^n;\ 0 \le z \le a_i,\ i = \overline{1,n}\}$ has been studied in [7]. Within the proof an essential role is played by the equality $\beta = \partial I_k$.

Problems of this type appear by discretization of parabolic variational inequalities and in the modelling of a class of physical problems.

2. Main results

We consider the following

Problem (P) Minimize the functional

$$\int_0^T (g(y(t)) + h(u(t)))dt + \varphi_o(y(T)), \quad 0 < T < \infty,$$

over the set of all functions $(y,u) \in W^{1,2}(0,T;R^n) \times U$ which satisfy the multivalued differential equation

(E) $$y'(t) + (F + \beta)(y(t)) \ni Bu(t) \text{ a.e. } t \in]0,T[,$$

with the initial condition

(IC) $$y(0) = x \in D(\beta).$$

The elements which appear in equation (E) have the following properties:

(i) $F:R^n \to R^n$ is a locally Lipschitz function which satisfies the condition: there are the constants $D_1, D_2 \in R_+^*$ such that

(1) $$\langle Fy,y\rangle \geq -D_1\|y\|^2 - D_2, \text{ for every } y \in R^n;$$

(ii) $\beta:D(\beta) \subset R^n \to R^n$ is a maximal monotone operator, with $0 \in D(\beta)$;

(iii) $B:R^m \to R^n$ is a linear continuous operator,

(iv) $U = \{u \in W^{1,2}(0,T,R^m); \|Bu\|^2_{L^2(0,T;R^n)} \leq A\}$ represents the set of admissible controls;

(v) the functions $g:R^n \to R_+$ and $\varphi_o:R^n \to R_+$ satisfy the condition: for every $\delta > 0$ there is $L_\delta > 0$ such that

$$|g(y) - g(z)| + |\varphi_o(y) - \varphi_o(z)| \leq L_\delta\|y - z\|,$$

for every $y,z \in R^n$ with $\|y\| + \|z\| \leq \delta$;

(vi) $h:R^m \to \bar{R}$ is a convex, lower semicontinuous function which satisfies the condition: there are $a_1 \in R_+^*$ and $a_2 \in R$ such that

$$h(u) \geq a_1\|u\|^2_m - a_2 \quad \text{for every } u \in R^m.$$

We make the supplementary, nonrestrictive, hypothesis

(H) $$S(0,C) \subset \text{int } D(\beta),$$

where

$$C = ((\|x\|^2 + 2D_2T + A)\, e^{(2D_1+1)T})^{1/2}$$

Thus, we take the initial data $x \in D(\beta)$, the constants $D_1, D_2 \in R_+^*$ and the set of admissible controls U such that the relation (H) be fulfilled.

We denote the set

$$X = \{x \in R^n;\ x \in D(\beta),\ x \text{ satisfies hypothesis (H)}\}$$

The basic results which are obtained there are

Theorem 2.1 For every $y_0 \in D(\beta)$ and $u \in W^{1,2}(0,T,R^m)$ the multivalued differential equation (E) with the initial condition $y(0)=y_0$ admits an unique solution $y \in W^{1,\infty}(0,T;R^n)$.

Moreover, y is differentiable to the right in every point of $[0,T[$ and we have

$$\frac{d^+y(t)}{dt} = (B(u(t)) - (F+\beta)(y(t)))^0;\quad t \in [0,T[,$$

where $(B(u(t))-(F+\beta)(y(t)))^0$ is the element of minimal norm of the set $B(u(t))-(F+\beta)(y(t))$ on $[0,T]$.

If y^1, y^2 there are two solutions to equation (E) corresponding to the pairs $(y_0^1, B(u^1)),\ (y_0^2, B(u^2)) \in D(\beta) \times W^{1,2}(0,T;R^n)$ then

$$\|y^1(t)-y^2(t)\| \le c(\|y_0^1-y_0^2\|^2 + \int_0^T \|B(u^1(t))-B(u^2(t))\|^2 dt) \quad \text{for} \quad 0 \le t \le T;$$

for details of the proof see [7].

Theorem 2.2 (Maximum Principle). Let $(y^*,u^*) \in W^{1,2}(0,T;R^n) \times W^{1,2}(0,T;R^m)$ be an optimal pair for problem (P) with the initial condition $y(0)=y_0 \in D(\beta)$.

