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PRIORITIES AND CONTRIBUTIONS OF F. VASILESCU IN THE THEORY OF MULTIFUNCTIONS

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1 Introduction

1. The theory of multifunctions is an useful tool for the most different branches of mathematics: differential equations, programming, game theory, optimal control etc.

This theory was appeared in closed relation with the theory of real functions of one or several real variables. The year 1925 is the beginning moment of the study for multifunctions concerning their continuity. In that year Florin Vasilescu presented two communications at the Société mathématique de France [29] and in the day of May 28 he presented his Thesis "*Essai sur les fonctions multiformes de variables réelles*" in front of a committee formed of E. Goursat (president) and P. Montel, J. Chazy (examiners). F. Vasilescu's thesis is a systematic and wide study of the continuity properties for real multifunctions. In the introduction of his thesis, he says: "*Elle fait apparaître la possibilité de généralisations qui puevent être intéressantes et utiles, tant que l'on n'a à s'occuper que des questions de voisinage, de continuité, de limite*" [30, p.3].

Indeed, in this thesis are introduced for the first time the following notions relative to the real multifunctions: upper and lower semicontinuity (u.s.c. and l.s.c.), the closed multifunctions (in the modern terminology), convergent, uniform convergent and quasi-convergent sequences of multifunctions etc. There are attacked some problems with origin in the real function theory: the extension of continuous multifunctions, approximation by multifunctions which are the union of a finite number of polynomials, Baire's classification, implicit definite multifunctions etc.

F. Vasilescu gives a reason for his study by the fact that multifunctions appear in a natural way in the classical analysis. For instance, if it is given a divergent series of functions we can define a multifunctions by associating to each point x the closed set of the limit points of the series of its values in x. Also, the system $f_1(x, y, ...) = 0, f_2(x, y, ...) = 0, ..., f_k(x, y, ...) = 0$ defines some of the variables as implicit functions of the others, and these are multifunctions, generally.

In [21], B.L. McAllister presents an interesting and remarkable historical exposition of the main topics in the theories of hyperspaces and multifunctions during the first half of the century (1900–1950). He states in Section 6, p.314, that the notions of upper and lower semicontinuity have been defined independently and nearly simultaneously by W.A. Wilson [33], L.S. Hill [15], and W. Hurewicz [16] in 1926. Unfortunately, he had not directly access to Vasilescu's thesis: "*Vasilescu's thesis (...) apparently also dealt with multifunctions. I have been unable to examine a copy of this thesis, but there is some evidence that he invented both upper and lower semicontinuity for multifunctions independently of Hill, Wilson, and Hurewicz, and perhaps even a year earlier. See* [6, p.1770]". All of these McAllister's suppositions are proved to be true.

In 1933, E. Blanc [6] makes precise the raport between Vasilescu's "*fermeture horizontale*" and "*fermeture extérieure*" (see below) and the notions of $\overline{SCI}$ (*upper semicontinuity of inclusion*) introduced by C. Bouligand [8] and $\underline{SCI}$ (*lower semicontinuity of inlusion*), introduced by itself [5], and also the Kuratowski's notions of semicontinuity [17].

The objective of this work is to point out Vasilescu's contribution in the theory of continuous multifunctions and his priority concerning some notions and results in this theory.

2 Fundamental notions

2. According to the proposed aim, F. Vasilescu considers only multifunctions F that assigns to each element of a given subset X of the Euclidean space $\mathbb{R}^n$ a nonempty subset of $\mathbb{R}$. At the same time, he notices that: "*On pourrait encore définir des opérations multiformes sur des ensembles abstraits*

[13], *mais c'est là une extension dont nous ne nous occuperons ici*" [30, p.12].

The tool of investigation presented everywhere in his thesis is the distance between two sets (*écart de deux ensembles*) [30, p.7]. Because the considered multifunctions have as values the subsets of real numbers, this distance is defined just for the subsets in $\mathbb{R}$ and it is mentioned that the extension to more general framework is evident. We shall adopt the context of the metric spaces.

