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Maximality Principles in Topological Ordered Structures

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Abstract. Some topological versions of the Zorn–Bourbaki maximality principle are discussed. The obtained facts refine some contributions in this area due to Brunner [4] and Manka [10].

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1 Introduction

The natural setting of our exposition is

(1D1) $\{ZF\}$ = the (axiomatic) Zermelo–Fraenkel system.

For a description of it, we refer to Cohen [5, ch.2, Sect.1]. A basic component of $\{ZF\}$ is the *Axiom of Choice*:

(AC) the cartesian product of any family of nonempty sets is nonempty.

There are many structural equivalents of (AC); see, e.g., Moore [11,Appendix] and the references therein. Here, we shall be interested especially in the ones involving *maximal elements* of partially ordered structures. A basic statement in this area is the 1935 Zorn Maximality Principle [18]:

(ZMP) each inductive ordered structure has at least one maximal element.

Now, the ambient ordered set may be endowed with *supplementary* algebraic/topological structures. And then, we may ask of **(i)** which is the specific form of (ZMP) in this case, and **(ii)** to what extent is this version of (ZMP) equivalent with (AC). Clearly, the second of these questions is to be treated in the framework of

(1D2) $\{ZF\}_0 = \{ZF\} \setminus \{(AC)\}$ (the *reduced* Zermelo–Fraenkel system).

It is our aim in the present exposition to discuss a lot of such facts. Section 2 develops a treatment for the *abstract* (non–topological) case. Further, Section 3 is devoted to some preliminary considerations involving the *topological* maximality principles emerging from the preceding ones. The first main result (over *topological trees*) is presented in Section 4; it may be viewed as a refinement of the one due to Brunner [4]. And, finally, the second main result (over *topological quasi-lattices*) is delineated in Section 5; it is comparable with certain developments in this area due to Manka [10].

We have to remark that the *linear algebraic* structures are not explicitly entering in our treatment. This is also true for their associated *linear topologies*; hence, maximality principles of the type indicated by Luc [9,ch.2,Sect.3] are not taken into consideration here. On the other hand, *non-maximal* topological statements related to the Tychonoff product theorem [15] which (cf. Kelley [8]) are also equivalent with (AC), were ignored as well in our exposition. We shall discuss these facts elsewhere.

2 Maximality principles in ordered structures

Let E be a nonempty set; and $\leq$, any *quasi-ordering* (i.e., a reflexive and transitive relation) over it. For each nonempty part X of E, put

(2D1) $\text{ubd}\,(X) = \{u \in E; x \leq u, \text{ for each } x \in X\}$

(the *upper bounds* of X). Denote further

(2D2) $\text{lst}\,(X) = X \cap \text{ubd}\,(X)$ (the *last elements* of X).

The sets $\text{lbd}(X)$ (of all *lower bounds* for X) and $\text{fst}(X)$ (the *first elements* of X) are introduced in a symmetric manner. We also put

(2D3) $\sup(X) = \text{fst}(\text{ubd}\,(X))$ (the *supremum elements* of X);

the notion of $\inf(X)$ (the *infimum elements* of X) is introduced in a *dual* manner. Finally, call the part X of E

(2D4) $\left\{\begin{array}{l}\text{a } \textit{semi chain} \text{ (modulo } \leq), \text{ when it is } \textit{self-directed} \\ \text{(each finite part of } X \text{ is bounded above in } X)\end{array}\right.$

(2D5) $\left\{\begin{array}{l}\text{a } \textit{standard chain} \text{ (modulo } \leq), \text{ if it is } \textit{totally ordered} \\ (u, v \in X \Longrightarrow \text{either } u \leq v \text{ or } v \leq u)\end{array}\right.$

(2D6) $\left\{\begin{array}{l}\text{a } \textit{super chain} \text{ (modulo } \leq), \text{ provided it is } \textit{well ordered} \\ \text{(each part of } X \text{ has a first element).}\end{array}\right.$

We note the generic implication

$$\text{super chain} \Longrightarrow \text{standard chain} \Longrightarrow \text{semi chain}. \tag{2.1}$$

Having this precised, call the point $z \in E$, *maximal* (modulo $\leq$), when

(2D7) $z \leq w \Longrightarrow w \leq z$; and put, for simplicity

(2D8) $\max(E, \leq) = \{z \in E;\ z \text{ is } (\leq) \text{ - maximal}\}$.

[The notion of *minimal* (modulo $\leq$) element is introduced in a symmetric manner; and the corresponding set will be denoted $\min(E, \leq)$]. We are interested in the following to determine (abstract) structural conditions upon $(E, \leq)$ so that this set be nonempty. To this end, put

(2D9) $\left\{\begin{array}{l}\mathcal{C}_1(E, \leq) \text{ (resp.: } \mathcal{C}_2(E, \leq), \mathcal{C}_3(E, \leq)) = \text{the class of} \\ \text{all semi (resp.: standard, super) chains in } (E, \leq);\end{array}\right.$

and consider the following types of *boundedness* (=inductivity) properties for $(E, \leq)$, where $i \in \{1, 2, 3\}$

(2D10) $\begin{cases} B_i^0(E, \leq)\text{: ubd}\,(X) \neq \emptyset, \text{ for each } X \in \mathcal{C}_i(E, \leq) \\ B_i^1(E, \leq)\text{: sup}(X) \neq \emptyset, \text{ for each } X \in \mathcal{C}_i(E, \leq). \end{cases}$

The relationships between these are (cf. (2.1) above)

$$B_s^u(E, \leq) \Longrightarrow B_t^v(E, \leq), \text{ if } s \leq t \text{ and } u \geq v. \tag{2.2}$$

In parallel to this, we must introduce different additional properties concerning the quasi–ordered structure $(E, \leq)$. The basic one is, as above said,

(2D11) $\begin{cases} \Omega_0(E, \leq)\text{: } E \text{ is } \textit{amorph} \text{ with respect to } (\leq) \\ \text{(no additional conditions are needed).} \end{cases}$

And, from the *non-amorph* ones, we shall use properties like

(2D12) $\begin{cases} \Omega_1(E, \leq)\text{: } E \text{ is a } \textit{tree} \text{ with respect to } (\leq) \\ (E(x, \geq) = \{u \in E;\ x \geq u\} \text{ is well ordered,for each } x \in E) \end{cases}$