Then, there is a function $p \in BV([0,T];R^n)$ and a measure $\mu \in ((L^\infty(0,T))^*)^n$ which satisfy the relations

$$p'(t) - \left[\frac{\partial F}{\partial y}(y^*(t))\right]^* p(t) - \mu \in \partial g(t, y^*(t)) \text{ a.e. } t \in]0,T[$$

$$p(T) + \partial\varphi_0(y^*(T)) \ni 0$$

$$B^{*}(p(t)) \in \partial h(u^{*}(t)) \text{ a.e. } t \in]0,T[$$

Proof. The initial problem (P) is approximated by a family of smooth problems (P^{ε}), $\varepsilon > 0$, for which the necessary optimality conditions are immediate then passing to the limit (ε tends to 0) we obtain the maximum principle.

For every $\varepsilon > 0$, we construct

Problem (P^{ε}) Minimize

$$\int_0^T (g^{\varepsilon}(y(t)) + h(u(t)) + \frac{1}{2}\| u(t) - u^{*}(t) \|)dt + \varphi_0^{\varepsilon}(y(T)), \quad 0 < T < \infty,$$

over the set of all functions $(y,u) \in W^{1,2}(0,T,R^n) \times L^2(0,T;R^m)$ which satisfy the approximate differential equation

$$y'(t) + (F^{\varepsilon} + \beta^{\varepsilon})(y(t)) = B(u(t)) \text{ a.e. } t \in]0,T[,$$

with the initial condition

$$y(0) = y_0 \in D(\beta).$$

For details see [7].

3. Fractional steps scheme

We associate to the initial problem (P) the following partial differential equation of Hamilton-Jacobi type

(4) $$\psi_t(t,x) - h^{*}(-B^{*}(\psi_x(t,x))) - \langle (F + \beta)(x), \psi_x(t,x) \rangle + g(x) = 0,$$

for every $t \in [0,T]$, $x \in X$, with the final condition

(5) $$\psi(T,x) = \varphi_0(x), \text{ for every } x \in X,$$

where ψ_t and ψ_x, represent the partial derivatives of the function ψ related to t and x, respectively.

If we denote by ψ the variational solution to the equation (4) with the condition (5) i.e.

$$\psi(t,x) = \inf\{ \int_t^T (g(y(s)) + h(u(s)))ds + \varphi_0(y(T)), \ u \in U,$$
$$y' + (F + \beta)y \ni Bu \text{ in } [t,T], \ y(t) = x\}$$

then ψ is a viscosity solution to (4) with the final condition (5).

If we set $\varphi(t,x)=\psi(T-t,x)$ then

(6) $$\varphi(t,x)=\inf\{\int_0^t(g(y(s))+h(u(s)))ds+\varphi_0(y(T)),\ u\in U,$$
$$y'+(F+\beta)y\ni Bu \text{ in } [0,t],\ y(0)=x\}$$

is a viscosity solution to forward Hamilton-Jacobi equation

(7) $\varphi_t(t,x)+h^*(-B^*(\varphi_x(t,x)))+\langle(F+\beta)x,\varphi_x(t,x)\rangle=g(x)$ for every $t\in[0,T]$, $x\in X$

with the initial condition

(8) $$\varphi(0,x)=\varphi_0(x) \text{ for every } x\in X.$$

The feedback optimal control is given by the relation

(9) $$u(t)=\partial h^*(-B^*(\psi_x(t,x))) \text{ for every } t\in[0,T]$$

where h^* is the conjugate function of h, i.e.

$$h^*(p)=\sup\{\langle p,u\rangle - h(u),\ u\in U\}.$$

In order to approximate the solution to the variationale quation (7) with the initial condition (8), we use the following approximation scheme (see [1], pp.156)

(10) $$\begin{cases} \varphi_t^\varepsilon(t,x)+h^*(-B^*(\varphi_x^\varepsilon(t,x)))+(F(x),\varphi_x^\varepsilon(t,x))=0 \\ \quad \text{for every } (t,x)\in\,]i\varepsilon,(i+1)\varepsilon[\,\times X,\ i=\overline{0,n-1}, \\ \varphi_+^\varepsilon(i\varepsilon,x)=\varphi_-^\varepsilon(i\varepsilon,S(\varepsilon)x)+\varepsilon g(x),\ x\in X,\ i=\overline{1,n-1}, \\ \varphi_+^\varepsilon(0,x)=\varphi_o(x),\ x\in X, \end{cases}$$

where $n=[T/\varepsilon]$ and φ_+, φ_- represent the right and left-hand side limits of the function $t\to\varphi(t,\cdot)$, respectively.

