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APPROXIMATION PROCEDURES FOR AN ABSTRACT LQ-OPTIMAL CONTROL PROBLEM

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Abstract

This paper deals with two approximation procedures for the LQ-optimal control problem described by the input-output equation

$$x = Tu \qquad (E)$$

and the cost functional

$$C(x, u) = \langle Px, x\rangle + \langle Qu, u\rangle \qquad (CF)$$

$x \in X$, and $u \in Y$, where X and Y are real separable Hilbert spaces and $T: Y \to X$ is a linear continuous operator. Find $\bar{u} \in U$ -a closed convex set in Y such that

$$C(\bar{x}, \bar{u}) = min\, C(x, u),\ u \in U. \qquad (MF)$$

In (CF), $P: X \to X$, $Q: Y \to Y$, P and Q are linear continuous self-adjoint operators. P is positive semidefinite and Q is positive definite.

The first approach is the classical Ritz-Garlerkin method while the second method is based on reduction to finite dimensional problems and uses convexity properties. Examples are discussed for illustration.

1. INTRODUCTION

Control problems of the form described by equations (E) and (CF) have various applications in the fields of science, engineering, and economics. In most cases of interest, (E) is posed as a differential equation (ordinary or functional), and the underlying spaces- the state space and the control space are infinite dimensional separable HIlbert spaces. For example the problem can be presented as follows:

Minimize:

$$\mathcal{C}(\mathrm{x}, \mathrm{u}) = \int_0^{T_1} \{<P\mathrm{x}(\mathrm{t}), \mathrm{x}(\mathrm{t})> + <Q\mathrm{u}(\mathrm{t}), \mathrm{u}(\mathrm{t})>\}\mathrm{dt} \tag{1.1}$$

Subject to

$$\dot{\mathrm{x}} = (\mathcal{L}\mathrm{x})(\mathrm{t}) + (\mathcal{B}\mathrm{u})(\mathrm{t}) \tag{1.2}$$

$$\mathrm{x}(0) = \theta \in R^n, \tag{1.3}$$

where $\mathcal{L}$ and $\mathcal{B}$ are linear Volterra operators acting on function spaces to be prescribed in the examples to be presented in this paper for illustration. Indeed, (1.2), (1.3) can be reduced to x = Tu, where T is a linear continuous operator. Similarly, if the underlying spaces are l^2-spaces (L^2 – spaces) , then, the problem can be put in the abstract form:

Minimize

$$\mathcal{C}(\mathrm{x,u}) = <\mathrm{Px,x}>+<\mathrm{Qu,u}>, \tag{1.4}$$

subject to

$$\mathrm{x} = \mathcal{A}\mathrm{x} + B\mathrm{u}, \tag{1.5}$$

where $\mathcal{A}$ and B are linear operators; $\mathcal{A} \in \mathrm{LB}\ (X, X)$, while $B: U \subset Y \to X$, $B \in \mathrm{LB}(Y, X)$.

Indeed, (1.5) can be put in the form

$$(I - \mathcal{A})\mathrm{x} = B\mathrm{u} \ \Rightarrow \ \mathrm{x} = (I - \mathcal{A})^{-1}B\mathrm{u} = T\mathrm{u} \tag{1.6}$$

with the property that the spectral radius of $\mathcal{A}$, $r(\mathcal{A}) < 1$.

Examples of this class of control problems are abound in the literature. For example, the linear regulator problem of electrical engineering can be described by the above abstract systems. In Okonkwo [13], the existence and uniqueness of solutions of an optimal control problem described by differential equations (both ordinary and functional) with a quadratic cost functional are discussed. Okonkwo[14], discussed control problems described by discrete dynamical systems on sequence spaces, while Corduneanu [4] dealt with LQ-optimal control problems with abstract Volterra operators. Rockafellar [15], [16] discussed various optimization problems with inequality constraints on finite dimensional spaces, while Valentine [17], discussed optimization problems on Minskonski spaces. It turns out that an important problem associated with the problem discussed in Okonkwo [13] is obtain an expression for the control that gives the minimum of the cost functional.