(2D13) $\begin{cases} \Omega_2(E, \leq)\text{: } E \text{ is a } \textit{quasi-lattice} \text{ with respect to } (\leq) \\ (\text{ubd}(x, y) \neq \emptyset, \text{ for each } x, y \in E). \end{cases}$

Now, by a *maximality principle* over quasi–ordered structures we mean the logical proposition

(2D14) $\begin{cases} \mathcal{M}^0(P)\text{: } (\forall (E, \leq)\text{=quasi–ordered structure}) \\ P(E, \leq) \Longrightarrow \max(E, \leq) \neq \emptyset; \end{cases}$

here, P is a certain "composed" property of $(E, \leq)$. The Zorn Maximality Principle enters in this scheme, for $P = B_2^0\Omega_0$. Another case of interest corresponds to the choice $P = B_3^1\Omega_1$; the associated statement will be referred to as the Bourbaki Maximality Principle [3]. Finally, the case $P = B_1^1\Omega_2$ gives the Felgner Maximality Principle; cf. Brunner [4].

A basic particular aspect of these constructions is related to the case of $(\leq)$ being an *order* (i.e., antisymmetric quasi–order) on E. Precisely,by a *maximality principle* over such structures we mean the logical proposition

(2D15) $\mathcal{M}^1(P)$=the principle $\mathcal{M}^0(P)$ associated to ordered structures;

where P is the same property as in (2D14). Clearly,

$$\mathcal{M}^0(P) \Longrightarrow \mathcal{M}^1(P), \text{ for all such } P. \tag{2.3}$$

The converse implication is also true, provided we *restrict* these properties to the above discussed ones. More exactly, we have

Proposition 1. *The following is valid*

$$\begin{cases} \mathcal{M}^1(P) \Longrightarrow \mathcal{M}^0(P) \textit{ (hence } \mathcal{M}^1(P) \Longleftrightarrow \mathcal{M}^0(P)) \\ \textit{when } P = B_i^r\Omega_j,\ i \in \{1, 2, 3\},\ r \in \{0, 1\},\ j \in \{0, 1, 2\}. \end{cases} \tag{2.4}$$

Proof. Assume $\mathcal{M}^1(P)$ is true and let $(E, \leq)$ be a quasi–ordered structure fulfiling (P). Denote by $\sim$ its associated *equivalence*

(2D16) $x \sim y$ iff $x \leq y,\ y \leq x$;

and put, for each $x \in E$,

(2D17) $x^\sim = \{y \in E;\ x \sim y\}$ (the *equivalence class* of x).
This makes possible the introduction of the *factor space*
(2D18) $E^\sim = E|_\sim$ $(= \{x^\sim;\ x \in E\})$;
as well as the *factor ordering* (over it)
(2D19) $x^\sim \preceq y^\sim$ iff $x \le y$ for some $x \in x^\sim,\ y \in y^\sim$.
The following property is clear (so, we do not give details):

$$P(E, \le) \Longrightarrow P(E^\sim, \preceq) \quad \text{(where } P \text{ is the above).} \tag{2.5}$$

This, along with $\mathcal{M}^1(P)$, tells us that $\max(E^\sim, \preceq) \neq \emptyset$. But, evidently,

$$z^\sim \in \max(E^\sim, \preceq) \Longrightarrow z^\sim \subseteq \max(E, \le). \tag{2.6}$$

This ends the argument. ■

We are now in position to discuss the relationships between such maximality principles and (AC). Note that, from Proposition 1, one may restrict only to principles like $\mathcal{M}^1(P)$, where P is as in (2.4). Concerning the mutual relations between these, one has (via (2.2)) for each $j \in \{0, 1, 2\}$

$$\mathcal{M}^1(B_s^u\Omega_j) \Longrightarrow \mathcal{M}^1(B_t^v\Omega_j), \quad \text{if} \quad s \ge t,\ u \le v. \tag{2.7}$$

On the other hand, for each $i \in \{1, 2, 3\},\ r \in \{0, 1\}$, it is trivial that

$$\mathcal{M}^1(B_i^r\Omega_0) \Longrightarrow \mathcal{M}^1(B_i^r\Omega_j), \quad j \in \{1, 2\}. \tag{2.8}$$

However [for (i, r) as before], the maximality principles $\mathcal{M}(B_i^r\Omega_1)$ and $\mathcal{M}(B_i^r\Omega_2)$ are not in general comparable. So, it will be useful treating separately the cases $j = 1,\ j = 2$.

(A) By the considerations above, the following relational diagram is true

$$\begin{cases} \mathcal{M}^1(B_3^0\Omega_0) \Longrightarrow \mathcal{M}^1(B_i^r\Omega_j) \Longrightarrow \mathcal{M}^1(B_1^1\Omega_1), \\ \text{for all } i \in \{1, 2, 3\},\ r \in \{0, 1\},\ j \in \{0, 1\}. \end{cases} \tag{2.9}$$

The remarkable fact to be aded here is contained in

Proposition 2. *Under these conventions,*

$$\mathcal{M}^1(B_1^1\Omega_1) \Longrightarrow \mathcal{M}^1(B_3^0\Omega_0) \quad [= \mathcal{M}^1(B_3^0)], \quad \textit{in } \{ZF\}_0. \tag{2.10}$$

So that, by (2.9) above,

$$\begin{cases} \textit{the maximality principles } \mathcal{M}^1(B_i^r\Omega_j),\ i \in \{1, 2, 3\},\ r \in \{0, 1\}, \\ j \in \{0, 1\} \textit{ are all equivalent in } \{ZF\}_0. \end{cases} \tag{2.11}$$

Proof. Assume $\mathcal{M}^1(B_1^1\Omega_1)$ is true,and let $(E, \le)$ be an ordered structure fulfiling B_3^0 (i.e.: any super chain is bounded above). Denote for simplicity $\mathcal{E} = C_3(E, \le)$. Let us introduce a (partial) order on this class by

(2D20) $\begin{cases} X \preceq Y \text{ iff } X \text{ is a } \textit{segment} \text{ of } Y \\ \text{(i.e.,a part of } Y \text{ with } Y(x, \ge) \subseteq X,\ \forall x \in X). \end{cases}$

