LIBERTAS MATHEMATICA,VOL.XVIII(1998)

# ON THE INTEGRAL EQUATION FOR AIRFOIL DESIGN IN THE CASE OF INVERSE FLUID JET PROBLEM

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**Abstract**

The paper deals with a special inverse boundary problem, when the boundary of the domain is completely unknown and a singular integral equation for the velocity angle is obtained. For the model of free plane symmetric incompressible jet forked by an airfoil, the boundary equations and airfoil shape are "a posteriori" determined, while the velocity along them is "a priori" prescribed . The airfoil is drawn through numerical computations and the drag coefficient determined.

## 1 INTRODUCTION

The stationary potential plane flow of an inviscid fluid is considered in the absence of mass forces *(Hyp)*. Relating the velocity field $w(z) = u(x,y) + i\,v(x,y)$ to the $xOy$ frame in the physical flow domain $D_z$, $z = x + i\,y$, then in the hypothesis *(Hyp)* formulated above, we have $\mathrm{rot}\nu = 0, (\nu = \mathbf{grad}\varphi(x,y)), \mathrm{div}\nu = 0$. The complex potential $f(z)$ and the complex velocity $w(z)$ are defined through the analytic functions:

$$f(z) = \varphi(x,y) + i\psi(x,y), \quad \overline{w} = \frac{df}{dz} = Ve^{-i\theta}, \tag{1}$$

$$u = V\cos\theta \quad v = V\sin\theta.$$

Here $\varphi = \varphi(x,y)$ is the velocity potential, $\psi = \psi(x,y)$ the stream function, $V = (u^2+v^2)^{\frac{1}{2}}$ and $\theta = \arctan(v/u)$, are the velocity magnitude and respectively its angle with the $Ox$ axis. Passing to the hodographic plane $(V,\theta)$, i.e. $W = V + i\theta$, using (1), $f$ is analytically generalized by $W$ [3], [4], [8]:

$$\theta_\psi = -\frac{1}{V}V_\varphi, \quad \theta_\varphi = \frac{1}{V}V_\psi, \quad \varphi_\theta = V\psi_V, \quad \varphi_V = -\frac{1}{V}\psi_\theta \tag{2}$$

In the case of the direct problem the flow will be determined using the hodographic method [3], [4], [8] and $f(W), z(W)$ will be obtained. In the case of a curvilinear domain $D_z$, it is generally difficult to obtain directly $f = f(z)$ and $w = w(z)$ by solving the Dirichlet or Volterra boundary problem, therefore a canonic auxiliary domain $D_\zeta$, $\zeta = \xi + i\,\eta$, $\eta \geq 0$; [2], [4] should be introduced.

To the domain $D_z, y \geq 0$, of the plane symmetrical jets it corresponds the domain $D_f, f = \varphi + i\,\psi, \varphi \in (-\infty,\infty), 0 \leq \psi \leq \frac{Q}{2} < \infty$ where $Q$ is the total flow mass. We try to determine the analytic function $f = f(z)$ which realizes the conformal mapping $D_f \leftrightarrow D_\zeta$ with:

$$\varphi_\xi = \psi_\eta; \quad \varphi_\eta = -\psi_\xi; \quad f_{\overline{\zeta}} = 0. \tag{3}$$

To obtain the analyticity conditions for the velocity $(V,\theta)$ in (2) we introduce the Jukovski function $\omega$, considering that along the stream lines (free lines), $V = V^0$. That is:

$$\omega = t + i\,\theta, \quad \overline{w} = V^0 e^{-\omega}, \quad t = \ln\frac{V^0}{V}, \quad 0 \leq V \leq V^0; \tag{4}$$

$$\theta_\psi = t_\varphi, \quad \theta_\varphi = -t_\psi; \quad \varphi_\theta = -\psi_t, \quad \varphi_t = \psi_\theta; \quad \omega_{\overline{f}} = 0, \quad f_{\overline{\omega}} = 0. \tag{5}$$

In the case of free surface flow, the flow domain $D_z$ is generally bounded by polygonal rigid walls, (curvilinear) obstacles and stream lines that diverge from the walls or the obstacle [5]. Along these free lines the velocity, pressure and density are respectively $V^0, p^0, r^0 = const.$, and in the point whose velocity is zero $V \equiv 0, p = p_0, \rho = \rho_0 = const.$ Applying Bernoulli's law in the case of incompressible flow along a stream line $\psi = const.$ we obtain:

$$\frac{1}{2}V^2 + \frac{p}{\rho} = \frac{1}{2}V^{0\,2} + \frac{p^0}{\rho^0}; \quad p = p_0 - \frac{\rho V^2}{2}; \quad p^0 = p_0 - \frac{\rho_0 V^{0\,2}}{2}. \tag{6}$$

Now we consider the theorems TI and TII [5],[4].