Let (X, d) be a metric space. Let $B(M, \varepsilon)$ denote the ε-closed sphere of center $M \subset X$. Let E and F be two closed bounded nonempty subsets in X. The sets E and F *differ at most with* ε (*diffèrent de* ε *au plus*) if $E \subset \bigcup\limits_{x \in F} B(x, \varepsilon)$ and $F \subset \bigcup\limits_{x \in E} B(x, \varepsilon)$. The infimum of all these ε is called the *distance between two sets* (*l'écart de deux ensembles*). Therefore, we have

$$D(E, F) = \inf\{\varepsilon > 0;\ E \text{ and } F \text{ differ at most with } \varepsilon\}. \tag{1}$$

This distance coincides with that one introduced in 1905 by D. Pompeiu in his thesis [25] for the case of the plane sets and which has been put in a form equivalent to (1) by F. Hausdorff in 1914 [14, p.293]. It is possible that F. Vasilescu had not knew about the existence of this concept because of the World War I which had broken the normal changes of information in all domains of activity (F. Vasilescu had taken part in this war during 1916–1918 [1, p.80]). Later, he took note of Pompeiu's priority, as it was resulted from a fragment of his manuscript reproduced in [1, p.82]: "*Il semble que la notion d'***écart** *que j'ai donné dans ma thèse équivale au plus grand des nombres* Δ_{hk} *et* Δ_{kh} *que M. Pompeiu emploie dans sa thèse p.18 pour définir l'***écart mutuel** " (F. Vasilescu's underlinenings).

The Sections 3–7 deal with the convergence of the sequences of subsets. The sequence (E_n) *tends to the set* E *up to* ε (*à* ε *près*) if

1° for any $\varepsilon' > \varepsilon$ and $x \in E$ the closed ball $B(x, \varepsilon')$ meets E_p for each $p \geq n$ (n depends on ε and x), and

2° $E_p \subset \bigcup\limits_{x \in E} B(x, \varepsilon')$ beginning with some index.

Also, (E_n) *tends to* E if for any $\varepsilon > 0$ (E_n) tends to E up to ε. By using the actual terminology, then $E_n \rightarrow E$ if $E \subset \lim E_n$ and $\lim e(E_n, E) = 0$, where

$$e(E_n, E) = \inf\{\varepsilon > 0;\ E_n \subset \bigcup_{x \in E} B(x, \varepsilon)\}. \tag{2}$$

Let X be a metric space and let $C(X)$ ($K(X)$) be the set of all nonempty bounded closed (compact) subsets of X (in particular, $C(\mathbb{R}) = K(\mathbb{R})$). It is observed that, if $E, E_n \in C(X)$ and $E_n \to E$ (in the sense of the precedent definition), then $E = \underline{\lim} E_n = \overline{\lim} E_n$, that is (E_n) tends to E in the sense of the topological convergence. On the other hand, in [30, p.9] it is proved that on the space $C(\mathbb{R})$ the convergence in the sense given by F. Vasilescu is equivalent with the convergence with respect to the metric (1) (this assertion is also true on $K(X)$). Hence, $E_n \to E \Longleftrightarrow D(E_n, E) \to 0$ $(E_n, E \in K(X))$. We mention that Theorem VI in [14, p.297] requires an additional condition for the coincidence on $K(X)$ of the topological convergence with that one in the metric (1), namely, it is required the compactness of the union $\bigcup_n E_n$.

The notion of *set defined by a sequence of sets* introduced at p.9 isn't something else than the lower limit of the sequence (see [18, p.241] and [21, p.310] for the $\overline{\lim} E_n$ and $\underline{\lim} E_n$). It is proved via this notion a Cauchy criterion for convergence of a sequence of sets.

3. In Sections 8–20 a few types of continuity are defined for the multifunctions. As a matter of fact, it is here that occur, for the first time, all the types of continuity usually introduced into the theory of multifunctions.

The multifunctions considered in [30] are defined on a subset Π of $\mathbb{R}^n$ and have as values subsets of $\mathbb{R}$. Generally, Π and the values of the multifunctions are consdered bounded closed nonempty sets (that is, compact sets). The set–values of such a multifunction F are imagined as being situated on an axis. That will mark the denomination of some types of continuity introduced: vertically (horizontally or externally) closed multifunctions. These notions of continuity are generally defined, at first, at one point and only then globally, at all the points where the multifunctions F is defined. Also, there can be distinguished two stages of generality in the presentation of these notions:

1° in specific terms of the topological structures, and

2° using the euclidean metric (more precisely, using the metric (1) on $C(\mathbb{R})$).

The multifunction F is *vertically closed* (*fermée verticalement*) if the set $F(P)$ is closed for any point $P \in \Pi$ [30, p.13].