The ordered structure $(\mathcal{E}, \preceq)$ is a tree with the extra property B_1^1. (This follows essentially from the fact that, in a well ordered part $(Y, \le)$,

i) each proper segment U of Y may be uniquely represented as $U = Y(a(U), >)$, for some $a(U) \in Y$;

ii) the map $U \vdash a(U)$ is an order isonorphism between the class of all such objects (ordered as before) and the set Y_0 of all points in Y distinct from its first element.
We do not give details). As a consequence,

$$\max(\mathcal{E}, \preceq) \neq \emptyset \text{ (because } \mathcal{M}^1(B_1^1\Omega_1) \text{ holds).} \tag{2.12}$$

Now, by the very definition of this order,

$$\begin{cases} X \in \max(\mathcal{E}, \preceq) \Longrightarrow \text{ubd}(X) \subseteq X \\ \text{(hence lst}(X) \text{ exists and belongs to } \max(E, \leq)). \end{cases} \tag{2.13}$$

Hence the conclusion. ■

The following essential completion of these facts is to be noted.

Theorem 1. *Each of the maximality principles* $\mathcal{M}^1(B_i^r\Omega_j)$*;* $i \in \{1,1,3\}$, $r \in \{0,1\}$, $j \in \{0,1\}$ *is equivalent with (AC) in* $\{ZF\}_0$.

PROOF. By the developments in Bourbaki [3], one has

$$(AC) \Longrightarrow \mathcal{M}^1(B_3^1\Omega_0) \; [\Longrightarrow \mathcal{M}^1(B_1^1\Omega_1)], \text{ in } \{ZF\}_0 . \tag{2.14}$$

On the other hand, let us consider the statement (referred to as the (Zermelo) *Well Ordering Principle*)

(WOP) each nonempty set can be well ordered;

(cf.the 1908 Zermelo's paper [17]). We claim that

$$\mathcal{M}^1(B_1^1\Omega_1) \Longrightarrow (WOP), \quad \text{in } \{ZF\}_0 . \tag{2.15}$$

Indeed, letting E be a nonempty set, put

(2D21) $\mathcal{E} = \{(X, R);\ X \subseteq E,\ \mathcal{R} = \text{well order on } X\}$.

Define a partial order $\preceq$ over E according to

(2D22) $(X, \mathcal{R}) \preceq (Y, \mathcal{S})$ iff $X \subseteq Y$, $\mathcal{R} = \mathcal{S}|X$ and X is a segment of Y.

(Here, $\mathcal{S}|X = \mathcal{S} \cap X^2$ is the *restriction* of $\mathcal{S}$ to X). By the same argument as in Proposition 2, one proves that $(\mathcal{E}, \preceq)$ is a tree with the property B_1^1. So, necessarily, (2.12) holds. Moreover, by the very definition of our order,

$$(X, \mathcal{R}) \in \max(\mathcal{E}, \preceq) \Longrightarrow X = E; \tag{2.16}$$

hence the claim. Finally, by the argument in Alexandrov [1,ch..3, Sect.5]

$$(WOP) \Longrightarrow (AC), \quad \text{in } \{ZF\}_0 . \tag{2.17}$$

We therefore closed the circle between these principles; and this, along with Proposition 2, gives all we need. ■

(B) By the relations (2.7)+(2.8) above, it is the case that

$$\begin{cases} \mathcal{M}^1(B_3^0\Omega_0) \Longrightarrow \mathcal{M}^1(B_i^r\Omega_j) \Longrightarrow \mathcal{M}^1(B_1^1\Omega_2), \\ i \in \{1,2,3\}, r \in \{0,1\}, j \in \{0,2\}. \end{cases} \tag{2.18}$$

A basic completion of these is contained in

Theorem 2. *The following is true*

$$M^1(B_3^1\Omega_2) \Longrightarrow (WOP) \quad \textit{in } \{ZF\}_0 . \tag{2.19}$$

So that (combining with the above)

$$\begin{cases} \textit{the maximality principles } \mathcal{M}^1(B_3^r\Omega_j),\ r\in\{0,1\},\ j\in\{0,2\} \\ \textit{are equivalent with } (AC) \ (\textit{in } \{ZF\}_0). \end{cases} \tag{2.20}$$

PROOF. Assume that $\mathcal{M}^1(B_3^1\Omega_2)$ is true; and let E be a nonempty set. We are intending to show that a hypothesis like

(2H1) E cannot be well ordered

leads us to a contradiction. Let the (nonempty) class $\mathcal{E}$ be introduced along with (2D21). Define a *strict order* $\prec$ over it, according to

(2D23) $(X,\mathcal{R}) \prec (Y,\mathcal{S})$ iff $X \subset Y$.

(Here, $\subset$ denotes the *strict inclusion*). Let $\preceq$ stand for its associated order

(2D24) $\begin{cases} (X,\mathcal{R}) \preceq (Y,\mathcal{S}) \text{ iff either } (X,\mathcal{R}) \prec (Y,\mathcal{S}) \\ \text{or } (X,\mathcal{R}) = (Y,\mathcal{S}) \text{ (i.e. } X=Y,\ \mathcal{R}=\mathcal{S}). \end{cases}$

Note that, in view of (2H1) above,

$$\text{for each } (X,\mathcal{R})\in\mathcal{E} \text{ there exists } (X^*,\mathcal{R}^*)\in\mathcal{E} \text{ with } (X,\mathcal{R})\prec(X^*,\mathcal{R}^*). \tag{2.21}$$

Indeed, by the quoted hypothesis, there exists $(Y,\mathcal{S})\in\mathcal{E}$ with $X\cap Y=\emptyset$. (Just take $Y=\{a\}$, $S=\{(a,a)\}$, where $a\in E\setminus X$). It will suffice putting

$$X^* = X\cup Y, \quad \mathcal{R}^* = \mathcal{R}\oplus\mathcal{S} \tag{2.22}$$

to establish our assertion. (Here, $\oplus$ stands for the *lexicographic sum*). We are going to establish that the ordered structure $(\mathcal{E},\preceq)$ is a quasi–lattice having the property B_3^1.