**Theorem 1.1** *In the hypothesis (Hyp), if there is a conformal mapping* $f = f(\zeta), f_{\overline{\zeta}} = 0$ *with* $D_f \leftrightarrow D_\zeta$ *then* $z = z(\zeta)$ *is analytic (conformal), with* $D_z \leftrightarrow D_\zeta$.

That can be easily proved: $f$ is analytic of $z, f_{\bar{z}} = 0$ and from *(Hyp)* $z$ is analytical of $f, z_{\bar{f}} = 0$. If $f$ is analytical of $\zeta$ then their composition $z = z(\zeta)$ is analytic, too. At this stage, we shall find $f = f(\zeta)$ so that the boundaries of the domains $D_z, D_f$ correspond to the boundary of $D_\zeta, \eta = 0, \xi \in (-\infty, \infty)$, on which we have the stream lines $\psi = const.$ As $x'Ox$ is the axis of symmetry, we shall prove that for the founded function $f = f(\zeta)$ the following conditions hold $\eta = 0, \psi = const.$ and $\frac{\partial\varphi}{\partial\eta}|_{\eta=0} = 0$.

In this case, the passing relations (1) $dz = \frac{1}{V}e^{i\theta}df$ become in $D_\zeta$, on $\eta = 0$:

$$dx + i\,dy = (\frac{\partial\varphi}{\partial\xi})\frac{1}{V}(\cos\theta + i\,\sin\theta)d\xi. \tag{7}$$

Performing the separation of the real and imaginary parts, the geometrical equations of the boundary (the $BOB'$ airfoil), are derived:

$$\frac{dx}{d\xi} = \frac{\cos\theta}{V}\varphi_\xi, \frac{dy}{d\xi} = \frac{\sin\theta}{V}\varphi_\xi; \tag{8}$$

whence

$$x(\xi) = \int_{\xi_0}^{\xi}\frac{\cos\theta}{V}d\xi + x_0, \qquad y(\xi) = \int_{\xi_0}^{\xi}\frac{\sin\theta}{V}d\xi + y_0.$$

In the case of the inverse problem, the boundary $D_z$ is completely unknown and using these formulae, it will be determined by TII. We only need to known $w(\eta)$, or $\omega(\zeta)$.

**Theorem 1.2** *If in the hypothesis (Hyp) there is $f$ analytic of $\zeta$, and it is the conformal mapping between $D_f \leftrightarrow D_\zeta$ then $\omega = t + i\,\theta = \omega(\zeta)$ is analytic of $\zeta, \omega_{\bar{\zeta}} = 0$ and it is the conformal mapping between $D_\omega \leftrightarrow D_\zeta$.*

As demonstrated above in TI, if $f$ is analytic of $\zeta$, and using (5), $\omega$ is analytic of $f$, then their composition $\omega = \omega(\zeta)$ will be analytic, too. Usually, in $D_\zeta, \eta \geq 0$, for $\omega = \omega(\zeta)$ there is a mixed boundary problem. Solving it, we obtain $\omega(\zeta)$ and $w(\zeta)$ to be used in (8).

## 2 THE INTEGRAL EQUATION OF THE INVERSE PROBLEM

In the previously stated conditions, we consider the plane flow of a symmetrical free fluid jet, bounded by the free lines $(AD)$ and $(A'D')$ along which the velocity is $V^0$. At infinite upstream, the jet width is $AA' = 2h$, and the velocity is $\mathbf{V}^0 = V^0 i$, while the total flow mass is $Q = 2hV^0$.