The function $y=S(t)x$ is the solution to Cauchy problem

(11) $$y(t)+\beta(y(t))\ni 0 \text{ a.e. } t\in\,]0,T[$$

with the initial condition

(12) $$y(0)=x\in D(\beta).$$

We recognize in the approximation scheme (10) a product formula of Lie-Trotter type associated to the Hamilton-Jacobi equation (7) with the initial condition (8).

To begin with, we construct the approximate scheme

$$(S_i)\qquad \begin{cases} y_\varepsilon'(t) + F(y_\varepsilon(t)) = B(u_\varepsilon(t)) \quad a.e. \quad t \in \,]i\varepsilon,(i+1)\varepsilon[, \\ y_\varepsilon^+(i\varepsilon) = S(\varepsilon)\, y_\varepsilon^-(i\varepsilon), \end{cases}$$

for $i=\overline{1,n-1}$, $n = [T/\varepsilon]$, with the particular case $(i = 0)$

$$(S_o)\qquad \begin{cases} y_\varepsilon'(t) + F(y_\varepsilon(t)) = B(u_\varepsilon(t)) \quad a.e. \quad t \in \,]0,\varepsilon[, \\ y_\varepsilon^+(0) = x \in D(\beta), \end{cases}$$

where $\{u_\varepsilon\}_{\varepsilon>0} \subset U$ is a given sequence.

Theorem 3.1. If $\{u_\varepsilon\}_{\varepsilon>0} \subset U$ is a weakly convergent sequence to u^* as $\varepsilon \to 0$ in U then

(13) $$y_\varepsilon(t) \to y^*(t), \text{ for every } t \in [0,T],$$

where y^* is the solution to equation (E), corresponding to u^*.

Proof. Multiplying the equation

$$y_\varepsilon'(t) + F(y_\varepsilon(t)) = B(u_\varepsilon(t)) \text{ a.e. } t \in \,](i-1)\varepsilon, i\varepsilon[$$

by y_ε and taking into account the properties of the elements which appear, after some calculation, we get

(14) $$\begin{aligned} \|y_\varepsilon^-(i\varepsilon)\|^2 &\le \Big(\|S(\varepsilon)y_\varepsilon^-((i-1)\varepsilon)\|^2 + 2D_2\varepsilon + \int_{(i-1)\varepsilon}^{i\varepsilon} \|B(u_\varepsilon(s))\|^2 ds\Big)e^{D\varepsilon} \le \\ &\le \Big(\|y_\varepsilon^-((i-1)\varepsilon)\|^2 + 2D_2\varepsilon + \int_{(i-1)\varepsilon}^{i\varepsilon} \|B(u_\varepsilon(s))\|^2 ds\Big)e^{D\varepsilon} \end{aligned}$$

for $i=\overline{2,n-1}$, $D = 2D_1 + 1$.

Analogously, we have

(15) $$\|y_\varepsilon^-(\varepsilon)\|^2 \le (\|y_o\|^2 + 2D_2\varepsilon + \int_o^\varepsilon \|B(u_\varepsilon(s))\|^2 ds)e^{D\varepsilon}.$$

Adding the inequalities from the relations (14) and (15) we get

(16) $$\|y_\varepsilon^-(i\varepsilon)\| \le C, \ i=\overline{1,n},$$

where $C^2 = (\|y_o\|^2 + 2D_2T + \|Bu_\varepsilon\|^2_{L^2(0,T;R^n)})e^{DT}.$

Also, we can show that

(17) $$\|y_\varepsilon^+(i\varepsilon)\| \le C, \ i=\overline{0,n-1}$$

and the sequence $\{y_\varepsilon\}_{\varepsilon>0}$ is bounded in $C([0,T];R^n)$.