(i) Let $(X,\mathcal{R})$, $(Y,\mathcal{S})$ be two elements of $\mathcal{E}$. Put $Z=X\cup Y$ and introduce a well order $\mathcal{T}$ on this set as

$$\mathcal{T} = \mathcal{R}|(X\setminus Y)\oplus\mathcal{Q}|(X\cap Y)\oplus\mathcal{S}|(Y\setminus X);$$

here, $\mathcal{Q}$ is one of the relations $\mathcal{R}$ or $\mathcal{S}$. Note that, in general, $(Z,\mathcal{T})$ does not appear as an upper bound of $\{(X,\mathcal{R}),(Y,\mathcal{S})\}$. (To verify this, it will suffice considering the choice $X=Y$ and $\mathcal{R}\neq\mathcal{S}$). However, by (2.21) above,

$$(Z,\mathcal{T})\prec(Z^*,\mathcal{T}^*), \quad \text{for some } (Z^*,\mathcal{T}^*)\in\mathcal{E};$$

and, in this case, $(Z^*,\mathcal{T}^*)$ is a majorant for both $(X,\mathcal{R})$ and $(Y,\mathcal{S})$.

(ii) Let $\mathcal{K}$ be an arbitrary fixed super chain in $(\mathcal{E},\preceq)$. Denoting by α its order type,it follows that this object may be represented as $\mathcal{K}=\{(X_\lambda,\mathcal{R}_\lambda);\ \lambda<\alpha\}$, where

$$\lambda<\mu<\alpha \Longrightarrow (X_\lambda,\mathcal{R}_\lambda)\prec(X_\mu,\mathcal{R}_\mu) \text{ (hence } X_\lambda\subset X_\mu). \tag{2.23}$$

If α is a first kind ordinal, ($\alpha-1$ exists) one has $(X_{\alpha-1},\mathcal{R}_{\alpha-1})=\sup\mathcal{K}$; and our assertion follows. It remains now to discuss the case

(2H2) α is a second kind (=limit) ordinal ($\alpha-1$ does not exist).

Let us construct a net $\{Y_\lambda;\ \lambda<\alpha\}$ according to

(2D25) $\begin{cases} Y_0 = X_0; \text{ and (for the remaining ranks)} \\ Y_\lambda = X_\lambda\setminus X_{\lambda-1}, \text{ if } \lambda-1 \text{ exists} \\ Y_\lambda = X_\lambda\setminus\cup\{X_\nu;\ \nu<\lambda\}, \text{ otherwise.} \end{cases}$

(Note that the case (=alternative) of

$$Y_\nu = \emptyset, \quad \text{for some (second kind) ordinal } \nu < \alpha \tag{2.24}$$

cannot be avoided). Clearly,

$$Y_\lambda \cap Y_\mu = \emptyset, \quad \text{whenever } \lambda \neq \mu \ \ (\lambda, \mu < \alpha). \tag{2.25}$$

Put $X = \cup\{X_\lambda; \lambda < \alpha\}$ and introduce a well ordering $\mathcal{R}$ over it according to

(2D26) $\mathcal{R} =$ the lexicographic sum of $\{\mathcal{R}_\lambda | Y_\lambda;\ \lambda < \alpha\}$.

The definition is meaningful, in view of $X = \cup\{Y_\lambda;\ \lambda < \alpha;\}$ and (2.25). Moreover, by (2.23) above,

$$X_\lambda \subset X \text{ (hence } (X_\lambda, \mathcal{R}_\lambda) \prec (X, \mathcal{R})), \quad \text{for each } \lambda < \alpha. \tag{2.26}$$

As a consequence, $(X, \mathcal{R}) = \sup \mathcal{K}$; and this proves our assertion.
Summing up, the maximality principle $\mathcal{M}^1(B_3^1\Omega_2)$ is applicable upon $(\mathcal{E}, \preceq)$; and so, necessarily,

$$\max(\mathcal{E}, \preceq) = \{\text{lst}(\mathcal{E}, \preceq)\} \neq \emptyset. \tag{2.27}$$

On the other hand,the existence of such an object is in contradiction with (2H1). So, the quoted alternative cannot be accepted; and this ends the argument. ■

Concerning the obtained facts, a natural question to be posed is of to what extent can the conclusion of Theorem 2 be valid for the remaining maximality principles of (2.18); i.e.,of whether or not are implications like

$$\mathcal{M}^1(B_i^r\Omega_2) \Longrightarrow (AC) \quad \text{in } \{ZF\}_0\,, \text{ for some } i \in \{1,2\},\ r \in \{0,1\} \tag{2.28}$$

retainable. Perhaps this may need some other equivalents of (AC) than (WOP) above. For a number of related aspects we refer to Sierpinski [13,ch.16] and the references therein.

3 Topological versions: preliminaries

Let E be a nonempty set and $(\leq)$, some quasi–ordering over it. In what follows,the notion of *topology* over E is to be taken in the usual manner,by starting from either
(i) the notion of *open set* (Bourbaki's construction); or
(ii) the notion of *closed set* (Kuratowski's construction).
Letting $\mathcal{T}$ be such an object, the triplet $(E, \leq, \mathcal{T})$ will be referred to as a *tolopogical quasi-ordered* space.

It is our aim in this section to introduce topological versions for the abstract maximality principles above. This requires new conventions. Letting X be a *semi chain* in E, put

(3D1) $\mathcal{B}_X = \{X(x, \leq);\ x \in X\}$ (the associated *filterbase*).

We say that $z \in X$ is a $(\leq)$–*accumulation point* of X, whenever

(3D2) $W \cap Y \neq \emptyset$, for all $W \in \mathcal{V}(z)$, $Y \in \mathcal{B}_X$ (i.e.: z is *adherent* to $\mathcal{B}_X$);

and a $(\leq)$–*limit point* of X, in case

(3D3) $\begin{cases} \text{for each } W \in \mathcal{V}(z) \text{ there exists } Y \in \mathcal{B}_X \text{ such that } Y \subseteq W \\ \text{(i.e.: } z \text{ is a limit point for the (filter generated by) } \mathcal{B}_X). \end{cases}$

(Here, $\mathcal{V}(z)$ stands for the *neighborhoods filter* of z). Denote for simplicity

(3D4) $\begin{cases} \text{acc}(X) \text{ (resp., } \lim(X))\text{=the class of all} \\ (\leq)\text{–accumulation (resp., } (\leq)\text{–limit) points of } X. \end{cases}$