In this paper therefore, we use direct methods of calculus of variation to obtain an approximate expression for the control that gives the minimum of the cost functional (CF). We also give some examples and applications.

In Section 2, we use the Ritz-Galerkin method to obtain a quadratic form for the cost functional (CF), while in Section 3, we construct a minimizing sequence for the cost functional. In Section 4, we outline a method of solution of the approximation problem in the case U is a closed ball of radius R. We also present a variant of this case when the underlying spaces are l^2-spaces. Section 5 deals with the case when the set U is a general convex set. Our problem is reduced to finite dimensions and convexity properties are exploited to deal with the problem. In Section 6, the case when the control problem is described by a system with Volterra operators is discussed.

Definition 1.1: Let Z be a subspace of a Hilbert space Y. For every u $\in Y$, the distance from u to Z is defined by

$$\text{dist}(u,Z) = \inf_{v\in Z} \|u-v\|.$$

Definition 1.2: Let Y be a Hilbert space. By a Galerkin scheme in Y we shall mean a sequence $\{Y_n\}, Y_n \subset Y_{n+1}, n \geq 1$ of finite dimensional subspaces of Y with the following property:

$$\lim_{n\to\infty} \text{dist}(u, Y_n) = 0 \text{ for all u in } Y.$$

The following property follows:

$$Y = \overline{\cup Y_n}.$$

Let us now deal with the problem of using Ritz-Galerkin approximation method in connection to the LQ-optimal control problem formulated above.

2. RITZ-GALERKIN METHOD

Let $\{\psi_1,\psi_2,\psi_3, \ldots, \psi_p, \ldots\}$ be an orthonormal basis for Y(assuming there exists one). Define $u_N \in Y$ by

$$u_N = \sum_{k=1}^{N} c_k\psi_k \tag{2.1}$$

where $c_k = \langle u_N, \psi_k\rangle$, the Fourier coefficient of u_N, $k = 1, 2, \ldots, N$, and N an arbitrary positive integer.

$$Qu_N = Q(\sum_{k=1}^{N} c_k\psi_k) = \sum_{k=1}^{N} c_k(Q\psi_k) = \sum_{k=1}^{N} c_k\phi_k \tag{2.2}$$

where

$$\phi_k = Q\psi_k, \; k = 1, 2, \ldots, N. \tag{2.3}$$

Also

$$x_N = Tu_N = T(\sum_{k=1}^{N} c_k\psi_k) = \sum_{k=1}^{N} c_k T\psi_k = \sum_{k=1}^{N} c_k\chi_k, \tag{2.4}$$

where

$$\chi_k = T\psi_k,\ i = 1, 2, ..., N. \tag{2.5}$$

Suppose

$$\mu_k = P\chi_k, \tag{2.6}$$

then

$$C(x_N,u_N) = \langle PTu_N, Tu_N\rangle + \langle Qu_N, u_N\rangle$$
$$= \left\{\langle\sum_{k=1}^{N} c_k\mu_k, \sum_{j=1}^{N} c_j\chi_j\rangle + \langle\sum_{k=1}^{N} c_k\phi_k, \sum_{j=1}^{N} c_j\psi_j\rangle\right\} = \sum_{k=1}^{N}\sum_{j=1}^{N}\beta_{kj}c_k c_j, \tag{2.7}$$

where

$$\beta_{kj} = \{\langle\mu_k,\chi_j\rangle + \langle\phi_k,\psi_j\rangle\}. \tag{2.8}$$

Since P and Q are self-adjoint operators, we have

$$\beta_{ij} = \beta_{ji},\ i,j = 1,2\ , \ldots, N. \tag{2.9}$$

Equation (2.7) clearly defines a quadratic form in $c_1, c_2, \ldots, c_N$, and we want to minimize it under constraint $u_N \in U$.