The variation of the function y_ε on $[0,T]$ is given by

(18) $$\overset{T}{\underset{o}{V}} y_\varepsilon = \sum_{i=1}^{n-1} \|y_\varepsilon^-(i\varepsilon) - y_\varepsilon^+(i\varepsilon)\| = \sum_{i=1}^{n-1} \|y_\varepsilon^-(i\varepsilon) - S(\varepsilon)y_\varepsilon^-(i\varepsilon)\| \le$$
$$\le \varepsilon\sum_{i=1}^{n-1} \|\beta^o(y_\varepsilon^-(i\varepsilon))\|;$$

(we have used lemma 2.1 pp.121 from [2]).

Relation (16) and hypothesis (H) imply $y_\varepsilon^-(i\varepsilon) \in int\,D(\beta)$. Since β is bounded on every compact subset of $int\,D(\beta)$, there results that

(19) $$\|\beta^o(y_\varepsilon^-(i\varepsilon))\| \le M.$$

Thus, there exists a constant such that

$$\overset{T}{\underset{o}{V}} y_\varepsilon \le C_1$$

Helly's Theorem assures the existence of a subsequence $\{\varepsilon_n\}_{n\in N}$, with $\varepsilon_n \to 0$, and a function y^*, with bounded variation on $[0,T]$ so that

(20) $$y_{\varepsilon_n}(t) \to y^*(t), \ \forall t \in [0,T].$$

Ussing the above assumptions, we can prove that y^* is the solution of equation (E) with (IC) (for the proof see [1]).

Theorem 3.2 For $\varepsilon \to 0$ we have the convergence

(21) $$\varphi^{\varepsilon}(t,y) \to \varphi(t,y), \ \forall\, t \in [0,T], \ y \in R^n,$$

where φ is the solution of the variational equation (8).

Proof is similar with that presented in [1] p.127 and theorem 3.1 is used as a partial result.

Remark 3.1 Theorem 3.2 permits us to approximate equation (7) by a sequence of more simple equations for which the solutions can be easily calculated.

4. Examples

Example 4.1 Maximize $\int_0^T y(t)dt$ over the set of all functions $(y,u) \in W^{1,2}(0,T) \times U$ that satisfy the scalar variational inequality

$$y'(t) - y(t) + \partial I_K(y(t)) \ni u(t) \ \text{a.e.} \ t \in]0,T[,$$

with the initial condition

$$y(0) = x \in K,$$

where $K = \{z \in R;\ |z| \le a,\ a \in R_+^*\}$.

$U = \{u \in L^2(0,T);\ |u(t)| \le \gamma,\ \gamma \ge 1,\ \forall\, t \in [0,T]\}$.

The multivalued function $\beta:[-a,a] \to 2^R$ is defined by

$$\beta(r) = \partial I_K(r) = \begin{cases} R^- & \text{for } r=-a \\ 0 & \text{for } -a<r<a \\ R^+ & \text{for } r=a \end{cases}$$

The above problem represents a particular case of problem (P) for $m = n = 1$, $F = -Id_R$, $\beta = \partial I_K$, $B = Id_R$, $g(y) = -y$, $h = I_U$, $\varphi_o = 0$; here Id_R and I_U stand for the identity function and the indicator function, respectively.

We have

$$h^*(u^*) = \gamma|u^*|.$$

Let ψ be the solution of Hamilton-Jacobi equation

$$\psi(t,x) = -\sup\left\{ \int_t^T y(s)ds; y'(s) - y(s) + \partial I_K(y(s)) \ni u(s) \ \text{a.e.} \ s \in]t,T[,\ y(t) = x \in K \right\}$$

Using the transformation $\varphi(t,x) = \psi(T-t,x)$ the Hamilton-Jacobi equation (7) becomes

$$\begin{cases} \varphi_t(t,x) + \gamma|\varphi_x(t,x)| - x\varphi_x(t,x) = -x, & (t,x) \in [0,T] \times K, \\ \varphi(o,x) = 0, & x \in K. \end{cases}$$

The fractional steps scheme leads to the equation

(E_1) $\quad \varphi_t^\varepsilon(t,x) + \gamma|\varphi_x^\varepsilon(t,x)| - x\varphi_x^\varepsilon(t,x) = 0 \quad (t,x) \in](i-1)\varepsilon, i\varepsilon[\times K; \quad i=\overline{1,n}$,

with the initial conditions

$$\begin{cases} \varphi_+^\varepsilon(0,x) = 0, \\ \varphi_+^\varepsilon((i-1)\varepsilon, x) = \varphi_-^\varepsilon((i-1)\varepsilon, Px) - \varepsilon x, \quad i = \overline{2,n-1}, \end{cases}$$

where $n = [T/\varepsilon]$ and $P: R \to K$ is the projection function.