The multifunction F is *horizontally closed at the point* P (*fermée horizontalement en* P) if, for any sequence of points $P_1, P_2, \ldots, P_n, \ldots$ convergent to P and any sequence of numbers $y_1, y_2, ..., y_n, ...$ convergent to y, so that $y_n \in F(P_n)$ $(n = 1, 2, 3, ...)$ it follows that $y \in F(P)$. It is said that F is *horizontally closed* if this property takes place for any point P [30, p.13]. It is also proved that F is horizontally closed if and only if its graph is a closed set (a well-known fact [4, p116]). In the current terminology such a multifunction is called a *closed* one.

The notion of upper semicontinuous multifunction is presented in F. Vasilescu's thesis without having a proper name, because it coincides, in the particular considered context with that of a closed multifunction (according to the theorem: if the multifunction F is defined on the topological space X and it has compact values in the separated Hausdorff topological space Y, then F is u.s.c. if and only if it is closed). More precisely, the property which usually serves to define u.s.c. multifunctions occurs in [30, p.14] as a condition characterizing the closed multifunctions: F is horizontally closed if and only if for any $\varepsilon > 0$ there is a domain of center P so that for each point P_1 in Π situated in this domain we have $F(P_1) = \cup\{B(y, \varepsilon); y \in F(P)\}$. Taking into account (2), this inclusion can be written $e(F(P), F(P_1)) \leq \varepsilon$.

The notion of lower semicontinuous multifunction is presented in [30] both with a definition of topological nature and with one of metric nature. The multifunction F is *continuous in the point P for a value $y_0 \in F(P)$* if for any $\varepsilon > 0$ there is a domain of center P so that for any point in Π situated in this domain there exists a value y of the multifunction for which $|y - y_0| < \varepsilon$ [30, p.15]. Obviously, if F is continuous in P for any of its values, then F is l.s.c. in P (in the current acception of this concept).

On the other hand, a multifunction F is *externally continuous in P up to ε(fermée extérieurement en P à ε près)* if there is a domain of center P so that for any point P_1 in Π situated in this domain we have $F(P) \subset \cup\{B(y, \varepsilon); y \in F(P_1)\}$. The multifunction F is *externally closed in P* if it is externally closed in P up to ε for any $\varepsilon > 0$ [30, p.14]. In the same definition one can find the notion of l.s.c. multifunction. This is not an accidental fact, in Section 12 of the thesis is demonstrated the following statement: in order that F be continuous in P for any of its values it is necessary and sufficient that it be externally closed in P. Finally, it must be noticed that the previous inclusion can be transcribed as $e(F(P), F(P_1)) \leq \varepsilon$.

F. Vasilescu is conducted to the notion of *continuous multifunction in a point* realizing that, in order to accomplish one's aim, it is important to bring together the notions of horizontally and externally closed multifunction in this point (i.e. F is continuous in P if it is u.s.c. and l.s.c. in P). The above considerations and the utilization of the metric (1) make possible the formulation: F is continuous in P if for any $\varepsilon > 0$ there is a domain of center P so that $D(F(P), F(P_1)) < \varepsilon$ for each point P_1 in Π situated in this domain.

It is also introduced the notion of uniform continuous multifunction in a sense easy to infer. It is shown that, since Π is a compact set, a continuous multifunction also is uniform continuous. Moreover, it is bounded and touches its extreme values. But even if the set Π is a continuum, the multifunction does not necessarily take any intermediate value between the two arbitrary values. A multifunction defined in the points of dense subset in Π and uniform continuous on this set can be extended to Π.

In the papers [15], [16] and [33], which was appeared nearly simultaneously in 1926, it is also introduced what we now call u.s.c. and l.s.c. multifunctions in the restrictive context of the Euclidean spaces. A more detailed analysis of the contributions of the authors of these papers in the theory of multifunctions can be find in [21, Section 6].

3 Principal results

4. F. Vasilescu points out the simplicity of continuous multifunctions in relation with the arbitrary ones and the analogy with the continuous functions. In Sections 21–25 are stated and demonstrated some theorems which make evident notable properties of the continuous multifunctions. We state here two of them (we remind that the multifunctions considered are defined on a compact set Π and they have compact values):

Let F and G be two continuous multifunctions, and A and B be two closed disjoint subset of Π. Then, there exists a continuous multifunction defined on Π which coincides with F on A and with G on B (p.23).

If F is a continuous multifunction on Π, there is a continuous multifunction on the entire space $\mathbb{R}^n$ which coincides with F on Π (p.27).