The next task is to construct a minimizing sequence $\{\hat{u}_N\}$, so that

$$\lim_{N\to\infty} C(\hat{x}_N,\hat{u}_N) = \min_{u\in U}\ C(x,u). \tag{2.10}$$

In order to accomplish this, we determine the constants c_k, k=1, 2,. . ., N, say the constants $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_N$, for which

$$\hat{u}_N = \sum_{k=1}^{N}\hat{c}_k\psi_k, \tag{2.11}$$

will be the minimizing element for each N, for $C(x_N,u_N)$.

It is essential at this stage to point out that the minimizing sequence can always be constructed for this problem, where U is a set of admissible controls u. It is therefore assumed that there are controls u for which $C(x,u)<\infty$ and that

$$\inf_{u\in U}\ C(x,u) = m \geq 0.$$

By the definition of m, there exists an infinite sequence $\{\hat{u}_N\}$, called a minimizing sequence, such that

$$\lim_{N\to\infty}\ C(\hat{x}_N,\hat{u}_N) = m.$$

Now if the sequence $\{\hat{u}_N\}$ has a limit $\hat{u}$ and if we can write

$$C(\hat{x},\hat{u}) = \lim_{N\to\infty}\ C(\hat{x}_N,\hat{u}_N)$$

or

$$C(\lim_{N\to\infty}\hat{x}_N, \lim_{N\to\infty}\hat{u}_N) = \lim_{N\to\infty}\ C(\hat{x}_N,\hat{u}_N),$$

then,

$$\mathcal{C}(\hat{x},\hat{u}) = m,$$

where $\hat{u}$ is the solution of the variational problem. See for example Gelfand and Fomin [9] for details.

Theorem 2.1: If $\{\hat{u}_N\}$ is a minimizing sequence of the cost functional $\mathcal{C}(x,u)$ with the limit function $\hat{u}$ and if $\mathcal{C}(x,u)$ is lower semicontinuous at $\hat{u}$, then

$$\mathcal{C}(\hat{x},\hat{u}) = \lim_{N \to \infty} \mathcal{C}(\hat{x}_N,\hat{u}_N).$$

Proof: It is clear that $\mathcal{C}(x,u)$ is a continuous function of u hence lower semicontinuous function of u. By definition we have

$$\mathcal{C}(\hat{x},\hat{u}) \geq \lim_{N \to \infty} \mathcal{C}(\hat{x}_N,\hat{u}_N) = \inf \mathcal{C}(x,u),\ u \in U \tag{2.12}$$

On the other hand given $\epsilon>0$ there exists an $N= N(\epsilon)$ large enough such that

$$\mathcal{C}(\hat{x}_N,\hat{u}_N) - \mathcal{C}(\hat{x},\hat{u}) \geq -\epsilon.$$

Letting $N \to \infty$ gives

$$\mathcal{C}(\hat{x},\hat{u}) \leq \lim_{N \to \infty} \mathcal{C}(\hat{x}_N,\hat{u}_N) + \epsilon, \tag{2.13}$$

Since $\epsilon>0$ is arbitrary we then have from (2.12) and (2.13) that

$$\inf_{u \in U} \mathcal{C}(x,u) = \lim_{N \to \infty} \mathcal{C}(\hat{x}_N,\hat{u}_N) \leq \mathcal{C}(\hat{x},\hat{u}) = \lim_{N \to \infty} \mathcal{C}(\hat{x}_N,\hat{u}_N).$$

Hence

$$\mathcal{C}(\hat{x},\hat{u}) = \lim_{N \to \infty} \mathcal{C}(\hat{x}_N,\hat{u}_N). \tag{2.14}$$

3. CONSTRUCTION OF A MINIMIZING SEQUENCE

Let $U_N \subset U$, where U_N is a closed convex set in the Galerkin subspace $Y_N \subset Y$ which is finite dimensional. Suppose $\hat{u}_N$ is the minimizing element of U_N; that is the element of U_N that gives the minimum value for the cost functional (2.7). We assume that $\theta \notin U_N$.