The jet encounters the symmetrical curvilinear obstacle $BOB'$. The stream lines $(BC)$ and $(B'C')$, along which the velocity is $V^0$, emanate from $B, B'$ and acquire an asymptotic direction at infinite downstream in $(CD),(C'D')$ at an angle $\pm\gamma\pi$ with the $x'Ox$ axis which is the symmetry axis $A_0O$ of the figure. It is sufficient to study the flow in the halfplane $D_z, y \geq 0$, and therefore the domain boundary is $(A_0OBCDA)$. The velocity angle in $O$ is $\theta(0) = \alpha\pi$ and $\theta(B) = \beta\pi$ where $0 \leq \gamma < \beta < \alpha \leq \frac{1}{2}$ and the velocity downstream $(CD)$ is $w = V^0e^{i\gamma\pi}$ (fig. 1). The direct boundary problem for analytic functions in $D_z$ is to find an analytic function in $D_z$ knowing the real (imaginary) part or mixed values along the boundary $\partial D_z$, which is also known. The inverse problem consists of determining an analytic function in $D_z$ which will fulfil the same conditions on the boundary, but in this case the boundary is only partially known (or completely unknown as is our case here) and it needs to be determined, too. Direct problems have been studied using these models — by V. Cisotti [1], H. Villat and C. Iacob [3].

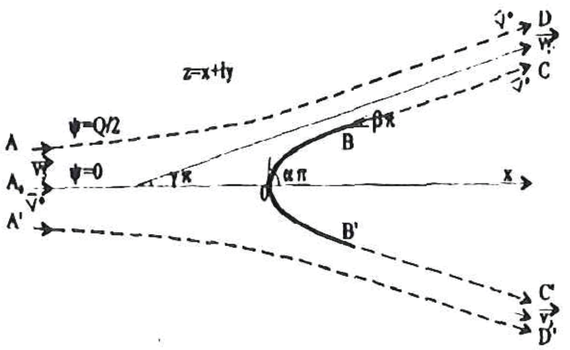



Fig.1. Motion range in the phisical plane $D_z$

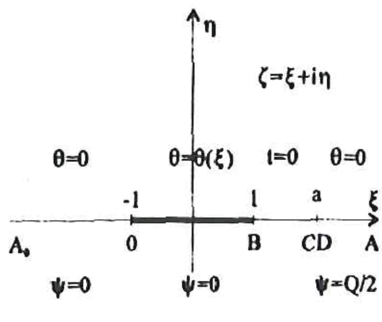



Fig.2.Corespondence of fields $D_f$, $D\omega$, with $D\zeta$

For the problem stated above, the boundary $(A_0A'D'C''OBCDAA_0)$ is unknown, but the stream function $\psi$, the velocity $V^0$ are known on this boundary and the distribution $V = V(\theta)(p = p(\theta))$ is given "apriori" on the $(B'OB)$ profile. Therefore, it is important to determine "a posteriori" the shape of this airfoil in the case.

Several papers, like [7] and [9], deal with the case of the flow with circulation.

In order to solve the mentioned problem, for $\theta(\xi)$, an integral equation will be derived. First the theorems TI and TII will be applied.

Let us consider the biunivocal correspondence between the domains $D_z, D_f$ with the halfplane $D_z, \eta \geq 0$, so that the boundary $(A_0OBCDAA_0)$ be placed upon the $\eta = 0$ axis, $\xi \in (-\infty, \infty) : A_0(-\infty); O(-1); B(1); C; D(a)$; $A(\infty)$ (fig. 2). The parameter $a > 1$ will be determined.

We shall determine $f(\zeta) = \varphi + i\,\psi$ analytic in $D_\zeta, \eta \geq 0$, such that $\Delta\psi = 0$ and with the boundary values $\psi = 0$ for $\xi \in (-\infty, a)$ and $\psi = \frac{Q}{2}$ for $\xi \in (a, \infty)$. The solution of the Dirichlet problem $(D_f \leftrightarrow D_\zeta)$ when $\varphi \in (-\infty, \infty), \psi \in [0, \frac{Q}{2}]$ is [3 ], [5]:

$$f(\zeta) = -\frac{Q}{2\pi}\ln(\zeta - a) + \frac{i\,Q}{2}; \quad \frac{\partial\varphi}{\partial\xi}|_{\eta=0} = -\frac{Q}{2\pi}\frac{1}{\xi - a}, \quad \frac{\partial\varphi}{\partial\eta}|_{\eta=0} = 0. \qquad (9)$$

