LIBERTAS MATHEMATICA,VOL.XVIII(1998)

NOTES ON THE EQUATION $Bx'(t) = Ax(t)$ IN BANACH SPACES

GASTON M. N'GUEREKATA

ABSTRACT. We are concerned with the abstract differential equation $Bx'(t) = Ax(t)$ in a Banach space. We prove uniqueness of solutions on the interval $[0, T]$.

1. Introduction

We are concered with the uniqueness of the Cauchy Problem (CP) for abstract differential equations in Banach spaces. We mention the interesting results from Agmon and Niremberg [1], Fattorini [2], Lybic, Prokopenko [4], Zaidman [8,9,10]. In his monograph [3], S.G. Krein presents the abstract CP for the equation $x'(t) = Ax(t)$ on $[0, T]$. He proves the uniqueness of the so-called weakened solution provided that the resolvant $R(\lambda) = (\lambda I - A)^{-1}$ exists and has some exponential growth for λ positive and sufficiently large.

Several authors ([4,5,8,9,10]) also studied the more general (implicit) equation $Bx'(t) = Ax(t)$ under appropriate conditions on the resolvant $R(\lambda; A, B) = (\lambda B - A)^{-1}$ in the complex plane and using different methods involving Fourier transforms. Our result below is a somewhat generalization of *Theorem 3.1* [3] to the case of $Bx'(t) = Ax(t)$ on $[0, T]$ but the equation needs to be satisfied at the origin which is not the case in [3]. We use simple tools from classical Analysis to prove uniqueness of some solutions.

2. Main results

Let X be a complex Banach space with norm $\|\ \|$ and A, B two closed linear operators with domains $D(A)$ and $D(B)$ respectively in X. We assume $D(A) \cap D(B)$ is dense in X. Consider in X the equation

$$Bx'(t) = Ax(t), \quad t \in [0,T]. \tag{2.1}$$

Integration of vector-valued functions throughout this paper is understood as in [3,10].

Hypothesis I: Assume $R(\lambda; A, B)$ exists in $L(X)$ for any $\lambda > K$ where K is a given positive real number and

$$\|R(\lambda; A, B)\|_{L(X)} \leq Me^{\beta\lambda}$$

holds true for any $\lambda > K$ with M and β some positive constants, $0 \leq \beta < T$.

Now we state and prove:

Theorem. Suppose Hypothesis I holds true and let $x(t) : [0,T] \to D(A) \cap D(B)$ with the properties:

i) $Ax(t)$ is continuous on $[0,T]$

ii) $x(t)$ is continuously differentiable and $x'(t) \in D(B)$ for any $t \in [0,T]$

iii) $x(0) = 0$.

Then $x(t) = 0$ for any $t \in [0, T - \beta]$.

Let us first mention the following lemma due to Titchmarsh:

Lemma 1: [[6] or [3] p.62]. If $g(t)$ is a summable function on $[0,T]$ and

$$\limsup_{\lambda\to\infty} \frac{\|\int_0^T e^{\lambda t} g(t)dt\|}{\lambda} \leq a$$

where a is a constant such that $0 \leq a < T$, then $g(t) = 0$ almost everywhere on $[a,T]$.

Lemma 2: [[10], p.153]. If $y(t) \in C([0,T]; D(B))$ with B a closed linear operator on X, then $\int_0^T By(t)\,dt$ exists in X and $\int_0^T By(t)\,dt = B\int_0^T y(t)\,dt$.

Proof of the Theorem: Let $x(t)$ be as in the Theorem with $x(0) = 0$. Then we have

$$x(t) = \int_0^t x'(s)\,ds, \quad t \in [0,T].$$

Since $Ax(t)$ is continuous and $Bx'(t) = Ax(t)$ on $[0,T]$, $\int_0^t Bx'(s)\,ds$ exists for any $t \in [0,T]$; then using Lemma 2, we get

$$Bx(t) = B\int_0^t Bx'(s)\,ds, \quad \text{for any} \quad t \in [0,T],$$

B being closed. This means that $(Bx(t))'$ exists on $[0,T]$ and equals $Bx'(t)$. We deduce that $Bx(t)$ is a continuous function on $[0,T]$, therefore it is integrable there. As $x(t) \in D(B)$ and is continuous on $[0,T]$,

we can say that $\int_0^T e^{-\lambda t}x(t)\,dt$ exists and is in $D(B)$. If we use again closedness of B we get

$$B\int_0^T e^{-\lambda t}x(t)\,dt = \int_0^T e^{-\lambda t}Bx(t)\,dt.$$

Integration by parts gives

$$\int_0^T e^{-\lambda t}x'(t)\,dt = e^{-\lambda T}x(T) + \lambda\int_0^T e^{-\lambda t}x(t)\,dt.$$