a) In the case $\varphi_x(t,x) \le 0$ the differential equation (E_1) becomes

$$\varphi_t^\varepsilon(t,x) - (\gamma + x)\varphi_x^\varepsilon(t,x) = 0 \quad (t,x) \in](i-1)\varepsilon, i\varepsilon[\times K, \quad i=\overline{1,n}.$$

with the initial condition

$$\varphi_+^\varepsilon((i-1)\varepsilon, x) = \varphi_-^\varepsilon((i-1)\varepsilon, Px) - \varepsilon x =$$

$$\begin{cases} \varepsilon((i-2)\gamma - e^{(i-1)\varepsilon}(\gamma-a)(\frac{1}{e^\varepsilon} + \frac{1}{e^{2\varepsilon}} + \dots + \frac{1}{e^{(i-2)\varepsilon}}) + \varepsilon a; & x \le -a, \\ \varepsilon((i-2)\gamma - e^{(i-1)\varepsilon}(\gamma + x)(\frac{1}{e^\varepsilon} + \frac{1}{e^{2\varepsilon}} + \dots + \frac{1}{e^{(i-2)\varepsilon}})) - \varepsilon x, & -a < x < a, \\ \varepsilon((i-2)\gamma - e^{(i-1)\varepsilon}(\gamma + a)(\frac{1}{e^\varepsilon} + \frac{1}{e^{2\varepsilon}} + \dots + \frac{1}{e^{(i-2)\varepsilon}})) - \varepsilon a, & x \ge a, \end{cases}$$

and has the solution

$$\varphi^\varepsilon(t,x) = \gamma\varepsilon(i-1) - \varepsilon(\gamma + x)e^{t-\varepsilon}\frac{1 - \frac{1}{e^{(i-1)\varepsilon}}}{1 - \frac{1}{e^\varepsilon}}.$$

Since $t \simeq (i-1)\varepsilon$ we obtain

$$\varphi^{\varepsilon}(t,x) = \gamma t - \varepsilon(\gamma + x)\frac{e^{t}-1}{e^{\varepsilon}-1}$$

Using Theorem 3.2 we have

$$\varphi(t,x) = \lim_{\varepsilon \to o} \varphi^{\varepsilon}(t,x) = t\gamma - (\gamma + x)(e^{t} - 1)$$

b) In the case $\varphi_x(t,x) > 0$ we have no solution

Example 4.2 Maximize $\int_o^T y^2(t)dt$ over the set of all the functions

$(y,u) \in W^{1,2}(0,T) \times U$ that satisfy the scalar variational inequality

$$y'(t) - y(t) + \partial I_K(y(t)) \ni u(t) \text{ a.e. } t \in]0,T[,$$

with the initial condition

$$y(0) = x \in K.$$

The significance of all elements is the same as in Example 4.1, except for $g(y)=-y^2$. Analougously, ψ denote the solution of Hamilton-Jacobi equation

$$\psi(t,x) = -\sup\left\{ \int_t^T y^2(s)ds; y'(s) - y(s) + \partial I_K(y(s)) \ni u(s) \text{ a.e. } s \in]t,T[,\ y(t) = x \in K \right\}$$

Using the transformation $\varphi(t,x) = \psi(T-t,x)$ the Hamilton-Jacobi equation (7) becomes

$$\begin{cases} \varphi_t(t,x) + \gamma|\varphi_x(t,x)| - x\varphi_x(t,x) = -x^2, & (t,x) \in [0,T] \times K, \\ \varphi_+(o,x) = 0, & x \in K. \end{cases}$$