Knowing the boundary values: $\theta = 0, \xi \in (-\infty, \infty); \theta = \theta(\xi), \xi \in (-1, 1); t = 0, \xi \in (1, a) \cup (a, \infty)$ we determine the analytic function in $D_\zeta, \eta \geq 0, \omega(\zeta) = t + i\,\theta, t = \ln\frac{V^0}{V}$. This is a mixed problem and we transform it into a Dirichlet problem for the function $S = R + iT = \frac{\omega(\zeta)}{\sqrt{\zeta - 1}}$, in which case along the boundary we have: $R = 0, \xi \in (-\infty, -1) \cup (1, \infty)$ and $R = \frac{\theta(\xi)}{\sqrt{1-\xi}}, \xi \in (-1, 1)$. Using the Cisotti-Villat formula for the $\eta \geq 0$ halfplane, we obtain $S = S(\zeta)$ and

$$\omega(\zeta) = \frac{\sqrt{\zeta - 1}}{\pi i}\left[\int_{-1}^{1} \frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s - \zeta} + C\right] = t + i\,\theta, \quad \zeta \in D_\zeta, \qquad (10)$$

where the constant $C$ is determined by specifying the velocity in $O$. If the function $V = V(\theta)$ is prescribed on the obstacle $(OB)$ then, using (4), $t = t[\theta(\xi)]$ is fixed for $\xi \in (-1, 1)$. Applying the Sohotski-Plemelj formula to the integral part of (10), we obtain the singular integral equation of the inverse problem [5]:

$$t[\theta(\xi)] = \frac{\sqrt{1 - \zeta}}{\pi i}\left[\int_{-1}^{\prime 1} \frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s - \zeta} + C\right], \quad \xi \in [-1, 1]. \qquad (11)$$

This equation is a non-linear one and the singularity of the integral is to be taken in Cauchy's principal value sense. Solving this we obtain $\theta = \theta(\xi)$, then $t = t(\xi)$ and $V = V(\xi)$ for $\xi \in (-1, 1)$ and returning to (10), we find $\omega = \omega(\zeta)$ and $W = W(\zeta)$ which determine the flow. Using (8), we obtain the parameter equations of the boundary of domain $D_z, x = x(\xi)$ and $y = y(\xi)$ taking for the $(OB)$ airfoil $\xi_0 = -1, \xi \in (-1, 1]$.

We shall study this equation in two important cases.

**Case 1.** - Along the $(OB)$ airfoil we prescribe "apriori" the distribution:

$$V(\theta) = (\frac{V_0}{V})^{\frac{\theta-\beta\pi}{\pi(\alpha-\beta)}} \Leftrightarrow t = \frac{\theta-\beta\pi}{\pi(\alpha-\beta)}\ln(\frac{V_0}{V}) \equiv m\theta + n, \tag{12}$$

where: $V(0) = V_0, \theta(0) = \alpha\pi, t_0 = \frac{V^0}{V_0}, \xi = -1$ and $V(B) = V^0, \theta(B) = \beta\pi, t_B = 0, \xi = 1$ with $V_0 << V^0, \alpha > \beta$. The meaning of these conditions is that in the neighborhood of $O$, the velocity $V_0$ has a very little value, fact that may be experimentally verified. In this case, the equation (11) is linear.

We impose in $O$ the condition $t(\xi = -1) = \ln\frac{V^0}{V_0}$, and using (11) we obtain: $C = \frac{\pi}{\sqrt{2}}\ln\frac{V^0}{V_0} - \int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s+1}$; next, considering again the linear equations (11) and (12), one gets:

$$m\theta + n = \frac{\sqrt{1-\xi}}{\pi}\int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{(1+\xi)ds}{(s+1)(s-\xi)} + \sqrt{\frac{1-\xi}{2}}\ln\frac{V^0}{V_0}; \tag{13}$$

$$m = \frac{1}{\pi(\alpha-\beta)}\ln\frac{V^0}{V_0}, \quad n = \beta\pi m.$$

Denoting $\theta(\xi) = (\xi+1)\sqrt{(1-\xi)}g(\xi)$, we obtain the following singular canonic integral equation:

$$ag(\xi) + \frac{b}{\pi}i\int_{-1}^{1}\frac{g(s)}{s-\xi}ds = \frac{\beta\pi m}{\sqrt{1-\xi}(1+\xi)} + \frac{1}{\sqrt{2}(1+\xi)}\ln\frac{V^0}{V_0} \equiv h(\xi) \tag{14}$$