It is easy to see that

$$\lim(X) \subseteq \text{acc}(X), \quad \text{for all such } X; \tag{3.1}$$

the reciprocal is not true, in general. We may now introduce the following types of convergence properties for $(E, \leq, \mathcal{T})$, where $i \in \{1, 2, 3\}$

(3D5) $\begin{cases} \mathcal{C}_i^0(E, \leq, \mathcal{T}): \ \text{acc}(X) \neq \emptyset, \quad \text{for all } X \in \mathcal{C}_i(E, \leq) \\ \mathcal{C}_i^1(E, \leq, \mathcal{T}): \ \lim(X) \neq \emptyset, \quad \text{for all } X \in \mathcal{C}_i(E, \leq). \end{cases}$

The relationships between these are (cf. (3.1) above)

$$\mathcal{C}_s^u(E, \leq, \mathcal{T}) \Longrightarrow \mathcal{C}_t^v(E, \leq, \mathcal{T}), \quad \text{if } s \geq t, u \geq v. \tag{3.2}$$

In parallel to this, we must introduce additional properties relating the (couple of) structures in $(E, \leq, \mathcal{T})$. The basic one is (cf.Section 2)

(3D6) $\begin{cases} \Delta_0(E, \leq, \mathcal{T})\text{: } (\leq) \text{ and } \mathcal{T} \text{ are amorph over } E \\ \text{(no additional conditions are needed).} \end{cases}$

And, from the non–amorph ones, we shall use properties like

(3D7) $\begin{cases} \Delta_1(E, \leq, \mathcal{T})\text{: } (\leq) \text{ is } \textit{semi-closed} \\ (E(x, \leq) \text{ is closed, for each } \ x \in E) \end{cases}$

(3D8) $\begin{cases} \Delta_2(E, \leq, \mathcal{T})\text{: } (\leq) \text{ is } \textit{double semi-closed} \\ \text{(both } (\leq) \text{ and } (\geq) \text{ are semi–closed)} \end{cases}$

(3D9) $\begin{cases} \Delta_3(E, \leq .\mathcal{T})\text{: } (\leq) \text{ is closed} \\ \text{(in } E^2\text{, endowed with the product topology).} \end{cases}$

These may be coupled with additional properties of the "partial" structures $(E, \leq)$ and $(E, \mathcal{T})$. The former ones are the properties $\{\Omega_0, \Omega_1, \Omega_2\}$ we already introduced in Section 2. And the latter ones start from

(3D10) $\begin{cases} \Theta_0(E, \mathcal{T})\text{: } \mathcal{T} \text{ is amorph over } E \\ \text{(no further conditions are needed).} \end{cases}$

As non–amorph properties of this type, we have

(3D11) $\begin{cases} \Theta_1(E, \mathcal{T})\text{: } \mathcal{T} \text{ is zero dimensional on } E \\ \text{(there is a } \textit{base} \text{ for } \mathcal{T} \text{ consisting of open–closed parts of } E) \end{cases}$

(3D12) $\begin{cases} \Theta_2(E, \mathcal{T})\text{: } \mathcal{T} \text{ is Frechet } (= T_1) \text{ separated} \\ (\{x\} \text{ is closed, for each } \ x \in E) \end{cases}$

(3D13) $\begin{cases} \Theta_3(E, \mathcal{T})\text{: } \mathcal{T} \text{ is Hausdorff } (= T_2) \text{ separated} \\ (x, y \in E, \ x \neq y \Longrightarrow \exists A \in \mathcal{V}(x), \ \exists B \in \mathcal{V}(y), \ A \cap B = \emptyset). \end{cases}$

Some remarks are in order. The list of all basic properties for $(E, \leq, \mathcal{T})$, $(E, \leq)$ and $(E, \mathcal{T})$ respectively is very large. But, as we shall see, only the written ones are effective. On the other hand, the properties in questiuon are not independent. For example, one has (cf. Nachbin [12,ch.1,Sect.1])

$$(\forall \ (E, \leq, \mathcal{T}) = \text{topological ordered space}) \quad \Delta_3(E, \leq, \mathcal{T}) \Longrightarrow \Theta_3(E, \mathcal{T}). \tag{3.3}$$

Finally, a similar construction is to be made for additional structures over E which are more general than topologies. This will be discussed elsewhere.

Now, by a *(global) maximality principle* on topological quasi–ordered structures we mean the logical proposition

(3D14) $\left\{ \begin{array}{l} \mathcal{M}_g^0(S;P,Q)\text{: } (\forall (E,\leq,\mathcal{T}) = \text{topological quasi–ordered structure}) \\ (S(E,\leq,\mathcal{T}); P(E,\leq), Q(E,\mathcal{T})) \Longrightarrow \max(E,\leq) \neq \emptyset; \end{array} \right.$

here, $(S;P,Q)$ are certain "composed" properties of $(E,\leq,\mathcal{T})$, $(E,\leq)$ and $(E,\mathcal{T})$ respectively. The (topological) maximality principle in Turinici [14] enters in this scheme, for $(S = C_2^0,; P = \Omega_0, Q = \Theta_0)$. And so does the statement in Brunner [4] for $(S = C_1^0\Delta_3; P = \Omega_2, Q = \Theta_1)$. A local version of such conventions may be depicted as follows. Denote

(3D15) $\text{lmax}(E,\leq,\mathcal{T}) = \cup\{\max(G,\leq);\ G \in \mathcal{T}\}$;

any point of this set will be referred to as a *local* (modulo $\mathcal{T}$) *maximal element* of E. Clearly,

$$\max(E,\leq) \subseteq \text{lmax}(E,\leq,\mathcal{T}),\ \text{foreach topology } \mathcal{T}; \tag{3.4}$$

but the converse is not in general true. Now, by a *local maximality principle* on topological quasi–ordered structures we mean the logical proposition

(3D16) $\left\{ \begin{array}{l} \mathcal{M}_l^0(S;P,Q)\text{: } (\forall (E,\leq,\mathcal{T}) = \text{topological quasi–ordered structure}) \\ (S(E,\leq,\mathcal{T}); P(E,\leq), Q(E,\mathcal{T})) \Longrightarrow \text{lmax}(E,\leq) \neq \emptyset. \end{array} \right.$

[Here, the meaning of $(S;P,Q)$ is the one of (3D14)]. By (3.4) above,

$$\mathcal{M}_g^0(S;P,Q) \Longrightarrow \mathcal{M}_l^0(S;P,Q), \quad \text{for all such } (S;P,Q). \tag{3.5}$$

The converse is not generally true; just take the discrete topology $\mathcal{T} = \mathcal{P}(E)$ to verify this assertion.

