LIBERTAS MATHEMATICA,VOL.XVIII(1998)

UNIDIRECTIONAL FLOWS OF SECOND GRADE FLUIDS IN SPHERICAL DOMAINS

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Summary. Exact solutions corresponding to some unidirectional flows of a second grade fluid filling spherical domains are obtained. The adequate steady flows appear as a limiting case for $t \to \infty$. The differences between our solutions and those corresponding to Newtonian fluids are delineated.

1. Introduction

Among the many constitutive assumptions that have been employed to study the behaviour of certain real fluids for which the Navier-Stokes equations seem to be an inappropriate model, one class that has gained support from both the experimentalists and the theoreticians is that of Rivlin-Ericksen fluids of second grade. For these fluids many exact solutions have been obtained during the last sixteen[1] years [1-8]. Recently, Bandelli and Rajagopal [7], for example, have treated some unsteady unidirectional flows, in Cartesian and cylindrical coordinates. They have also considered an unidirectional motion in spherical coordinates but no solution was given.

The aim of this paper is to present some exact solutions corresponding to this last flow of a second grade fluid filling spherical domains. The existence and the uniqueness of a such flow were prove in [11]. For some interesting studies, regarding the motion of a viscous medium in a spherical domain see [12-13] where different exact solutions

[1] As early as, Ting [9] and Markovitz and Coleman [10] have studied unsteady flows of fluids of second grade.

were obtained, or [14-19] where the numerical simulation results were compared with experimental observations.

The arrangement of the paper is as follows. In Sec.2, we present the governing equations which are followed, in Sec. 3, by a general discussion. The exact solutions corresponding to some spherical gap flows and to a flow within a sphere are obtained in Sec. 4 and 5, respectively. For each case, the velocity profile is easily derivable from the corresponding one to a Newtonian fluid. In steady flows these profiles coincide.

2. Governing equations

For a homogeneous incompressible fluid of second grade, the Cauchy stress $\mathbf{T}$ and the fluid motion are assumed to be related as follows [1-11]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli, $-p\mathbf{I}$ is the spherical stress due to the constraint of incompressibility and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T \text{ and } \mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1. \tag{2.2}$$

Here $\mathbf{L}$ represents the spatial gradient of the velocity $\mathbf{v}$ and the overdot denotes material time differentiation. Of course, all motions are restricted to be isochoric, so that

$$div\ \mathbf{v} = 0, \tag{2.3}$$

and, thus, $\mathbf{A}_1$ is traceless. Further, Dunn and Fosdick [20] showed that if all motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy be a minimum in equilibrium, then the material coefficients obey the restrictions

$$\mu \geq 0, \quad \alpha_1 \geq 0 \text{ and } \alpha_1 + \alpha_2 = 0. \tag{2.4}$$

The types of flows to be considered here, have the contravariant components of the velocity field $\mathbf{v}$

$$\dot{r} = 0, \quad \dot{\theta} = 0, \quad \dot{\varphi} = \omega(r, t), \tag{2.5}$$

in the spherical co-ordinate system r, θ and φ. For such flows, from (2.1) and (2.2), it results that the physical components of the extra-stress tensor $\mathbf{S} = p\mathbf{I} + \mathbf{T}$ are functions of r, θ and t only. By using this result we can easily show that the motion equations reduce to[2]

$$T_{rr} = \rho\psi + a(t) + b(t)\varphi + c(r, \theta, t), \tag{2.6}$$

and

$$(\mu + \alpha_1\partial_t)(\partial_r^2 + \frac{4}{r}\partial_r)\omega(r,t) - \frac{b(t)}{r^2 sin^2\theta} = \rho\partial_t\omega(r,t), \tag{2.7}$$

where

$$\frac{\partial c}{\partial r} + \frac{1}{r}\partial_\theta T_{r\theta} + \frac{1}{r}(2T_{rr} - T_{\theta\theta} - T_{\varphi\varphi} + T_{r\theta}ctg\theta) = -\rho r\omega^2 sin^2\theta,$$

$$\frac{\partial c}{\partial \theta} + \partial_\theta(T_{\theta\theta} - T_{rr}) + r\partial_r T_{r\theta} + 3T_{r\theta} + (T_{\theta\theta} - T_{\varphi\varphi})ctg\theta = -\rho r^2\omega^2 sin\theta cos\theta.$$

Here ψ is the body force potential, $a(\cdot)$ is an arbitrary function of t, and the function $b(\cdot)$ must vanish because T_{rr} has to be a single valued function of position.