Applying B to both sides and using closedness of B yields

$$\int_0^T e^{-\lambda t}Bx'(t)\,dt = e^{-\lambda T}Bx(T) + \lambda B\int_0^T e^{-\lambda t}x(t)\,dt. \tag{2.2}$$

Now recall that $Bx'(t) = Ax(t)$ for any t in $[0,T]$ and A is closed. So the left hand member of (2.2) becomes

$$\int_0^T e^{-\lambda t}Ax(t)\,dt = A\int_0^T e^{-\lambda t}x(t)\,dt \tag{2.3}$$

for $\int_0^T e^{-\lambda t}x(t)\,dt$ is in $D(A)$. Combine (2.2) and (2.3) to get

$$A\int_0^T e^{-\lambda t}x(t)\,dt = e^{-\lambda T}Bx(T) + \lambda B\int_0^T e^{-\lambda t}x(t)\,dt$$

or

$$(A-\lambda B)\int_0^T e^{-\lambda t}x(t)\,dt = e^{-\lambda T}Bx(T).$$

Multiplying by $e^{\lambda T}$ we obtain

$$(A-\lambda B)\int_0^T e^{\lambda(T-t)}x(t)\,dt = Bx(T).$$

Take now $\lambda > K$ as in the Theorem, then we get

$$\int_0^T e^{\lambda(T-t)}x(t)\,dt = R(\lambda; A, B)Bx(T). \tag{2.4}$$

Put $s = T - t$ in the integral, then (2.4) becomes

$$\int_0^T e^{\lambda s}x(T-s)\,ds = R(\lambda; A, B)Bx(T).$$

Therefore, we have the inequality

$$\left\|\int_0^T e^{\lambda s}x(T-s)\,ds\right\| \le \|R(\lambda; A, B)\|_{L(X)}\|Bx(T)\|$$
$$\le \|Bx(T)\|Me^{\lambda\beta} = Ce^{\lambda\beta}$$

where $C = M\|Bx(T)\| > 0$ does not depend on λ nor β. Take now the natural logarithm of both sides of the inequality, divide by λ and get

$$\frac{\|\int_0^T e^{\lambda s}x(T-s)\,ds\|}{\lambda} \le \frac{\ell n C}{\lambda} + \beta.$$

Since $\limsup_{\lambda\to\infty}\left(\frac{\ell n C}{\lambda} + \beta\right) = \beta$, we get

$$\limsup_{\lambda\to\infty}\frac{\ell n\|\int_0^T e^{\lambda s}x(T-s)\,ds\|}{\lambda} \le \beta.$$

Then we can apply Lemma 1 to the function $g(s) = x(t-s)$, $s \in [\beta, T]$, and say that $x(t) = 0$ almost everywhere on $[0, T-\beta]$. And since $x(t)$ is continuous on $[0, T]$ we obtain the desired result $x(t) = 0$ for any $t \in [0, T-\beta]$.

Corollary. If Hypothesis I is replaced in the Theorem by

Hypothesis II: Assume $R(\lambda; A, B)$ exists in $L(X)$ for any $\lambda > K$ where K is a given positive real number and we have the uniform bound $\|R(\lambda; A, B)\| \leq M$ for all such λ.

Then $x(t) = 0$, for any t in $[0, T]$.

Proof: Obvious.

References

1. S. Agmon and L. Niremberg, *Properties of solutions of ordinary differential equations in Banach spaces*, Comm. Pure and Appl. Math. **16** (1963), pp.121–239.
2. H.O. Fattorini, *The Cauchy Problem*, Addison-Wesley, London-Amsterdam-Tokyo, 1983.
3. S.G. Krein, *Linear Differential Equations in Banach Spaces*, Translations of Monographs, **vol. 29**, A.M.S. Providence, R.I., 1971.
4. L.N. Prokopenko, *Uniqueness of the Cauchy problem for operator-differential equations*, Soviet Math. Dokl. **3** (1963), pp.227–230.
5. A.R. Rutkas, *Cauchy's problem for the equations $Ax'(t) + Bx(t) = f(t)$*, Differential Equations, 1975, XI, No.11, pp.1996-2010.
6. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, 1937.
7. K. Yosida, *Functional Analysis*, Springer-Verlag, 1995.
8. S. Zaidman, *Uniqueness of the Cauchy problem for the equation $Bx'(t) = Ax(t)$ in Hilbert spaces*, Ann. Sci. Math. Quebec, **13**(20), 1989, pp.93–96.
9. S. Zaidman, *Topics in Abstract Differential Equations*, Pitman Research Notes in Math. Ser. I & II, John Wiley and Sons, New York, 1994.
10. S. Zaidman, *Well-posed Cauchy problem and related semi-groups of operators for the equation $Bu'(t) = Au(t) \geq 0$, in Banach spaces*, Libertas Mathematica, **vol.12** (1992), pp.147–159.