where: $a = m, b = -i$, and $h(\xi)$ is Hölderian. Since $a$ and $b$ are constant and the index of the equation is zero, [6], [5] we can compute directly the solution. We introduce Schwarz's operator $S[f(\xi)] = \frac{1}{\pi i}\int_{-1}^{1}\frac{f(s)}{s-\xi}ds$ where $S^2 = I$, the identical operator $I(f) = f$ and then the regularization operator will be $\frac{1}{a^2-b^2}[aI - bS], a^2 - b^2 = m^2 + 1$. Since the regularity conditions are fulfilled, we shall apply this operator to the equation (14). Finally, like in [6], [5], the solution is obtained as:

$$g(\xi) = \frac{a}{a^2-b^2}h(\xi) - \frac{b}{a^2-b^2}\frac{1}{\pi i}\int_{-1}^{1}\frac{h(s)}{s-\xi}ds. \tag{15}$$

Computing the singular integral directly or through numeric methods, we obtain $g = g(\xi), \theta = \theta(\xi), t = t(\xi), V = V(\xi)$ and then $w = w(\zeta)$ is found. Using (8), as mentioned above, we can obtain the boundary and the shape of the airfoil.

**Case 2.** - Using the "semi-inverse" method, we shall prescribe a practical model of the velocity angle:

$$\theta(\xi) = \pi(\beta-\alpha)\sqrt{\frac{1+\xi}{2}} + \alpha\pi, \quad \beta < \alpha; \quad \xi \in [-1, 1]. \tag{16}$$

$$\theta(0) = \theta(-1) = \alpha\pi; \qquad \theta(B) = \theta(1) = \beta\pi.$$

If we impose the condition $V(O) = 0, t(\xi = -1) \to \infty$, then it follows that $C \equiv 0$ in (11) and the velocity is obtained through the computation of the Glauert singular integral [3], [5].

$$\begin{aligned} I_1 &= \int_{-1}^{1} \sqrt{\frac{1+s}{1-s}} \frac{ds}{s-\xi} = \pi; \\ I_2 &= \int_{-1}^{1} \frac{ds}{\sqrt{1-s}(s-\xi)} = 2\ln\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{1+\xi}}, \ \xi \in (-1,1) \\ t(\xi) &= \pi\sqrt{\frac{1-\xi}{2}}(\beta-\alpha) + 2\alpha\sqrt{1-\xi}\ln\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{1+\xi}}, \\ V(\xi) &= V^0 e^{\pi(\alpha-\beta)\frac{\sqrt{1-\xi}}{2}}\left[\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{1+\xi}}\right]^{2\alpha\sqrt{1-\xi}} \end{aligned} \tag{17}$$

where the conditions $V(O) = V(\xi = -1) = 0, V(B) = V(\xi = 1) = V^0$ are fulfilled. It is obvious that eliminating $\xi$, from (16) and (17), $V = V(\theta), t = t(\theta)$ are obtained. In this case, the integral equation is a nonlinear one.

Returning to (10), we have:

$$\omega(\zeta) = t(\zeta) + i\theta(\zeta) = \frac{\sqrt{\zeta-1}}{\pi i}\int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s-\zeta}. \tag{18}$$

Using (16) on the $\eta = 0$ axis and computing the integral, we shall obtain the distribution of the velocity along the boundary:

- along the free lines $(BCDA), V = V^0, t \equiv 0$ from (18) for $\xi \in (1,a) \cup (a,\infty)$ we obtain:

$$\theta(\xi) = \frac{\pi(\beta-\alpha)}{\sqrt{2}}(\sqrt{\xi+1} - \sqrt{\xi-1}) + 2\alpha\arctan\sqrt{\frac{2}{\xi-1}}; \tag{19}$$

$$u = V^0\cos\theta \qquad v = V^0\sin\theta.$$

We have used the quadrature formulas:

$$I_3 = \int_{-1}^{1}\sqrt{\frac{1+s}{1-s}}\frac{ds}{s-\xi} = \pi(1 - \sqrt{\frac{\xi+1}{\xi-1}}),$$

$$I_4 = \int_{-1}^{1}\frac{ds}{\sqrt{1-s}(s-\xi)} = -\frac{2}{\sqrt{\xi-1}}\arctan\sqrt{\frac{2}{\xi-1}}.$$