3. General discussions

In order to determine an unsteady unidirectional flow (2.5) of a second grade fluid, filling a spherical domain, we have to solve the following equation with partial derivatives

$$(\mu + \alpha_1\partial_t)(\partial_r^2 + \frac{4}{r}\partial_r)\omega(r,t) = \rho\partial_t\omega(r,t). \tag{3.1}$$

The coefficient α_2 does not appear in this equation. Consequently, the unknown function $\omega(r,t)$ will not depend of this coefficient, though, of course, the surface tractions that must be applied in order to produce the motion, will vary according to its value.

In the case of a Reiner-Rivlin fluid, when $\alpha_1 = 0$, the equation (3.1) reduces to

$$\mu(\partial_r^2 + \frac{4}{r}\partial_r)\omega(r,t) = \rho\partial_t\omega(r,t), \tag{3.2}$$

which is identical with that resulting from the Navier-Stokes theory. Therefore, the unsteady unidirectional flows (2.5) of an incompressible Reiner-Rivlin fluid are the same as those of an incompressible Newtonian fluid, although, again, certain normal stresses can depend on α_2.

[2]The motion being isochoric, the continuity equation is identically verified.

In the steady case when ω in (2.5) is a function of r only, the equation (3.1) reduces to[3]

$$(\partial_r^2 + \frac{4}{r}\partial_r)\omega(r) = 0, \tag{3.3}$$

which is again identical with that resulting from the Navier-Stokes theory. Consequently, the steady unidirectional flows (2.5) of an incompressible second grade fluid or of an incompressible Reiner-Rivlin fluid are the same as those in an incompressible Newtonian fluid.

4. Flow between two rotating spheres

In the following we consider an unsteady unidirectional flow of a homogeneous incompressible fluid of second grade, filling the gap between two concentric rotating spheres of radii R_1 and R_2. We also assume that the motion is due to the two spheres, which at the moment $t = 0$ suddenly begin to rotate about the same axis $\theta = 0$ with the angular velocities $\Omega_1(t)$ and $\Omega_2(t)$, respectively. Assuming that the fluid adheres to the walls and that it is at rest up to the moment $t = 0$, we have to solve the equation (3.1) with boundary and initial conditions

$$\omega(R_1, t) = \Omega_1(t), \quad \omega(R_2, t) = \Omega_2(t); \quad t \geq 0, \tag{4.1}$$

and

$$\omega(r, 0) = 0; \quad r \in (R_1, R_2). \tag{4.2}$$

Making a change of unknown function

$$\omega(r,t) = \Omega_2(t) + \frac{\Omega_2(t) - \Omega_1(t)}{R_2^3 - R_1^3} \cdot \frac{R_1^3}{r^3}(r^3 - R_2^3) + r^{-3/2}w(r,t), \tag{4.3}$$

our problem reduces to the next equation

$$(\mu + \alpha_1\partial_t)\Delta w(r,t) = \rho\partial_t w(r,t) + d(r,t); \quad r \in (R_1, R_2), \quad t > 0, \tag{4.4}$$

with boundary and initial conditions

$$w(R_1, t) = w(R_2, t) = 0; \quad t \geq 0, \tag{4.5}$$

[3]The relations (3.3) and (3.2) can be obtained from the equations (3.1c) of [12] and (1.7) of [13] for $v(r,\theta) = r sin\theta\omega(r)$ and $v(r,\theta,t) = r^2 sin^2\theta\omega(r,t)$, respectively.

respectively,

$$w(r,0)=r^{3/2}[-\Omega_2+\frac{\Omega_2-\Omega_1}{R_2^3-R_1^3}\cdot\frac{R_1^3}{r^3}(R_2^3-r^3)];\quad r\in(R_1,R_2). \tag{4.6}$$

Here the operator $\Delta=\partial_r^2+\frac{1}{r}\partial_r-\frac{9}{4r^2}$, the function $d(r,t)$ is given by the relation

$$d(r,t)=\rho r^{3/2}[\Omega_2'(t)+\frac{\Omega_2'(t)-\Omega_1'(t)}{R_2^3-R_1^3}\cdot\frac{R_1^3}{r^3}(r^3-R_2^3)], \tag{4.7}$$

and the pair $(\Omega_1,\Omega_2)=\lim_{t\searrow 0}(\Omega_1(t),\Omega_2(t))$.