This implies

$$\mathcal{C}(\hat{x}_N,\hat{u}_N) = m_N \geq \min_{u \in U} \mathcal{C}(x,u). \tag{3.1}$$

Suppose v_{N+1} is the minimizing element of U_{N+1}. Since $U_N \subset U_{N+1}$, there are two possibilities:

Either

(i) $\mathcal{C}(\hat{x}_N,\hat{u}_N) = \mathcal{C}(\hat{x}_{N+1},v_{N+1})$, or (ii) $\mathcal{C}(\hat{x}_N,\hat{u}_N) \geq \mathcal{C}(\hat{x}_{N+1},v_{N+1})$, $\hat{x}_{N+1} = Tv_{N+1}$

.

Suppose w_{N+2} is the minimizing element of U_{N+2}, since $U_{N+1} \subset U_{N+2}$, either (iii) $\mathcal{C}(\widehat{x}_{N+1}, v_{N+1}) = \mathcal{C}(\widehat{x}_{N+2}, w_{N+2})$, or (iv) $\mathcal{C}(\widehat{x}_{N+1}, v_{N+1}) \geq \mathcal{C}(\widehat{x}_{N+2}, w_{N+2})$, $\widehat{x}_{N+2} = Tw_{N+2}$. Clearly,

$$m_N \geq m_{N+1} \geq \ldots \underline{m} > 0.$$

We therefore have a non-increasing sequence which is bounded from below and converges to the minimum. $\{\mathcal{C}(\widehat{x}_N, \widehat{u}_N)\}$ will be a minimizing sequence if we prove that

$$\lim_{N \to \infty} m_N = m.$$

The proof of this statement can be found in Gelfand and Fomin [9].

Having stated how a minimizing sequence can be formed, let us select concrete sets in the Hilbert space Y to illustrate the problem. We now apply the procedure described in Section 2 and 3 in the case $U = B(u_0, R)$ is assumed to be closed and convex in Y. We assume that $\theta \notin U$.

4. TWO EXAMPLES

Example 1: The Ball $B(u_0, R)$:

Suppose

$$U = \{ u : \| u - u_0 \| \leq R \} \subset Y.$$

Y_N is the subspace of Y generated by $\psi_1, \psi_2, \ldots, \psi_N$, which are elements of an orthonormal sequence. Let

$$u_N = c_1\psi_1 + c_2\psi_2 + \ldots + c_N\psi_N.$$

Then $U_N \subset U$ means

$$\mathcal{F}_1(c) = \sum_{k=1}^{N} c_k^2 - 2\sum_{k=1}^{N} \langle u_0, \psi_k \rangle c_k + |u_0|^2 - R^2 \leq 0, \quad (|u_0|_{L^2} > R). \qquad (4.1)$$

Observe that $\mathcal{F}_1(c)$, as defined by equation (4.1) is a quadratic constraint on $(c_1, c_2, \ldots, c_N)^T$. For the purpose of explicit notation, let $B = (\beta_{ij})$, then $B = B^*$. B is clearly a positive matrix, and $F(c) = \langle Bc, c \rangle$.

At this point different methods can be used to show the existence of the optimal control. For example, ordinary convex programming technique can be used with Kuhn-Tucker theorem as the main tool. The application of this method to minimization problems in finite dimensional spaces can be found in the book of Rockafellar [15].

F(c) is clearly a continuous function of $c_1, c_2, \ldots, c_N$ and hence continuous on the ball U_N which is closed, convex and bounded (in fact compact). U_N by definition is a subset of a finite dimensional subspace Y_N. Since U_N is compact, we invoke Weierstrass theorem and assert that F(c) attains its maximum and minimum in the ball U_N.

Let $\lambda>0$, and $\Lambda>0$ be the smallest and largest eigenvalues of the operator B. Then

$$\sqrt{\lambda}|c| \leq \sqrt{<Bc,c>} \leq \sqrt{\Lambda}|c|. \qquad (4.2)$$

That is, $\sqrt{F(c)}$ is equivalent to the original Euclidean norm. (The proof of this inequality can be found in Kreyszig [12]).