The fractional steps scheme leads to the equation

$$(E_2) \qquad \varphi_t^{\varepsilon}(t,x) + \gamma|\varphi_x^{\varepsilon}(t,x)| - x\varphi_x^{\varepsilon}(t,x) = 0 \quad (t,x) \in](i-1)\varepsilon, i\varepsilon[\times K; \ i=\overline{1,n-1},$$

with the initial conditions

$$\begin{cases} \varphi_+^{\varepsilon}(0,x) = 0, \\ \varphi_+^{\varepsilon}(i-1)\varepsilon,x) = \varphi_-^{\varepsilon}((i-1)\varepsilon, Px) - \varepsilon x^2, \quad i=\overline{2,n-1} \end{cases}$$

where $n = [T/\varepsilon]$ and $P:R \to K$ is the projection function.

a) If $\varphi_x^\varepsilon < 0$ then the differential equation

$$\varphi_t^\varepsilon(t,x) - (\gamma + x)\varphi_x^\varepsilon(t,x) = 0 \qquad (t,x) \in]i\varepsilon, (i+1)\varepsilon[\times K, \quad i = \overline{0,n-1}$$

with the initial conditions

$$\varphi_+^\varepsilon(i\varepsilon,x) = \varphi_-^\varepsilon(i\varepsilon, Px) - \varepsilon x^2 =$$

$$= \begin{cases} -\varepsilon\{[(\gamma-a)e^\varepsilon-\gamma]^2+[(\gamma-a)e^{2\varepsilon}-\gamma]^2+\ldots+[(\gamma-a)e^{(i-2)\varepsilon}-\gamma]^2\}-\varepsilon a^2, & x<-a \\ -\varepsilon\{[(\gamma+x)e^\varepsilon-\gamma]^2+[(\gamma+x)e^{2\varepsilon}-\gamma]^2+\ldots+[(\gamma+x)e^{(i-2)\varepsilon}-\gamma]^2\}-\varepsilon x^2, & -a\le x\le a, \\ -\varepsilon\{[(\gamma+a)e^\varepsilon_{-\gamma}]^2+[(\gamma+a)e^{2\varepsilon}-\gamma]^2+\ldots+[(\gamma+a)e^{(i-2)\varepsilon}-\gamma]^2\}-\varepsilon a^2, & x>a, \end{cases}$$

$i=\overline{1,n-1}$ has the solution

$$\varphi^\varepsilon(t,x) = -\varepsilon(\gamma+x)^2e^{2t}\left[\frac{1}{e^{2\varepsilon(i-1)}}+\frac{1}{e^{2\varepsilon(i-2)}}+\ldots+\frac{1}{e^{2\varepsilon}}\right]+$$

$$+2\varepsilon\gamma(\gamma+x)e^t\left[\frac{1}{e^{\varepsilon(i-1)}}+\frac{1}{e^{\varepsilon(i-2)}}+\ldots+\frac{1}{e^\varepsilon}\right]-\varepsilon(i-1)\gamma^2=$$

$$=-\varepsilon(\gamma+x)^2e^{2t-2\varepsilon}\frac{1-\frac{1}{e^{2\varepsilon(i-1)}}}{1-\frac{1}{e^{2\varepsilon}}}+2\varepsilon\gamma(\gamma+x)e^{t-\varepsilon}\frac{1-\frac{1}{e^{\varepsilon(i-1)}}}{1-\frac{1}{e^\varepsilon}}-\varepsilon(i-1)\gamma^2$$

Since $t \simeq (i-1)\varepsilon$ we get the approximate solution

$$\varphi_1^\varepsilon(t,x) = -\varepsilon(\gamma+x)^2\frac{e^{2t}-1}{e^{2\varepsilon}-1}+2\varepsilon\gamma(\gamma+x)\frac{e^t-1}{e^\varepsilon-1}-\gamma^2 t$$

From

$$(\varphi_1^\varepsilon)_x(t,x) = 2\varepsilon\frac{e^t-1}{e^\varepsilon-1}\left(\gamma-(\gamma+x)\frac{e^t+1}{e^\varepsilon+1}\right)<0$$

it results

$$x > \gamma \frac{(e^{\varepsilon} - e^{t})}{e^{t} + 1} = -x_1^{\varepsilon}$$

Theorem 3.2 lends to

$$\varphi_1(t,x) = \lim_{\varepsilon \to 0} \varphi_1^{\varepsilon}(t,x) = -\frac{1}{2}(\gamma + x)^2(e^{2t} - 1) + 2\gamma(\gamma + x)(e^{t} - 1) - \gamma^2 t$$

