LIBERTAS MATHEMATICA, VOL. XVIII (1998)

A CRITERION OF RELATIVE COMPACTNESS IN THE SPACE $B_{1,s}(\mathbb{R}^n)$

S. ZAIDMAN

Summary

In this paper we establish a criterion of relative compactness in the space $B_{1,s}(\mathbb{R}^n)$ which is similar to a criterion of relative compactness in the space $H^s(\mathbb{R}^n)$ (this one apparently well-known).

Introduction. The Banach space of(tempered) distributions $B_{1,s}(\mathbb{R}^n)$ - where s is any real number - is defined for instance in [3]:

$B_{1,s}(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n)$, such that $\hat{T} \in L^1_{loc}(\mathbb{R}^n)$ and $(1+|\xi|^2)^{s/2}\hat{T}(\xi) \in L^1(\mathbb{R}^n)$

It is in fact a complete normed space under the norm

$$\|T\|_{B_{1,s}} = \int_{\mathbf{R}^n} (1+|\xi|^2)^{s/2} |\hat{T}(\xi)| d\xi \tag{0.1}$$

(here $\xi = (\xi_1, \dots, \xi_n), |\xi|^2 = \xi_1^2 + \dots \xi_n^2$, $\wedge$ means the Fourier transformation).

In this work we shall prove a $B_{1,s}(\mathbb{R}^n)$ - version of a criterion for relative compactness of sets in $H^s(\mathbb{R}^n)$ which appears for example in [1] as "Lemma 8.1".

The special case of $B_{1,0}(\mathbb{R}^n)$ appears as "Proposition 2.1" in [4] - page 68.

1. Let us now state the

Theorem 1.1. *Let κ be a subset in $B_{1,s}(\mathbb{R}^n)$ satisfying conditions:*

α) For any $\Lambda > 0$ it results that

$$\lim_{|\tau|\to 0} \int_{|\xi|\le\Lambda} |\hat{u}(\xi+\tau) - \hat{u}(\xi)| d\xi = 0 \tag{1.1}$$

holds, uniformly for $u \in \kappa$

β) there exists a continuous function $\psi(t), [0,\infty) \to]0,\infty)$, such that $\lim_{t\to\infty}\psi(t) = \infty$ and

$$\int_{\mathbf{R}^n} (1+|\xi|^2)^{s/2} \psi(|\xi|) |\hat{u}(\xi)| d\xi \le C \quad \forall u \in \kappa \tag{1.2}$$

Then κ is relatively compact in $B_{1,s}(\mathbb{R}^n)$.

Proof. We shall use the well-known criterion of compactness in $L^1(\mathbb{R}^n)$ which appears in [4]- p. 68 as a) -b) -c).

Note also that from (0.1) it readily follows that a set κ is relatively compact in $B_{1,s}(\mathbb{R}^n)$ if and only if the set of Fourier transforms

$$\kappa^\wedge = \{\hat{u}, u \in \kappa\} \tag{1.3}$$

has the property that the set

$$\{(1+|\xi|^2)^{s/2}\hat{u}(\xi), u \in \kappa\} \tag{1.4}$$

is relatively compact in $L^1(\mathbb{R}^n)$.

Therefore, our proof will consist in the fact that from the above assumptions α) and β), the properties $a) - b) - c)$ are true for the set in (1.4).

As $\psi(t) > 0 \ \forall\, t \geq 0$ and $\psi(\cdot) \in C[0,\infty)$, it results $\inf_{0\leq t\leq \bar{t}} \psi(t) = \bar{\gamma} > 0$.

Also, as $\psi(t) \to +\infty$ at $+\infty$, we find $\bar{t} > 0$ such that $\psi(t) > 1$ for $t \geq \bar{t}$. Thus, $\psi(t) \geq \gamma = \inf\,(\bar{\gamma}, 1), \forall\, t \geq 0$ where $\gamma > 0$.

Then from (1.2) we derive the estimate

$$\gamma \int_{\mathbf{R}^n} (1+|\xi|^2)^{s/2}\, |\hat{u}(\xi)|\; d\xi \leq C \ \forall\, u \in \kappa \tag{1.5}$$

which is a) for the set $\{(1+|\xi|^2)^{s/2}\ \hat{u}(\xi), u \in \kappa\}$.