A basic particular aspect of these constructions is related to the case of $(\leq)$ being an *ordering* over E. Precisely, by a global/local maximality principle on topological ordered structures we mean the logical proposition (for $a \in \{g,l\}$)

(3D17) $\left\{ \begin{array}{l} \mathcal{M}_a^1(S;P,Q)\text{=the principle } \mathcal{M}_a^0(S;P,Q), \\ \text{restricted to topological ordered structures.} \end{array} \right.$

The relation between these may be written as

$$\mathcal{M}_a^0(S;P,Q) \Longrightarrow \mathcal{M}_a^1(S;P,Q), \quad a \in \{g,l\}. \tag{3.6}$$

A natural question is that of the reciprocal being or not valid. For an appropriate answer, we need some auxiliary facts. Let $(E,\leq,\mathcal{T})$ be a topological quasi–ordered space. Further, let $E^\sim = E|_\sim$ stand for the factor space; and $\preceq$, for the factor ordering (cf. Section 2). The natural topology over $E^\sim$ to be considered is the quotient one, $\mathcal{T}^\sim$, constructed according to

(3D18) $D^\sim \in \mathcal{T}^\sim \quad \text{iff } \cup D^\sim = \cup\{x^\sim;\ x^\sim \in D^\sim\} \in \mathcal{T}$.

Now, it is clear that, whenever the properties $(S;P,Q)$ are maintained in passing from $(E,\leq,\mathcal{T})$ to $(E^\sim,\preceq,\mathcal{T}^\sim)$, the converse to (3.6) takes place; but,in the opposite case, this cannot be true (in general).

The logical constructions we just performed are, until now, formal ones. So, it is legitimate asking of to what extent are these deductible in $\{ZF\}_0$. As we shall see, this is retainable for combined properties of $(E,\leq,\mathcal{T})$ like $S = C_i^r\Delta_j$, $i,j \in \{1,2,3\}$, $r \in \{0,1\}$. Precisely,let P,Q be arbitrary properties of $(E,\leq)$ and $(E,\mathcal{T})$ respectively. The following relations are available,for $a \in \{g, l\}$, $h \in \{0, 1\}$

$$\mathcal{M}_a^h(C_i^s\Delta_u; P, Q) \Longrightarrow \mathcal{M}_a^h(C_j^t\Delta_v; P, Q), \quad \text{when } i \geq j,\ s \leq t,\ u \leq v. \tag{3.7}$$

So, combining with (3.6) above,

$$\begin{cases} \mathcal{M}_g^0(C_3^0\Delta_1; P, Q) \Longrightarrow \mathcal{M}_a^h(C_i^r\Delta_j; P, Q), \\ \text{for } a \in \{g, l\}, i, j \in \{1, 2, 3\},\ h, r \in \{0, 1\}. \end{cases} \tag{3.8}$$

On the other hand, a standard argument assures us that

$$\Delta_1 (E, \leq, \mathcal{T}) \Longrightarrow [\operatorname{acc}(X) \subseteq \operatorname{ubd}(X), \forall X \in C_1 (E, \leq)]. \tag{3.9}$$

This immediately gives the (formal) implication

$$[\mathcal{M}^0(B_3^0\Omega_0) =] \quad \mathcal{M}^0(B_3^0) \Longrightarrow \mathcal{M}_g^0 \left(C_3^0\Delta_1; P, Q\right). \tag{3.10}$$

But (cf. Theorem 1), the maximality principle in the left side is deductible from (AC). We therefore proved

Proposition 3. *Under the precised conventions, one has*

$$\begin{cases} \textit{the maximality principles } \mathcal{M}_a^h(C_i^r\Delta_j; P, Q),\ a \in \{g, l\}, \\ i, j \in \{1, 2, 3\},\ h \in \{0, 1\} \textit{ are all deductible from (AC) in } \{ZF\}_0. \end{cases} \tag{3.11}$$

Some technical remarks are in order. The obtained statement may be viewed as a refinement of the one in Turinici [op.cit.] which,in turn,extends the result in Dancs, Hegedus and Medvegyev [6]. But, as explicitly stated by the quoted authors, their contribution appears as an abstract metrical version of the variational Ekeland's principle [7]; hence Proposition 3 also has this character. Further aspects may be found in Ward [16] and the references therein.

4 Main results (in topological trees)

By the developments above, it results that the implication below holds (in $\{ZF\}_0$)

$$(AC) \Longrightarrow \mathcal{M}_a^1(C_i^r\Delta_j; P, Q), \quad a \in \{g, l\},\ i, j \in \{1, 2, 3\},\ r \in \{0, 1\}, \tag{4.1}$$

for each couple of properties (P, Q) of $(E, \leq)$ and $(E, \mathcal{T})$ respectively. To complete the diagram, we have to analyze the converse question. From the results in Section 2, it is to be expected that a positive answer to this be available for a restricted class of properties (P, Q) in $\{\Omega_1, \Omega_2\} \times \{\Theta_1, \Theta_2\}$. It is our aim in the following to confirm such a claim for the case of $(P = \Omega_1, Q = \Theta_1)$. The starting point of it is the implication below, holding for $a \in \{g, l\}$, $i, j \in \{1, 2, 3\}$, $r \in \{0, 1\}$

$$\begin{cases} \mathcal{M}_a^1(C_i^r\Delta_j; \Omega_s\Theta_u) \Longrightarrow \mathcal{M}_a^1(C_i^r\Delta_j; \Omega_t, \Delta_v), \\ \text{for } s, t, u, v \in \{0, 1\},\ s \leq t,\ u \leq v. \end{cases} \tag{4.2}$$

This, added to (3.5)–(3.7) and (4.1), yields (in $\{ZF\}_0$)

$$\begin{cases} \text{(AC)} \Longrightarrow \mathcal{M}_a^1(C_i^r\Delta_j; \Omega_k, \Theta_h) \Longrightarrow \mathcal{M}_l^1(C_1^1\Delta_3; \Omega_1, \Theta_1), \\ \text{for all } \ a \in \{g, l\},\ i, j \in \{1, 2, 3\},\ r, k, h \in \{0, 1\}. \end{cases} \tag{4.3}$$

The following completion of it – which is our first main result – gives the announced answer to the question we deal with.