These theorems have their roots in well-known results of the theory of real functions [7], [19], [28]. On the other hand, we observe that even in 1925 P. Urysohn [27, pp.290 and 293] has given an extension of these results to the normal topological spaces (obviously, only in the case of functions), which are precisely so–called now Urysohn's Lemma and Tietze's Extension Theorem.

5. The union of a finite number of multifunctions is considered as an operation inverse in some regards to that one from classical analysis which sets off the branches of a multiform function.

In Section 29 it is established the following result which is analogous with the Weierstrass Approximation Theorem: *a continuous multifunction F can be represented up to ε by a multifunction Φ which is finite union of polynomials* (i.e. $D(F(P), \Phi(P)) < \varepsilon,\ P \in \Pi$). In [26] it can be found more recent generalizations of the same classical theorem.

In Section 30, F. Vasilescu proves a theorem about the structure of the graph of a continuous multifunction. He deduces that, in general, a continuous multifunction cannot be a union of continuous functions.

These results are related to the theory of selection initiated by E. Michael in [22], [23] (see also [12],[34]).

6. Regarding to a sequence of multifunctions defined on Π, in Section 34 are introduced the notion of simple, unifrom and quasi–uniform convergence.

The sequence $(F_n)_n$ *converges quasi–uniformly* to F if

1° $(F_n)_n$ converges to F, and

2° for any $\varepsilon > 0$ and any natural number N we can fixed $N' \geq N$ such that for every $P \in \Pi$ it is an index n between N and N' which to that F_n and F differ at most with ε at the point P.

It is known that the concept of quasi–uniform convergence has been introduced in 1899 by C. Arzelà for the function. Next, E. Borel has given a more simple form for this concept in [7, p.41] which has been utilized by F. Vasilescu to obtain the dfinition for multifunctions given above. A study of the quasi–uniform convergence in the more general context of the multifunctions from a topological space into a uniform space is made later in [3], [11]. Following E. Borel [7, p.42], F. Vasilescu has proved the next generalization of *Arzelà's Theorem:*

If F is the limit of a sequence $(F_n)_n$ of continuous multifunctions, then F is continuous if and only if $(F_n)_n$ converges quasi–uniformly to F.

It is also proved the following version of *Weierstrass Approximation Theorem*:

A continuous multifunction F on the closed interval $[a,b]$ is the uniform limit of a sequence $(F_n)_n$, where F_n is the finite union of continuous functions defined on $[a,b]$, and, moreover, these functions can be polynomials.

7. The aim of Sections 38–45 is to define two type of *oscillations* of a multifunction F and to give a few properties of them . Let $P \in \Pi$, $y_0 \in F(P)$, and δ a domain containing P in his interior. Let y', y'' be un interval such that $y' < y_0 < y''$ and having the property that for each $P_1 \in \delta$ there is a value $y_1 \in F(P_1)$ belonging to this interval. It is denoted $\omega(P, y_0, \delta)$ the greatest lower bound of the lengths of all intervals $y'y''$. Further, the number $\omega(P, y_0) = \inf\{\omega(P, y_0, \delta) : \delta \text{ tends to } P\}$ is called *the oscillation of the multifunction F at P for the value y_0*. Finally, it is called the *oscillation of F* the multifunction Ω having as values in P the numbers $\omega(P, y_0)$ for each $y_0 \in F(P)$.

Now, let $\omega(\delta) = \sup\{D(F(P), F(P')); \ P, P' \in \delta \cap \Pi\}$. Then, the *oscillation of F at the point P* it is said to be the number $\omega(P) = \inf\{\omega(\delta); \ P$ interior of $\delta\}$ [30, p.55].

It is proved in [30] that the application $\omega : P \to \omega(P)$ is a u.s.c. function. Also, the multifunction F is continuous in P for the value y_0 if and only if $\omega(P, y_0) = 0$. The multifunction F is continuous in P if and only if $\omega(P) = 0$.

With regard to the oscillation Ω and ω there exists the relation: the values of Ω at a point P are at most equal to $2\omega(P)$, i.e. $\omega(P, y) \leq 2\omega(P)$ for any $y \in F(P)$. In order that $\omega(P) = 0$ it is necessary and sufficient that $\omega(P, y) = 0$ for all the values $y \in F(P)$ and, moreover, F will be u.s.c. at the point P.