We now invoke the well known theorem which states the following: every closed convex set in a Hilbert space has a unique element of minimum norm. Suppose $\widehat{u}_N \in U_N$ is the element such that

$$\mathcal{C}(\widehat{x}_N,\widehat{u}_N) = m_N = \min_{u \in U} \mathcal{C}(x,u).$$

Then $\widehat{u}_N$ is unique, and $\widehat{u}_N = \widehat{c}_1\psi_1 + \widehat{c}_2\psi_2 + \ldots + \widehat{c}_N\psi_N$. At each stage N therefore we have a unique element which gives the minimum of the quadratic form (4.1). This obviously means that $\{\widehat{u}_N\}$ is a minimizing sequence in U. Consequently,

$\left\{ \sum_{k=1}^{N} \sum_{j=1}^{N} \beta_{kj}\widehat{c}_k \widehat{c}_j \right\}$ is a minimizing sequence converging to the minimum of the quadratic functional $\mathcal{C}(x,u)$, $u \in U$.

$$\lim_{N\to\infty} \sum_{k=1}^{N} \sum_{j=1}^{N} \beta_{kj}\widehat{c}_k \widehat{c}_j = \lim_{N\to\infty} m_N = m. \qquad (4.3)$$

Remark 4.1: Notice that even though Y is an infinite dimensional space, the completeness of the orthonormal basis $\{\psi_k\}$ guarantees that elements of U can be approximated arbitrarily close by elements of a subset U_N in a finite dimensional space, for large N.

Consequently

$$\lim_{N \to \infty} U_N = U. \qquad (4.4)$$

Hence $\{u_N\}$ is a minimizing sequence.

Remark 4.2: Even though we have selected a ball of radius R for this analysis, an ellipsoid in
N-dimensions will also be adequate. The only difference in this later case will be the nature of quadratic constraint inequalities.

Example 2: Approximations in l^2 – spaces.

Let us consider the problem:

Minimize

$$<x,x>_g + <Qu,u> \qquad (4.5)$$

subject to

$$x(k+1) = A(k)x(k) + B(k)u(k), \; k \geq 0 \qquad (4.6)$$

$$x(0) = x^0 \in R^m. \qquad (4.7)$$

The state space is $X_g = l_g^2(Z_+, R^m)$, the space of m − dimensional vector sequences with the norm

$$\sqrt{<x,x>_g} = \left(\sum_{k=0}^{\infty} |x(k)|^2 g_k \right)^{\frac{1}{2}} < \infty,$$

where $g = \{g_k\}$ is a weight, $g_k>0$, $\sum_{k=0}^{\infty} g_k < \infty$. In the case Q is an infinite matrix, Q must satisfy the conditions outlined by Crone [7]. The control space is $Y=l^2(Z_+, R^q)$. $\{A(k)\}$ is a sequence of invertible matrices and $\{B(k\}$ is a sequence of $m \times q$ matrices whose entries are bounded in $l(Z_+, R)$, where $Z_+ = \{0, 1, 2, 3, \cdots\}$.

Observe that the solution of (4.6), (4.7) can be put in the form

$$x(n+1) = \mathcal{U}(n+1)x(0) + \sum_{j=0}^{n} \mathcal{U}(n+1)\mathcal{U}^{-1}(j+1)\, B(j)u(j) \tag{4.8}$$

where

$$\mathcal{U}(n+1) = A(n) \cdot A(n-1) \cdot \cdot \cdot A(1)A(0).$$

We impose the condition

$$det(\lambda I - A(k)) = 0 \Rightarrow |\lambda| < \delta < 1.$$

If we choose $x(0) = \theta \in l_g^2(Z_+, R^m)$, then

$$x(n+1) = \sum_{j=0}^{n} \mathcal{U}(n+1)\mathcal{U}^{-1}(j+1)\, B(j)u(j) \tag{4.9}$$

or in abstract form

$$x=Tu, \text{ where } T: (l^2)^q \rightarrow (l_g^2)^m \text{ is linear and continuous.}$$