- along $(A_0O)$, $\xi \in (-\infty, -1)$, because $\theta = 0$, using (16) in (18) we obtain:

$$t(\xi) = \ln\frac{V^0}{V} = \frac{(\beta-\alpha)\pi}{\sqrt{2}}(\sqrt{1-\xi}-\sqrt{-1-\xi}) + \ln\left(\frac{\sqrt{1-\xi}+\sqrt{2}}{\sqrt{1-\xi}-\sqrt{2}}\right)^{\alpha}; \quad (20)$$

$$u = V = V^0 e^{-t(\xi)}, \qquad v \equiv 0.$$

The following results were used:

$$I_5 = \int_{-1}^{1}\sqrt{\frac{1+s}{1-s}}\frac{ds}{s-\xi} = \pi\left(1-\sqrt{\frac{-1-\xi}{1-\xi}}\right),$$

$$I_6 = \int_{-1}^{1}\frac{ds}{\sqrt{1-s}(s-\xi)} = \frac{1}{\sqrt{1-\xi}}\ln\frac{\sqrt{1-\xi}+\sqrt{2}}{\sqrt{1-\xi}-\sqrt{2}}.$$

- along the airfoil $OB; u = V(\xi)\cos\theta(\xi), v = V(\xi)\sin\theta(\xi), \xi \in (-1,1)$ where $\theta(\xi)$ and $V(\xi)$ are given by (16) and (17). It may be verified that $V(A_0) = V(A) = V^0; V(0) = 0, V(B) = V^0$ and from $V(C,D) = V^0$ when $\xi \to a, \theta(C,D) = \gamma\pi$, the parameter $a$ may be computed. At the same time, using (6), the pressure along $(A_0O)$ and $(OB)$ can be determined.

## 3 DETERMINING THE SHAPE AND GEOMETRICAL PARAMETERS OF THE AIRFOIL

We study the problem in the 2-nd case. From the distribution (16), we note that along the airfoil $\theta\prime(\xi) < 0, \frac{dy}{dx} = \tan\theta > 0, \frac{d^2y}{dx^2} = \frac{V^2\theta\prime(\xi)}{u\cos^2\theta} < 0$ i.e. the arc $(OB)$ is convex and symmetrical relative the $Ox$ axis. From (8), (16) and (17) we obtain the coordinates of a point $P(x(\xi), y(\xi))$ describing the airfoil $(OB)$:

$$x(\xi) = \frac{Q}{2\pi}\int_{-1}^{\xi}\frac{\cos\theta(s)}{V(s)}\frac{ds}{(a-s)}, \qquad \xi \in (-1,1] \quad (21)$$

$$y(\xi) = \frac{Q}{2\pi}\int_{-1}^{\xi}\frac{\sin\theta(s)}{V(s)}\frac{ds}{(a-s)}. \quad (22)$$

The length of the elementary arc in $xOy$ plane is $dS = \frac{\varphi_\xi}{V(\xi)}d\xi$, and therefore the length of the OB arc is:

$$l_{OB} = l = \frac{Q}{2\pi}\int_{-1}^{1}\frac{d\xi}{V(\xi)(a-\xi)} = \frac{Q}{2\pi}I(a, V^0, \alpha, \beta). \quad (23)$$

If $Q = 2V^0 h$, $a = a(\alpha, \beta, V^0)$ are given, then $l$ can be determined or conversely, if $l$ is given, then $a$ can be determined.

In (21) the coordinates functions $X(\xi) = \frac{x(\xi)}{l}, Y(\xi) = \frac{y(\xi)}{l}$ are normalized and the airfoil may be drown, see Table 1. Using (8) we compute the airfoil curvature. The airfoil curvature is $k(\xi) = \frac{1}{R(\xi)} = \frac{|x'y''-y'x''|}{(x'^2+y'^2)^{3/2}} = \frac{\theta'(\xi)}{\varphi'(\xi)}V(\xi)$. If one denotes $\overline{V} = \frac{R}{h} = \frac{2R}{Q}V^0$, using (16) and (9), we find:

$$\overline{k}(\xi) = \frac{(\alpha-\beta)(a-\xi)}{\sqrt{2(1+\xi)}}\left[\frac{\sqrt{1+\xi}}{\sqrt{2}+\sqrt{1-\xi}}\right]^{2\alpha\sqrt{1-\xi}} e^{\frac{\pi}{\sqrt{2}}(\alpha-\beta)\sqrt{1-\xi}}, \tag{24}$$

$$\xi \in [-1, 1].$$

Introducing $\theta(\xi)$ in (19), the equations of the free lines $(BC) \cup (DA)$ are derived. On these free lines, the curvature has the expression $k = |\frac{\theta'}{\varphi'}|V^0$, i.e. $\overline{k}(BC, AD) = \frac{(\alpha-\beta)}{2\sqrt{2(1+\xi)}}|\xi - a|, \xi \in (1, a) \cup (a, \infty)$. It may be observed that along the asymptotic direction $\xi \to a, \overline{K}(CD) \to 0$. Since the curvature sign does not change, the free line $(AD)$ is convex, while $(OB)\cup(BC)$ is concave and there are no inflexion points along them.

**Remark.** If the direct problem is examined, i.e. the flow around a given curvilinear airfoil $(BOB')$, then the curvature $k(\xi)$ is known and from $t = \ln\frac{V^0}{V}, V = k(\xi)\frac{\varphi'}{\theta'}$ using (11) or (18) we obtain - for the first time in this form - the singular integral - differential equation:

$$\ln\frac{\pi\theta'(\xi - a)}{k(\xi)} = \frac{\sqrt{1-\xi}}{\pi}\int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s-\xi}. \tag{25}$$

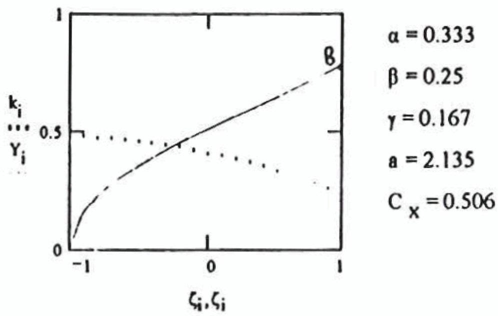
From this equation, we find $\theta = \theta(\xi)$, then $V(\xi)$ and $\omega(\zeta)$ may be determined (in the particular case of a circle arc, $k(\xi) \equiv 1$).

Using (19) for $\xi = a$, we find the equation:

$$\theta(B, C) = \gamma\pi = \frac{\pi(\beta-\alpha)}{\sqrt{2}}(\sqrt{a+1} - \sqrt{a-1}) + 2\alpha \arctan\sqrt{\frac{2}{a-1}}. \tag{26}$$

Specifying $\alpha, \beta$ and $\gamma(0 \leq \gamma < \beta < \alpha \leq 1/2)$, we solve the above algebrical equation finding $a(\alpha, \beta, \gamma)$. (Table 2). The result is used in (21), (23), (24). For an example the results are presented in Table 1.

| $\zeta_i$ | $X_i$ | $Y_i$ | $\theta(\zeta_i)$ | $V_i$ | $k_i$ |
|---|---|---|---|---|---|
| -1 | 0 | 0 | 1.047 | 0 | 0 |
| -0.9 | 0.098 | 0.158 | 0.989 | 0.174 | 0.487 |
| -0.8 | 0.143 | 0.225 | 0.964 | 0.252 | 0.481 |
| -0.7 | 0.179 | 0.277 | 0.946 | 0.316 | 0.476 |
| -0.6 | 0.211 | 0.32 | 0.93 | 0.374 | 0.47 |
| -0.5 | 0.24 | 0.358 | 0.916 | 0.428 | 0.464 |
| -0.4 | 0.267 | 0.392 | 0.904 | 0.479 | 0.456 |
| -0.3 | 0.292 | 0.425 | 0.892 | 0.529 | 0.447 |
| -0.2 | 0.317 | 0.455 | 0.882 | 0.577 | 0.438 |
| -0.1 | 0.341 | 0.484 | 0.872 | 0.623 | 0.427 |
| 0 | 0.365 | 0.511 | 0.862 | 0.669 | 0.415 |
| 0.1 | 0.388 | 0.538 | 0.853 | 0.713 | 0.402 |
| 0.2 | 0.411 | 0.565 | 0.844 | 0.757 | 0.389 |
| 0.3 | 0.435 | 0.591 | 0.836 | 0.799 | 0.374 |
| 0.4 | 0.458 | 0.617 | 0.828 | 0.84 | 0.358 |
| 0.5 | 0.482 | 0.643 | 0.82 | 0.88 | 0.342 |
| 0.6 | 0.507 | 0.67 | 0.813 | 0.918 | 0.324 |
| 0.7 | 0.533 | 0.696 | 0.806 | 0.953 | 0.305 |
| 0.8 | 0.559 | 0.724 | 0.799 | 0.985 | 0.285 |
| 0.9 | 0.587 | 0.752 | 0.792 | 1.011 | 0.263 |
| 1 | 0.617 | 0.783 | 0.785 | 1 | 0.233 |