Theorem 3. *Under the precised facts, we have*

$$\mathcal{M}_l^1(C_1^1\Delta_3;\Omega_1,\Theta_1) \Longrightarrow \mathcal{M}^1(B_3^0)\ [=\mathcal{M}^1(B_3^0\Omega_0)], \quad \textit{in } \{ZF\}_0. \tag{4.4}$$

Hence, combining with Theorem 1 and the preceding relation

$$\left\{\begin{array}{l} \textit{the maximality principles } \mathcal{M}_a^1(C_i^r\Delta_j;\Omega_k,\Theta_h),\ a\in\{g,l\},\ i,j\in \\ \{1,2,3\},\ r,k,h\in\{0,1\} \ \ \textit{are all equivalent with (AC) in } \{ZF\}_0. \end{array}\right. \tag{4.5}$$

PROOF. Assume that $\mathcal{M}_l^1(C_1^1\Delta_3;\Omega_1,\Theta_1)$ is true; and let $(E,\leq)$ be an ordered structure fulfiling B_3^0; [i.e.: any super chain is bounded above]. We associate it the structure $(\mathcal{E},\preceq)$, where $\mathcal{E}=\mathcal{C}_3(E,\leq)$ and $(\preceq)$ is the segment order introduced as in Proposition 2. Remember that, by the developments performed there,

$$(\mathcal{E},\preceq) \text{ is a tree } [i.e.:\ \text{the property } (\Omega_1) \text{ holds}]. \tag{4.6}$$

We now introduce a topological structure $\mathcal{T}$ over $\mathcal{E}$ under the way below. Denote

(4D1) $[U,V]=\{X\in\mathcal{E};\ U\subseteq X \text{ and } V\subseteq X^c\},\quad U,V\in\mathcal{P}_f(E)$.

[Here, $\mathcal{P}_f(E)$ stands for the class of all *finite* parts of E; and $(.)^c$, for the *absolute complement* in E]. The family of subsets in $\mathcal{E}$

(4D2) $\mathcal{B}=\{[U,V];\ U,V\in\mathcal{P}_f(E)\}$

is a topology basis (in the usual sense). Indeed

$$\left\{\begin{array}{l} \emptyset\in\mathcal{B} \text{ (since } [U,U]=\emptyset,\ \forall U\in\mathcal{P}_f(E),\ U\neq\emptyset\text{); and} \\ \mathcal{E}\in\mathcal{B} \text{ (in view of } [\emptyset,\emptyset]=\mathcal{E}\text{);} \end{array}\right. \tag{4.7}$$

$$\left\{\begin{array}{l} [U_i,V_i]\in\mathcal{B},\ \forall i\in I \text{ (=finite index set)} \Longrightarrow \\ \bigcap_{i\in I}[U_i,V_i]=\left[\bigcup_{i\in I}U_i,\bigcup_{i\in I}V_i\right]\in\mathcal{B}. \end{array}\right. \tag{4.8}$$

The topology in question is introduced as:

(4D3) $\mathcal{X}\in\mathcal{T} \Longleftrightarrow \mathcal{X}=$ (arbitrary) union of members in $\mathcal{B}$.

It remains to show that $\mathcal{T}$ has all the properties we need. This will be done in a number of steps.

(i) We firstly claim that

$$\text{each member of } \mathcal{B} \text{ is open-closed (i.e.: } (\Theta_1) \text{ holds over } (E,\mathcal{T})). \tag{4.9}$$

The former part is trivial, by definition. For the latter,let $U,V\in\mathcal{P}_f(E)$ be arbitrary fixed. Clearly,

$$[U,V]=\cap\{[u,v];\ u\in U,\ v\in V\} \quad (cf.\ (4.8)). \tag{4.10}$$

On the other hand, for each $u\in U,\ v\in V$,

$$[u,v]^c=[\emptyset,u]\cup[v,\emptyset]\in\mathcal{T} \text{ (where from } [u,v] \text{ is closed).} \tag{4.11}$$

Putting these facts together, we are done.

(ii) Secondly, we intend to show that

$$(\preceq) \text{ is closed in } \mathcal{E}^2 \text{ (i.e.: } (\Delta_3) \text{ hold sover } (\mathcal{E},\preceq,\mathcal{T})). \tag{4.12}$$

Indeed, let X,Y in $\mathcal{E}$ be such that

(4H1) $X \preceq Y$ does not hold (hence $X \neq Y$).

The following possibilities occur:

a) X is not included in Y. Then, if we take some $a \in X \setminus Y$, the neighborhood $[a, \emptyset] \times [\emptyset, a]$ of (X, Y) is disjoint from $(\preceq)$.

b) X is included in Y. We show that, necessarily,

$$\text{there exist } b \in X,\ c \in Y \setminus X \text{ with } b > c. \tag{4.13}$$

For, otherwise, a hypothesks like

(4H2) for each $b \in X,\ c \in Y \setminus X$ one has $b \leq c$ (hence $b < c$)

yields $X \preceq Y$; in contradiction to (4H1). But then, the neighborhood $[b, c] \times [c, \emptyset]$ of (X, Y) cannot intersect $(\preceq)$.

Having explored all possible cases, the assertion follows.

(iii) Further, we claim that

$$\text{any semi chain of } \mathcal{E} \text{ has limit points (i.e.,: } \left(C_1^1\right) \text{ holds over } (E, \preceq, \mathcal{T})). \tag{4.14}$$

Indeed, letting $\mathcal{X}$ be such an object, put $Z = \sup(\mathcal{X})$ $(= \cup\mathcal{X})$. For the moment, Z is an element of $\mathcal{E}$; this follows at once from the semi chain property of $\mathcal{X}$. Moreover, Z is a limit point of $\mathcal{X}$. To verify this, let $[U, V]$ be some neighborhood of Z. Again by the semi chain property of $\mathcal{X}$, there must be some $X_U \in \mathcal{X}$ with $U \subseteq X_U \subseteq Z$ (because U is finite). This immediately gives $\mathcal{X}(X_U, \preceq) \subseteq [U, V]$; and the assertion is proved.