Now, if we multiply both sides of the equation (4.4) by $rB_{3/2}(rr_n)$, where [4]

$$B_{3/2}(rr_n)=J_{3/2}(rr_n)J_{-3/2}(R_1r_n)-J_{3/2}(R_1r_n)J_{-3/2}(rr_n),$$

and r_n are the positive roots of the transcendental equation

$$J_{3/2}(R_2r)J_{-3/2}(R_1r)-J_{3/2}(R_1r)J_{-3/2}(R_2r)=0,$$

and integrate over r from R_1 to R_2, we find that

$$w_n'(t)+a_nw_n(t)+b_n(t)=0;\quad t>0,\quad n\in N^*, \tag{4.8}$$

and

$$w_n(0)=w_{n0};\quad n\in N^*. \tag{4.9}$$

In the last two relations $a_n=\mu r_n^2/(\rho+\alpha_1r_n^2)$, $b_n(\cdot)=d_n(\cdot)/(\rho+\alpha_1r_n^2)$ and $w_n(\cdot)$, w_{n0} and $d_n(\cdot)$ are the finite Hankel transforms [21] (Sect.98) of the functions $w(r,\cdot)$, $w(r,0)$ and $d(r,\cdot)$, respectively.

As the solutions of the equations (4.8), with the conditions (4.9), are of the form

$$w_n(t)=e^{-a_nt}[w_{n0}-\int_0^t b_n(s)e^{a_ns}ds],$$

the solution of our problem will be given by (see the inversion theorem for the finite Hankel transform [21] and the relation (4.3))

$$\omega(r,t)=\Omega_2(t)+\frac{\Omega_2(t)-\Omega_1(t)}{R_2^3-R_1^3}\cdot\frac{R_1^3}{r^3}(r^3-R_2^3)+$$

$$+\frac{\pi^2}{2r^{3/2}}\sum_{n=1}^{\infty}\frac{r_n^2J_{3/2}^2(R_2r_n)}{J_{3/2}^2(R_1r_n)-J_{3/2}^2(R_2r_n)}\cdot w_{n0}B_{3/2}(rr_n)e^{-a_nt}- \tag{4.10}$$

[4]Here $J_{3/2}(\cdot)$ and $J_{-3/2}(\cdot)$ are Bessel functions of the first kind.

$$-\frac{\pi^2}{2r^{3/2}}\sum_{n=1}^{\infty}\frac{r_n^2 J_{3/2}^2(R_2 r_n)}{J_{3/2}^2(R_1 r_n)-J_{3/2}^2(R_2 r_n)}\cdot B_{3/2}(rr_n)e^{-a_n t}\int_0^t b_n(s)e^{a_n s}ds.$$

If the two spheres rotate with constant velocities Ω_1 and Ω_2 , then $d(r,t)$ from (4.7) and therefore $d_n(t)$ respectively $b_n(t)$ will be zero, while the relation (4.10) reduces to

$$\omega(r,t)=\Omega_2+\frac{\Omega_2-\Omega_1}{R_2^3-R_1^3}\cdot\frac{R_1^3}{r^3}(r^3-R_2^3)+ \\ +\frac{\pi^2}{2r^{3/2}}\sum_{n=1}^{\infty}\frac{r_n^2 J_{3/2}^2(R_2 r_n)}{J_{3/2}^2(R_1 r_n)-J_{3/2}^2(R_2 r_n)}\cdot w_{n0}B_{3/2}(rr_n)e^{-a_n t}, \tag{4.11}$$

from which the solution corresponding to the steady flow, i.e.

$$\omega(r)=\Omega_2+\frac{\Omega_2-\Omega_1}{R_2^3-R_1^3}\cdot\frac{R_1^3}{r^3}(r^3-R_2^3);\ \ r\in(R_1,R_2), \tag{4.12}$$

appears as a limiting case for $t\to\infty$.

As regards the steady case, we shall now consider the next particular cases:

Case 1. The both spheres are rotating with the same angular velocity, i.e. $\Omega_1 = \Omega_2 = \Omega$. In this case all particles of the fluid are moving with the same angular velocity

$$\omega(r)=\Omega;\ \ r\in(R_1,R_2). \tag{4.13}$$

Case 2. The both spheres are rotating with the same angular velocity but in opposite directions, i.e. $\Omega_1 = -\Omega_2 = \Omega$. In this case, all particles belonging to the surface $r = R_1R_2\sqrt[3]{2}\ /\ \sqrt[3]{R_1^3+R_2^3}$ are at rest.

Case 3. One of the sphere is at rest and the other one is rotating with the angular velocity Ω, i.e. $\Omega_1 = 0,\ \Omega_2 = \Omega$ or $\Omega_1 = \Omega,\ \Omega_2 = 0$. In these two last cases all particles belonging to the surface $r = R_1R_2\sqrt[3]{2}/\sqrt[3]{R_1^3+R_2^3}$ are rotating with the same angular velocity $\omega = \Omega/2$.