Let L and ℓ be functions defined by $L(P) = \sup\{y; \ y \in F(P)\}$ and $\ell(P) = \inf\{y; \ y \in F(P)\}$. Then:

If the multifunction F verifies the condition that at every $P \in \Pi$ there is a value of $\Omega(P)$ inferior to ε, for any $\varepsilon > 0$ then there is a continuous function situated between L and ℓ.

8. In Section 46–58 F. Vasilescu first makes a classification of the multifunctions (classification (B)) following the example of Baire's classification

for the functions (classification (b)) [28]. Obviously, the classification (B) includes the classification (b).

The class 0 is formed from the continuous multifunctions defined on Π. The class 1 contains the multifunctions which are not continuous, but they are limits of sequences of continuous multifunctions on Π. It is generalized a well-known Baire's Theorem [28, p.124]:

A multifunction F is in the class 0 or 1 if and only if F is punctual discontinuous on every perfect subset (F is *punctual discontinuous* on Π if the subset of the point P with $\omega(P) = 0$ is dense in Π).

Then F. Vasilescu makes precise the relation between the two classifications (b) and (B):

A uniform function of the class α in (b) *is of the same class in* (B) *and conversely.*

The multifunctions from the classification (B) are called *analytically representable multifunctions* (*fonctions multiformes représentables analytiquement*), since they can be deduced by passing to the limit from the functions with a finite number of branches which are polynomials. Otherwise, all the multifunctions obtained in this way belong to the classification (B).

Now, one considers the equation $F(x_1, x_2, ..., x_{n+1}) = 0$, where F is an analytically representable multifunction. F. Vasilescu approaches the problem of the analytic representation of the implicit multifunction defined by this equation. He proves that, if F is in the class 0, then the implicit multifunctions are in the class 0 or 1.

Today the implicit function theory for multifunctions is particularly active (see, for example, [2]).

4 Final remarks

Florin Vasilescu's thesis represents a unitary theory about the continuity of the real multifunctions of several real variables. F. Vasilescu generalizes in the case of multifunctions some important and well-known theorems from the classical theory of continuity. The absence of arithmetical operations with multifunctions often makes that these generalizations are not simple transpositions.

By the novelty of the subject and the results obtained in his thesis, F. Vasilescu turns the attention of some great mathematicians of the time. H. Lebesgue says: "*Les fonctions à plusieurs déterminations ou fonctions multiformes, ont jusqu'ici été peu étudiées. Le seul travail de quelque étendue les concernant est la Thèse soutenue récemment ((...)) par M.F. Vasilesco*" [19, p.21]. N. Lusin observes: "*M.F. Vasilesco considère les fonctions multiformes comme des organismes indécomposables et il tâche de les étudier et de les classer; tandis que nous considérons une fonction multiforme comme un être très compliqué composé de fonctions uniformes*" [20, p.266]. This estimation shows that at that time F. Vasilescu broached in his thesis an advanced subject.

F. Vasilescu is preoccupied by the application of the obtained results. So, even in his thesis he demonstrates a result that belongs to the theory of real functions using a theorem established for multifunctions (see also [32]). G. Bouligand [10, p.114] points out the utilization in [31] of the multifunctions in problems concerning the limit points of the harmonic functions.

A succint valuation of F. Vasilescu's contribution in the theory of multifunctions is made by G. Andonie in [1, p.88]: "*a pus noi verigi pentru progres în studiul funcţiilor multiforme de o variabilă reală*" (he puts new links for progress in the study of multiform function of a real variable). In fact, F. Vasilescu makes the first steps in a series of actual directions of the theory of multifunctions as selection, apaproximation and implicit functions theory.

Bibliography

1. G.Şt. Andonie - *Istoria matematicii în România*, v. II, Editura Ştiinţifică, Bucureşti, 1966, p.79-91.

2. J.-P. Aubin, H. Frankowska - *Set-Valued Analysis, Systems and Control: Foundations and Applications*, v. 2, Birkhäuser Verlag, Boston/ Basel/ Berlin, 1990.

3. T. Bânzaru - *La convergence uniforme et quasi-uniforme et la continuité des applications multivoques*, C.R. Acad. Sci. Paris, 277(1973), 91-92.

4. C. Berge – *Espaces topologiques. Fonctions multivoques*, Dunod, Paris, 1966.

5. E. Blanc – *Sur une propriété différentielle des continus de Jordan*, C.R. Acad. Sci. Paris, 196(1933), 600–602.

6. E. Blanc – *Sur la structure de certaines lois générales régissant des correspondances multiformes*, C.R. Acad. Sci. Paris, 196(1933), 1769–1771.