Next, again from $\psi(t) \to +\infty$ as $t \to +\infty$ we find, $\forall\, n \in \mathbb{N}$, a number $t_n > 0$ such that

$$\psi(t) \geq n \ \forall\, t \geq t_n \tag{1.6}$$

Therefore, again from (1.2) we derive

$$n \int_{|\xi|\geq t_n} (1+|\xi|^2)^{s/2}\, |\hat{u}(\xi)| d\xi \leq C, \ \forall\, n \in \mathbb{N}, \forall\, u \in \kappa \tag{1.7}$$

Now $\displaystyle\int_{|\xi|\geq \wedge} (1+|\xi|^2)^{s/2}\, |\hat{u}(\xi)| d\xi \leq \frac{c}{n}$ if $\wedge \geq t_n, \forall\, u \in \kappa$ (1.8)

and this implies that

$$\lim_{\wedge\to\infty} \int_{|\xi|\geq \wedge} (1+|\xi|^2)^{s/2}\, |\hat{u}(\xi)| d\xi = 0\,, \text{ uniformly for } u \in \kappa\,, \tag{1.9}$$

as requested in b) - p. 68 of [4].

It remains (as the main step) to establish the condition c) in [4]- p. 68, that is the $L^1(\mathbb{R}^n)$ equicontinuity of the set $\{(1+|\xi|^2)^{s/2}\hat{u}(\xi), u \in \kappa\}$ which amounts to the relation

$$\lim_{|\tau|\to 0}\int_{\mathbf{R}^n} |(1+|\xi+\tau|^2)^{s/2}\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\hat{u}(\xi)|d\xi = 0 \quad \text{uniformly for } u\in\kappa\,. \tag{1.10}$$

We write (taking any $\wedge > 0$), the equality

$$\int_{\mathbf{R}^n} |(1+|\xi+\tau|^2)^{s/2}\,\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\,\hat{u}(\xi)|d\xi = \int_{|\xi|\le\wedge+1} |(1+|\xi+\tau|^2)^{s/2}$$
$$\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\,\hat{u}(\xi)|d\xi + \int_{|\xi|\ge\wedge+1} |(1+|\xi+\tau|^2)^{s/2}\,\hat{u}(\xi+\tau) \tag{1.11}$$
$$-(1+|\xi|^2)^{s/2}\,\hat{u}(\xi)|\,d\xi = I_1+I_2\,.$$

Then, the inequality

$$I_1 \le \int_{|\xi|\le\wedge+1} (1+|\xi+\tau|^2)^{s/2}\,|\hat{u}(\xi+\tau)-\hat{u}(\xi)|\,d\xi + \int_{|\xi|\le\wedge+1} (1+|\xi+\tau|^2)^{s/2}$$
$$-(1+|\xi|^2)^{s/2}|\;|\hat{u}(\xi)|d\xi = I_3+I_4 \text{ holds true (as readily seen)}\,. \tag{1.12}$$

For I_3 we note the trivial estimate (when - say - $|\tau|\le 1$)

$$I_3 \le C_{\wedge,s}\int_{|\xi|\le\wedge+1} |\hat{u}(\xi+\tau)-\hat{u}(\xi)|d\xi \tag{1.13}$$

In order to estimate I_4 we first handle (as in [2] - p. 369) the expression $|(1+|\xi+\tau|^2)^{s/2}-(1+|\xi|^2)^{s/2}|$ which, in view of the mean-value theorem is

$$\le |\tau|\;|\text{ grad }(1+|\xi|^2)^{s/2}\,\xi=\zeta|., \text{ with } \zeta\in[\xi,\xi+\tau]\,. \tag{1.14}$$

Actually

$$|\text{ grad }(1+|\xi|^2)^{s/2}| = |s|\;|\xi|(1+|\xi|^2)^{\frac{s}{2}-1} = |s|\left(\frac{|\xi|^2}{1+|\xi|^2}\right)^{\frac{1}{2}}(1+|\xi|^2)^{s/2-\frac{1}{2}}$$
$$\le |s|(1+|\xi|^2)^{\frac{s-1}{2}} \le |s|(1+|\xi|^2)^{s/2} \tag{1.15}$$

Combining (1.14) with (1.15) we thus get the estimate

$$|(1+|\xi+\tau|^2)^{s/2}-(1+|\xi|^2)^{s/2}| \le |s|\;|\tau|\,(1+|\zeta|^2)^{s/2} \quad \text{where } \zeta\in[\xi,\xi+\tau]\,. \tag{1.16}$$

Therefore $\zeta = t(\xi+\tau)+(1-t)\xi = \xi+t\,\tau$ where $0<t<1$

We apply then Peetre's inequality and obtain accordingly

$$(1+|\xi+t\,\tau|^2)^{s/2} \le 2^{\frac{|s|}{2}}(1+|\xi|^2)^{s/2}\,(1+|t\,\tau|^2)^{\frac{|s|}{2}} \le C_s(1+|\xi|^2)^{s/2} \quad \text{for } 0<t<1, |\tau|\le 1 \tag{1.17}$$