Remark. As $R_1R_2\sqrt[3]{2}/\sqrt[3]{R_1^3+R_2^3} < (R_1+R_2)/2$, if $0 < R_1 < R_2$, we can affirm that the outer sphere has a greater influence upon the motion of the fluid. After all, if the inner sphere is rotating with the angular velocity $\Omega_1 = -\Omega$, the outer sphere has to rotate with the angular velocity

$$\Omega_2=\frac{R_1^3}{R_2^3}\cdot\frac{8R_2^3-(R_1+R_2)^3}{(R_1+R_2)^3-8R_1^3}\Omega<\Omega$$

in order that the particles of the surface $r = (R_1+R_2)/2$ to be at rest.

5. The spin-up flow within a sphere

Let us now consider the unsteady motion of a homogeneous incompressible fluid of second grade filling a domain bounded by a rotating sphere of radius R. If we assume that the motion is due to the sphere, which at the moment $t = 0$ suddenly begins to rotate about the axis $\theta = 0$ with the angular velocity $\Omega(t)$, we have to solve the equation (3.1) with the boundary and initial conditions

$$\omega(R,t) = \Omega(t); \quad t > 0, \tag{5.1}$$

and

$$\omega(r,0) = 0; \quad r \in [0,R). \tag{5.2}$$

Moreover, the natural condition of boundedness in $r = 0$, i.e.

$$|\,\omega(0,t)\,| < \infty; \quad t \geq 0, \tag{5.3}$$

has to be carried out.

Making the change of unknown function

$$\omega(r,t) = \Omega(t) + r^{-3/2} w(r,t), \tag{5.4}$$

and following the same way as before, i.e. multiplying both sides of the obtained equation by $rJ_{3/2}(rr_n)$ and integrating over r from 0 to R, we find our solution under the form[5]

$$\begin{aligned} \omega(r,t) = \Omega(t) + \frac{2\Omega R^{1/2}}{r^{3/2}} \sum_{n=1}^{\infty} \frac{J_{3/2}(rr_n)}{r_n J'_{3/2}(Rr_n)} e^{-a_n t} + \\ + \frac{2\rho R^{1/2}}{r^{3/2}} \sum_{n=1}^{\infty} \frac{J_{3/2}(rr_n)}{r_n J'_{3/2}(Rr_n)} \cdot \frac{e^{-a_n t}}{\rho + \alpha_1 r_n^2} \int_0^t \Omega'(s) e^{a_n s} ds. \end{aligned} \tag{5.5}$$

In this last relation $\Omega = \lim_{t \searrow 0} \Omega(t)$, r_n are positive roots of the transcendental equation $J_{3/2}(Rr) = 0$ or equivalent $\mathrm{tg}(Rr) = Rr$, and the boundary and initial conditions (5.1) and (5.2) appear easily verified.

If our sphere rotates with the constant velocity Ω, then the relation (5.5) reduces to[6]

$$\omega(r,t) = \Omega + \frac{2\Omega R^{1/2}}{r^{3/2}} \sum_{n=1}^{\infty} \frac{J_{3/2}(rr_n)}{r_n J'_{3/2}(Rr_n)} e^{-a_n t}, \tag{5.6}$$

[5]Here we have used the known relation $J'_{3/2}(Rr_n) = -J_{5/2}(Rr_n)$ and the finite Hankel transform $R^{5/2}J_{5/2}(Rr_n)/r_n$ of the function $r^{3/2}$.

[6]If in (5.6) we make $\alpha_1 = 0$, this relation fall over (1.9) of [13]. There, $\nu = \mu/\rho$ and $a_n = Rr_n$.

from which the solution corresponding to the steady flow[7], i.e.

$$\omega(r) = \Omega; \quad r \in (0, R), \tag{5.7}$$

appears, again, as a limiting case for $t \to \infty$.

6. Conclusions.

The velocity fields corresponding to spherical gap flows and to spin-up flows within a sphere are given by the relations (4.10)-(4.12) and (5.5)-(5.7), respectively. The extra-stress components S_{ij} $(i, j = 1, 2, 3)$ can be obtained, in each case, using these relations as well as (2.1) and (2.2). The hydrostatic pressure p is also determined, up to an arbitrary function of t, by (2.6).

The unsteady solutions corresponding to a *Newtonian fluid* can be obtained making $\alpha_1 = 0$ in the relations (4.10), (4.11), (5.5) and (5.6), while *the steady ones*, given by (4.12) and (5.7), are the same for both *the Newtonian* and *the second grade fluid.*

In the case of *a steady spherical gap flow, the outer sphere* has *a greater influence* upon the motion of the fluid.

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