If we choose

$$U_N = \{ u_N = \sum_{i=1}^{N} c_i \psi_i : \sum_{i=1}^{N} \{ \frac{c_i^2 - 2 <h_i, \psi_i> c_i + |h_i|^2}{a_i^2} \} \leq 1 \} \tag{4.10}$$

where $h_i = (h_1, h_2, \ldots, h_N)$ is the center, then U_N is an N-dimensional ellipsoid in U − a closed convex set in $l^2(R^q)$. Our problem collapses to the minimization of the quadratic cost functional (4.5) subject to (4.9) and for each N, u_N is restricted to the ellipsoid (4.10). Under these outlined conditions, the Garlekin scheme can be constructed for this type of problem. The quadratic constraint inequalities will however be distinct in this case, see for example Webster [18].

Remark 4.3: Young and Dahleh [19] discussed infinite dimensional convex optimization involving linear matrix inequalities, and some of their results are worthy of note here. Since for each N we are dealing with u_N being an N-vector, the operators P and Q can be viewed as $N \times N$ matrices, the inner product $<PTu_N, Tu_N>$ can be written in the form $(PTu_N)^T(Tu_N) \geq 0\,(PTu_N)^T(Tu_N) < \infty$. Similarly $<Qu, u>$ can be viewed in the same way.

5. AN APPROXIMATION PROCEDURE FOR THE LQ-OPTIMAL CONTROL PROBLEM IN THE CASE U IS A GENERAL CONVEX SET

Consider the problem described by (E) and (CF). Namely given an input-output equation x=Tu, find $\bar{u} \in U$ -a closed convex set in Y such that

$$C(\bar{x}, \bar{u}) = min\, C(x, u), u \in U. \qquad (MF)$$

where

$$C(x, u) = <\mathcal{R}x,x>+<\mathcal{S}u,u>. \qquad (CF)$$

x $\in X$, and u $\in Y$, where X and Y are real separable Hilbert spaces and T:$Y \to X$ is a linear continuous operator. $\mathcal{R}$: $X \to X$, S: $Y \to Y$, $\mathcal{R}$ and S are linear continuous self-adjoint operators. $\mathcal{R}$ is positive semidefinite and S is positive definite.

Let us deal with problem under the condition that U is a general convex set. The case where X and Y are finite dimensional Hilbert spaces have been dealt with in the literature using various approximation techniques. In the case X and $Y$are infinite dimensional and U is only closed and convex, no results of this problem are known. However, in the case $U \subset Y$ is compact and C(x,u) is lower semicontinuous, the Krien-Milman theorem is applicable. The statement and proof of this theorem is available in the book of Dunford and Schwartz [8]. Since Y is infinite dimensional, in general the set of all extremal points of U is uncountable. If we assume that U is compact, then, U is the closed convex hull of its extremal points by Krein-Milman theorem. Without that assumption therefore, we have to make another relevant assumption that will not lead to too much restriction to the problem. Assume that the closed convex set U is the convex hull of countable dense subset $\mathcal{G}$ of its extremal points. That is

$$\mathcal{G} = \{ \xi_1, \xi_2, \xi_3, \cdots, \xi_p, \cdots \}, \qquad (5.1)$$

and

$$\overline{\text{conv}\mathcal{G}} = U.$$

For any fixed positive integer N, consider the convex combination

$$u_N = \sum_{i=1}^{N} \alpha_i \xi_i, \ \alpha_i \geq 0, \ i = 1, 2, \cdots, N \ \sum_{i=1}^{N} \alpha_i = 1. \qquad (5.2)$$

$u_N \in U_N \subset U$.

If the positive integer N is fixed, then

$$U_N = \{ u_N: u_N = \sum_{i=1}^{N} \alpha_i \xi_i, \ \alpha_i \geq 0, \ \sum_{i=1}^{N} \alpha_i = 1 \}$$

a polytope. It is closed, convex and bounded. It is a closed convex set in an N-dimensional space.