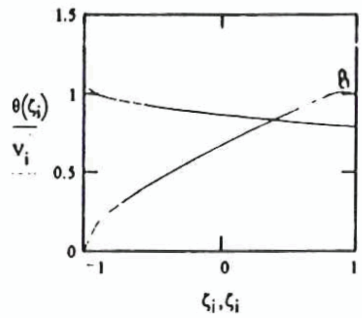




Table 1. The graphics for $\theta(\zeta_i)$, $V(\zeta_i)$, the curvature $k_i$ and the design of the profile $Y_i$.

## 4 THE COMPUTATION OF THE AERODYNAMIC FORCE

Using Bernoulli's law (6), with $P = \int_{OB}(p - p^0)ds$, along the symmetric airfoil $(B'OB)$, the resultant of the pressure forces is $P = \int_O^B \frac{\rho V^{0\,2}}{2}[1 - (\frac{V}{V^0})^2]dy$ and passing on the $\xi \in [-1, 1]$ segment we obtain:

$$P = \int_{-1}^{1} \frac{\rho V^{0\,2}}{2}[1 - (\frac{V(\xi)}{V^0})^2]\frac{\partial y}{\partial \xi}d\xi. \tag{27}$$

To compute $P$, the derivate $\frac{\partial y}{\partial \xi}$ results from (8), where $\theta(\xi)$ and $V(\xi)$ are defined through (16) and (17), respectively. Using (27) and (23) the expression of the drag coefficient $C_x = \frac{Pl}{\frac{\rho V^{0\,2}}{2}}$ becomes:

$$C_x = \frac{\int_{-1}^{1} [1 - (\frac{V(\xi)}{V^0})^2] \frac{\sin\theta(\xi)}{V(\xi)} \frac{d\xi}{a-\xi}}{\int_{-1}^{1} \frac{d\xi}{V(\xi)(a-\xi)}}. \tag{28}$$

In Table 1, the parameters $\alpha = 1/3, \beta = 1/4, \gamma = 1/6, V^o \equiv 1$ are given; $a$ and $C_x$ are computed and the functions $\theta(\xi), V(\xi), k(\xi), Y(\xi)$ are plotted.

| $\alpha$ | $\beta$ | $\gamma$ | $a$ | $C_x$ |
|---|---|---|---|---|
| 1/4(0.25) | 1/6(0.167) | 1/12(0.083) | 4.483 | 0.325 |
| 1/3(0.333) | 1/4(0.25) | 1/6(0.167) | 2.135 | 0.506 |
| 1/3(0.333) | 1/6(0.167) | 1/12(0.083) | 6.336 | 0.423 |
| 4/9(0.444) | 1/4(0.25) | 1/12(0.083) | 12.775 | 0.709 |
| 5/12(0.417) | 1/4(0.25) | 1/12(0.083) | 11.843 | 0.673 |
| 4/9(0.444) | 1/6(0.167) | 1/12(0.083) | 9.298 | 0.598 |

Table 2. The values of the parameter $a$ and the drag coefficient $C_x$.

In Table 2 for several values of the parameters $\alpha, \beta, \gamma$ the correspondent values of $a$ and $C_x$ are reported. The results are comparable to these obtained for direct problems by C. Iacob [3], [8] when the profile is curvilinear or dihedral, $(\alpha = \beta = \gamma)$. These inverse problem results can be subject to further researches to achieve the optimal control of the minimal drag or they can be used as a first approximation for axially-symmetrical problems in the case of compressible flow [4], [2].

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