7. E. Borel – *Leçons sur les fonctions de variables réelles*, Gauthier–Villars, Paris, 1905.

8. G. Bouligand – *Sur l'idée d'ensemble d'accumulation*, Ens. Math., 30(1931), 243–247.

9. G. Bouligand – *Sur la semi-continuité d'inclusion et quelques sujets connexes*, Ens. Math., 31(1932), 14–22.

10. G. Bouligand – *La première phase de l'évolution ensembliste du problème de Dirichlet (À la mémoire de Florin Vasilescu)*, Bull. des Sci. Math., 84(1960), 111–115.

11. N. Crivăţ, T. Bânzaru – *On the quasi-uniform convergence for multifunctions*, Lucrările Sem. Mat. Fiz., I.P. "Traian Vuia" Timişoara, 1986, 27–32.

12. W.M. Fleishman (ed.) – *Set-Valued Mappings, Selection and Topological Properties of* 2^X, Lectures Notes in Math., nr. 171, Springer–Verlag, 1970.

13. M. Fréchet – *Sur quelques points du calcul fonctionnel* (Thèse), Rend. Circ. Mat. Palermo, 22(1906), 1–74.

14. F. Hausdorff – *Grundzüge der Mengenlehre*, Verlag von Veit, Leipzig, 1914.

15. L.S. Hill – *Properties of certain aggregate functions*, Bull. Amer. Math. Soc. 32(1926), p.596 (Abstract nr. 52); Amer. J. Math., 49(1927), 419–432.

16. W. Hurewicz – *Über stetige Bilder von Punktmengen,* Proc. Kon. Ned. Akad. Wet., 29(1926), 1014–1017.

17. C. Kuratowski – *Les fonctions semi-continues dans l'espace des ensembles fermés*, Fund. Math. 18(1932), 148–159.

18. C. Kuratowski – *Topologie*, I, Warszawa, 1958.

19. H. Lebesgue – *Leçons sur l'intégration et la recherche des fonctions primitives*, 2-ème éd., Gauthier–Villars, Paris, 1928.

20. N. Lusin – *Leçons sur les ensembles analytiques et leurs applications*, Gauthier–Villars, Paris, 1930.

21. B.L. McAllister – *Hyperspaces and multifunctions, the first half century* (1900–1950), Nieuw Arch. Wisk. (3), 26(1978), 309–329.

22. E. Michael – *Topologies on spaces of subsets,* Trans. Amer. Math. Soc., 71(1951), 152–182.

23. E. Michael – *Continuous selection* I, Ann. of Math. 63(1956), 361–382.

24. T. Parthasarathy – *Selection Theorems and Their Applications*, Lectures Notes in Math., nr. 263, Springer–Verlag, 1972.

25. D. Pompeiu – *Sur la continuité des fonctions de variables complexes*, (Thèse), Ann. Fac. Sci. Univ. Toulouse (2) 7(1905), 265–315.

26. W.T. Spears – *A generalization of the Stone–Weierstrass theorem to multivalued function*, J. Natur. Sci. and Math. (Lahore), 9(1969), 201–208.

27. P. Urysohn – *Über die Machtigkeit der zusammenhängenden Mengen,* Math. Ann. 94(1925), 262–295.

28. Ch. de la Vallée Poussin – *Intégrales de Lebesgue. Fonctions d'ensembles. Classes de Baires*, Gauthier–Villars, Paris, 1916.

29. F. Vasilesco – *Sur les fonctions multiformes de variables réelles* (deux communications), C.R. des scéances de la Soc. Math. de France, Paris, 1925.

30. F. Vasilesco – *Essai sur les fonctions multiformes de variables réelles* (Thèse), Gauthier–Villars, Paris, 1925 (84 p).

31. F. Vasilesco – *Sur les valeurs limites des fonctions harmoniques*, C.R. Acad. Sci. Paris, 184(1927), 434–436.

32. F. Vasilescu – *Funcţiile multiforme de variabile reale şi analiza clasică*, Bull. Math. Soc. Roumaine des Sci., 30(1927), 110–112.

33. W.A. Wilson – *On the structure of a continuum, limited and irreducible between two points*, Amer. J. Math., 48(1926), 147–168.