From

$$(\varphi_1)_x(t,x) = (e^{t} - 1)(\gamma(1 - e^{t}) - x(1 + e^{t})) < 0$$

it results

$$x > \gamma \frac{1 - e^{t}}{1 + e^{t}} = -x_0$$

Case b. If $\varphi_x^{\varepsilon}(t,x) > 0$ then the differential equation

$$\varphi_t^{\varepsilon}(t,x) + (\gamma - x)\varphi_x^{\varepsilon}(t,x) = 0 \quad (t,x) \in]i\varepsilon, (i+1)\varepsilon[\times K, \quad i = \overline{0, n-1}.$$

with the initial condition

$$\varphi_+^{\varepsilon}(i\varepsilon, x) = \varphi_-^{\varepsilon}(i\varepsilon, Px) - \varepsilon x^2 =$$

$$= \begin{cases} -\varepsilon\{[\gamma - (a+\gamma)e^{\varepsilon}]^2 + [\gamma - (a+\gamma)e^{2\varepsilon}]^2 + \ldots + [\gamma - (a+\gamma)e^{(i-2)\varepsilon}]^2\} - \varepsilon a^2, & x < -a \\ -\varepsilon\{[\gamma + (x-\gamma)e^{\varepsilon}]^2 + [\gamma + (x-\gamma)e^{2\varepsilon}]^2 + \ldots + [\gamma + (x-\gamma)e^{(i-2)\varepsilon}]^2\} - \varepsilon x^2, & -a \le x \le a, \\ -\varepsilon\{[\gamma + (a-\gamma)e^{\varepsilon}]^2 + [\gamma + (a-\gamma)e^{2\varepsilon}]^2 + \ldots + [\gamma + (a+\gamma)e^{(i-2)\varepsilon}]^2\} - \varepsilon a^2, & x > a, \end{cases}$$

$i = \overline{1, n-1}$ has the solution

$$\varphi_2^{\varepsilon}(t,x) = -\varepsilon(x-\gamma)^2 \frac{e^{2t} - 1}{e^{2\varepsilon} - 1} - 2\varepsilon\gamma(x-\gamma)\frac{e^{t} - 1}{e^{\varepsilon} - 1} - \gamma^2 t$$

From

$$(\varphi_2^{\varepsilon})_x(t,x) = -2\varepsilon \frac{e^{t} - 1}{e^{\varepsilon} - 1}\left[\gamma + (x-\gamma)\frac{e^{t} + 1}{e^{\varepsilon} + 1}\right] > 0$$

it results

$$x < \gamma \frac{e^{t} - e^{\varepsilon}}{e^{t} + 1} = x_1^{\varepsilon}$$

Theorem 3.2 leads to

$$\varphi_2(t,x) = \lim_{\varepsilon\to 0}\varphi_2^{\varepsilon}(t,x) = -\frac{1}{2}(x-\gamma)^2(e^{2t}-1) - 2\gamma(x-\gamma)(e^t-1) - \gamma^2 t$$

From

$$(\varphi_2)_x(t,x) = -(e^t-1)(x(e^t+1) - \gamma(e^t-1)) > 0$$

it results

$$x < \gamma\frac{e^t-1}{e^t+1} = x_0.$$

In conclusion

$$\varphi^{\varepsilon}(t,x) = \begin{cases} \varphi_2^{(\varepsilon)}(t,x) & x \in (-a, -x_1^{\varepsilon}) \\ \varphi_1^{(\varepsilon)}(t,x) \quad \text{or} \quad \varphi_2(t,x) & x \in (-x_1^{\varepsilon}, x_1^{\varepsilon}) \\ \varphi_1^{(\varepsilon)}(t,x) & x \in (x_1^{\varepsilon}, a) \end{cases}$$

$t < \ln\dfrac{a+e^{\varepsilon}\gamma}{\gamma-a}$; $\gamma > a$

$$\varphi(t,x) = \begin{cases} \varphi_2(t,x) & x \in (-a, -x_0) \\ \varphi_1(t,x) \quad \text{or} \quad \varphi_2(t,x) & x \in (-x_0, x_0) \\ \varphi_1(t,x) & x \in (x_0, a) \end{cases}$$

$t < \ln\dfrac{\gamma+a}{\gamma-a}$; $\gamma > a$.

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