Now

$$Su_N = S(\sum_{i=1}^{N}\alpha_i\xi_i) = \sum_{i=1}^{N}\alpha_i(S\xi_i) = \sum_{i=1}^{N}\alpha_i s_i \tag{5.3}$$

where

$$s_i = S\xi_i,\ 1, 2, ..., N. \tag{5.4}$$

Also

$$x_N = Tu_N = T(\sum_{i=1}^{N}\alpha_i\xi_i) = \sum_{i=1}^{N}\alpha_i T\xi_i = \sum_{i=1}^{N}\alpha_i t_i \tag{5.5}$$

where

$$t_i = T\xi_i,\ i = 1, 2, ..., N. \tag{5.6}$$

Suppose

$$l_i = \mathcal{R}t_i \tag{5.7}$$

then

$$\mathcal{C}(x_N,u_N) = <\mathcal{R}x_N,x_N> + <Su_N,u_N>$$

$$= \left\{<\sum_{i=1}^{N}\alpha_i l_i, \sum_{j=1}^{N}\alpha_j t_j> + <\sum_{k=1}^{N}\alpha_i s_i, \sum_{j=1}^{N}\alpha_j\xi_j>\right\} = \sum_{i=1}^{N}\sum_{j=1}^{N}\omega_{ij}\alpha_i\alpha_j, \tag{5.8}$$

where

$$\omega_{ij} = < l_i, t_j> + < s_i, \xi_j>. \tag{5.9}$$

Since $\mathcal{R}$ and S are self-adjoint operators, we have

$$\omega_{ij} = \omega_{ji},\ i,j = 1,2, \ldots, N. \tag{5.10}$$

Equation (5.8) clearly defines a quadratic form in $\alpha_1, \alpha_2, \ldots, \alpha_N$, and we want to minimize it under constraint $u_N \in U$.

Let us notice here that for each N, U_N is a polyhedral convex set. Every $u_N \in U_N$ must satisfy the following system of inequalities (Rockafeller [16], Webster [18])

$$< u_N, b_i> \le \beta_i,\ i = 1,2, \ldots, N, \tag{5.11}$$

where $b_i \in U_N$ and $\beta_i \ge 0$, $i = 1, 2, \ldots, N$ (5.11) is clearly a finite system of linear equations and inequalities.

For each N therefore, the ordinary convex program reduces to the following problem:

Minimize

$$\mathcal{C}(x_N,u_N) = \sum_{i=1}^{N}\sum_{j=1}^{N}\omega_{ij}\alpha_i\alpha_j = \mathcal{F}(\alpha), \tag{5.12}$$

subject to

$$f_i = < u_N, b_i> - \beta_i \le 0,\ i = 1,2, \ldots, N. \tag{5.13}$$

The above problem has an immediate answer. Since U_N is a polyhedral convex set, it is finitely generated. The cost functional $\mathcal{F}(\alpha)$ is continuous on the closed convex set U_N. Inequalities (5.13) show that U_N is bounded. By Weierstrass theorem, $\mathcal{F}(\alpha)$ attains its maximum and minimum on the set U_N. Any linear or nonlinear programming method can be used to obtain $\widehat{u}_N$ that gives the minimum for the quadratic form for $u_N \in U_N$. Such methods include the simplex method, Kuhn-Tucker theorem etc.

Suppose $\widehat{u}_N$ is the minimizing element in U_N, then

$$\widehat{u}_N = \sum_{i=1}^{N} \widehat{\alpha}_i \xi_i, \quad \widehat{\alpha}_i \geq 0, \quad i = 1, 2, \cdots, N \tag{5.14}$$

and

$$\mathcal{C}(\widehat{x}_N, \widehat{u}_N) = m_N = \min_{u \in U} \mathcal{C}(x,u).$$

As in Section 3, one can easily see that

$$\mathcal{C}(\widehat{x}_N, \widehat{u}_N) \geq \mathcal{C}(\widehat{x}_{N+1}, u_{N+1}), \quad N = 1, 2, \cdots;$$

This process can be continued and $\{u_N\}$ is the minimizing sequence for the functional $\mathcal{C}(x,u)$, $u \in U$. The sequence $\{m_N\} = \{\mathcal{C}(\widehat{x}_N, \widehat{u}_N)\}$ converges to the minimum value for the cost functional for the same reason we have seen in Section 3. Clearly

$$\mathcal{C}(\widehat{x}_N, \widehat{u}_N) \geq \mathcal{C}(\widehat{x}_{N+1}, u_{N+1} \geq \cdots > 0.$$

Hence we have a non-increasing sequence which is bounded from below and converges to the minimum if $\{\mathcal{C}(\widehat{x}_N, \widehat{u}_N)\}$ is a minimizing sequence.