Introducing in (1.16) one gets

$$|(1+|\xi+\tau|^2)^{s/2}-(1+|\xi|^2)^{s/2}| \leq C_{1,s}|\tau|\,(1+|\xi|^2)^{s/2}, \quad \text{for } |\tau|\leq 1 \tag{1.18}$$

and accordingly we obtain

$$I_4 \leq \int_{|\xi|\leq\Lambda+1} C_{1,s}|\tau|(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi \leq C_{1,s}|\tau|\int_{\mathbf{R}^n}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi \tag{1.19}$$

Thus we now have - from 1.11 - 1.12 - 1.13 - 1.19

$$\begin{aligned}&\int_{\mathbf{R}^n}|(1+|\xi+\tau|^2)^{s/2}\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\hat{u}(\xi)|d\xi \leq \\ &C_{\Lambda,s}\int_{|\xi|\leq\Lambda+1}|\hat{u}(\xi+\tau)-\hat{u}(\xi)|d\xi + C_{1,s}|\tau|\int_{\mathbf{R}^n}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi + \\ &\int_{|\xi|\geq\Lambda+1}|(1+|\xi+\tau|^2)^{s/2}\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\hat{u}(\xi)|d\xi\end{aligned} \tag{1.20}$$

Looking at the last term in right-hand side of (1.20) we note that the set $\{\xi\in\mathbb{R}^n, |\xi|\geq \Lambda+1\}$ is contained in the set $\{\xi\in\mathbb{R}^n, |\xi+\tau|\geq\Lambda\}$ provided that $|\tau|<1$; it follows that

$$\begin{aligned}&\int_{|\xi|\geq\Lambda+1}|(1+|\xi+\tau|^2)^{s/2}\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\hat{u}(\xi)|d\xi \leq \int_{|\xi|\geq\Lambda+1}(1+|\xi+\tau|^2)^{s/2} \\ &|\hat{u}(\xi+\tau)|d\xi + \int_{|\xi|\geq\Lambda+1}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi \leq \int_{\{\xi\in\mathbf{R}^n,|\xi+\tau|\geq\Lambda\}}(1+|\xi+\tau|^2)^{s/2} \\ &|\hat{u}(\xi+\tau)|d\xi + \int_{|\xi|\geq\Lambda}(1+|\xi|^2)^{s/2}\hat{u}(\xi)|d\xi = 2\int_{|\xi|\geq\Lambda}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi\end{aligned} \tag{1.21}$$

Therefore (1.20) now becomes

$$\begin{aligned}&\int_{\mathbf{R}^n}|(1+|\xi+\tau|^2)^{s/2}\hat{u}(\xi+\tau)-(1+|\xi|^2)^{s/2}\hat{u}(\xi)|d\xi \leq C_{\Lambda,s}\int_{|\xi|\leq\Lambda+1} \\ &|\hat{u}(\xi+\tau)-\hat{u}(\xi)|\,d\xi + C_{1,s}|\tau|\int_{\mathbf{R}^n}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi + \\ &2\int_{|\xi|\geq\Lambda}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi\,.\end{aligned} \tag{1.22}$$

Finally, if $u\in\kappa$ we know that, $\forall\Lambda>0$,

$\lim_{|\tau|\to 0}\int_{|\xi|\leq\Lambda+1}|\hat{u}(\xi+\tau)-\hat{u}(\xi)|\,d\xi = 0$ uniformly on κ (from assumption (1.1)).

$\int_{\mathbf{R}^n}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi \leq \frac{c}{\gamma}, \forall u\in\kappa$ (from estimate (1.5)).

$\lim_{\Lambda\to\infty}\int_{|\xi|\geq\Lambda}(1+|\xi|^2)^{s/2}|\hat{u}(\xi)|d\xi = 0$, uniformly for $u\in\kappa$ (from (1.8)).

Introducing these properties in (1.22) we get (1.10) - uniformly on κ. ■

REFERENCES

[1] J.J. Kohn-L. Nirenberg: An algebra of pseudo-differential operators, Comm. Pure Appl. Math., 18, p. 267-305, 1965.

[2] S. Zaidman: Pseudo-Differential Operators, Ann. Mat. pura ed. appl. (IV) vol. XCII, pp. 345-399, 1972.

[3] S. Zaidman: Distributions and Pseudo-Differential Operators, Pitman RNM vol. 248, Longman Scientific and Technical, 1991.

[4] S. Zaidman: Topics in pseudo-differential operators, Pitman RNM vol. 359, Longman, 1996.