Remark 5.1: Notice that at each stage N, we are solving the problem in finite dimensional space. Due to strict convexity, $\{\widehat{u}_N\}$ is unique.

6. AN APPROXIMATION TO LQ-OPTIMAL CONTROL PROBLEM WITH ABSTRACT VOLTERRA OPERATORS

In this section, we are concerned with the following problem: Given the functional differential equation

$$\dot{x}(t) = (Lx)(t) + (Nu)(t), \tag{6.1}$$

with the initial condition

$$x(0) = x^0 \in R^n, \tag{6.2}$$

and the cost functional

$$\mathcal{C}(x,u) = \int_0^T \{<(Px)(t),x(t)> + <(Qu)(t),u(t)>\} dt, \tag{6.3}$$

find the control $\widetilde{u} \in U \subset L^2([0,T], R^m)$ – the control space such that

$$\mathcal{C}(\tilde{x},\tilde{u}) = \min_{u \in U} \mathcal{C}(x,u).$$

Here, L is a linear continuous abstract operator of Volterra type,

$$L: X = L^2([0,T], R^n) \to L^2([0,T], R^n), \quad T < \infty.$$

$L^2([0,T], R^n)$ is the state space. N is a linear continuous operator of Volterra type

$$N: Y = L^2([0,T], R^m) \to L^2([0,T], R^n).$$

Indeed if we let $x^0 = \theta \in R^n$, (6.1), (6.2) is equivalent to

$$x(t) = \int_0^t (Lx)(s)ds + \int_0^t (Nu)(s)ds,$$

and on application of Riesz representation we have (see Kantorovich and Akilov [10])

$$x(t) = \int_0^t g(t,s)x(s)ds + \int_0^t n(t,s)u(s)ds. \tag{6.4}$$

Then the solution of (6.4) can be put in the form (see Corduneanu [4])

$$x(t) = \int_0^t \mathcal{H}(t,s)(N_1 u)(s)ds, \tag{6.5}$$

where

$$(N_1 u)(t) = \int_0^t n(t,s)(u)(s)ds.$$

$\mathcal{H}(t,s)$ is the transition operator associated with the abstract operator L. Indeed,

$$\mathcal{H}(t,s) = I_{n\times n} + \int_s^t \tilde{g}(t,w)dw. \tag{6.6}$$

$\tilde{g}(t,w)$ is the resolvent kernel associated with the Volterra kernel g(t,w). Using again a result of Corduneanu [4], x(t;u) can be put in the form

$$x(t;u) = (Tu)(t) = \int_0^t H(t,s)u(s)ds. \tag{6.7}$$

where H(t,s) is an $n \times m$ matrix valued function assumed to be measurable for $\{0 \le s \le t \le T < \infty\}$ in the sense of Bukhvalov, (see Kantorovich and Akilov [10]). H(t,s) is determined by the Volterra operators L and N.

The following condition is necessary and sufficient for H(t,s) to take L^2 into L^2:

$$\int_0^T dt \int_0^t |H(t,s)|^2 ds = h_0 < \infty. \tag{6.8}$$

Let $\{\psi_1, \psi_2, \psi_3, \ldots, \psi_p, \ldots\}$ be an orthonormal basis for $L^2([0,T], R^m)$. The Ritz-Galerkin approximation procedure outlined in Section 2 can then be applied. If we choose U to be a ball of radius R centered at $u_0 \in L^2([0,T], R^m)$, the problem is reduced to the one discussed in Sections 3 and 4. Hence, the conclusion of Section 3 remains valid.

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