(iv) Finally, we show that the following apriori evaluation is true, in $\{ZF\}_0$:

$$\left\{ \begin{array}{l} \max(\mathcal{E}, \preceq) \text{ is cofinal in } \operatorname{lmax}(\mathcal{E}, \preceq) \quad \text{(for each} \\ X \in \operatorname{lmax}(\mathcal{E}, \preceq) \text{ there exists } Y \in \max(\mathcal{E}, \preceq) \text{ with } X \preceq Y). \end{array} \right. \tag{4.15}$$

In fact, let X be some point of $\operatorname{lmax}(\mathcal{E}, \preceq)$. Without any loss, one may assume that (cf. the construction of the ambient topology)

(4H3) $X \in \max([U, V], \preceq)$, for some $U, V \in \mathcal{P}_f(E)$.

By the working hypothesis about $(E, \leq)$, we have $\operatorname{ubd}(X) \neq \emptyset$. Denote for simplicity $V_X = V \cap \operatorname{ubd}(X)$. If $V_X = \emptyset$, then $X \in \max(\mathcal{E}, \preceq)$; and we are done. Otherwise,

$$\mathcal{E}(V_X) = \mathcal{C}_3(V_X, \leq) \text{ is finite (as } V_X = \text{finite)}. \tag{4.16}$$

Denote again by $\preceq$ the restriction to $\mathcal{E}(V_X)$ of the initial order $(\preceq)$ on $\mathcal{E}$. By (4.16) one derives (in the $\{ZF\}_0$ setting)

$$\max(\mathcal{E}(V_X), \preceq) \quad \text{is (finite and) nonempty.} \tag{4.17}$$

Fix some $\mathcal{Z} = \mathcal{Z}(X; V)$ in this (finite) set; and put $X^* = X \cup Z$. Clearly,

$$X \preceq X^*; \quad \text{and, moreover, } X^* \in \max(\mathcal{E}, \preceq). \tag{4.18}$$

(It will suffice noting that, whenever $Y \in \mathcal{E}$ is such that $X^* \preceq Y$, then

$$X \quad \preceq \quad Y_1 = Y \setminus V \in [U, V] \quad (\text{hence } X = Y_1) \tag{4.19}$$

$$Z \quad \preceq \quad Y_2 = Y \cap V \in \mathcal{E}(V_X) \quad (\text{hence } Z = Y_2); \tag{4.20}$$

we do not give details). This proves our claim.

Having these established, it is now easy to complete the argument. By (i)–(iii) above, the topological ordered structure $(\mathcal{E}, \preceq, \mathcal{T})$ fulfils all premises of $\mathcal{M}_l^1(C_1^1\Delta_3; \Omega_1, \Theta_1)$. So, by the underlying principle,

$$\max(\mathcal{E}, \preceq) \neq \emptyset \quad \text{(if one takes (iv) into account).} \tag{4.21}$$

This, along with the arguments in Proposition 2 [concerning (2.13)] and the working hypothesis about $(E, \leq)$, yields the conclusion we need. ■

The obtained statement may be viewed as a refinement of the one due to Brunner [op.cit.] This is especially related to the fact that (cf. the remarks in Section 3) the Hausdorff ($= T_2$) separation property (Θ_3) is not necessary in (4.4). It would be interesting to determine of to what extent is the topological property (Θ_1) the best one so as to solve the posed question. We conjecture that the answer is negative.

5 Further aspects (in topological quasi-lattices)

As explicitly results from its formulation, Theorem 3 above is a topological version of Theorem 1, related to the choice (Ω_1, Θ_1) for the couple (P, Q) (of properties attached to $(E, \leq)$ and $(E, \mathcal{T})$ respectively). So, it is natural to ask whether a similar device is applicable to Theorem 2 as well. It is our aim in the following to indicate a positive answer to this, under the choice (Ω_2, Θ_2) for the couple (P, Q). The starting point of it is the implication below,holding for $i, j \in \{1, 2, 3\}$, $r \in \{0, 1\}$;

$$\begin{cases} \mathcal{M}^1_g(C^r_i\Delta_j; \Omega_s, \Theta_u) \Longrightarrow \mathcal{M}^1_g(C^r_i\Delta_j; \Omega_t, \Theta_v), \\ \text{for } s, t, u, v \in \{0, 2\},\ s \leq t,\ u \leq v. \end{cases} \tag{5.1}$$

This, added to (3.5)–(3.7) and Proposition 3 yields (in $\{ZF\}_0$)

$$\begin{cases} \text{(AC)} \Longrightarrow \mathcal{M}^1_g(C^r_3\Delta_1; \Omega_k, \Theta_j) \Longrightarrow \mathcal{M}^1_g(C^1_3\Delta_1; \Omega_2, \Theta_2), \\ \text{for all } r \in \{0, 1\},\ k, h \in \{0, 2\}. \end{cases} \tag{5.2}$$

The following completion of this is our second main result in the present exposition.

Theorem 4. *Under the precised facts,we have*

$$\mathcal{M}^1_g(C^1_3\Delta_1; \Omega_2, \Theta_2) \Longrightarrow \mathcal{M}^1(B^0_3\Omega_2),\ \textit{in}\ \{ZF\}_0\,. \tag{5.3}$$

Hence, combining with Theorem 2 and the preceding relation,

$$\begin{cases} \textit{the maximality principles } \mathcal{M}^1_g(C^r_3\Delta_1; \Omega_k, \Theta_h),\ r \in \{0, 1\}, \\ k, h \in \{0, 2\} \textit{ are all equivalent with (AC) in } \{ZF\}_0. \end{cases} \tag{5.4}$$

Proof. Assume that $\mathcal{M}^1_g(C^1_3\Delta_1; \Omega_2, \Theta_2)$ is true; and let $(E, \leq)$ be a quasi–lattice fulfiling B^0_3 (i.e.: any super chain is bounded above). We construct a topology $\mathcal{T}$ over E in the way below: declare a part X of E, *closed* when either

(5D1) X is finite; or (when X is infinite)

(5D2) each infinite super chain G of E with $G \subseteq X$ fulfils $\mathrm{ubd}(G) \subseteq X$.

The consistency of this definition is to be proved as follows.

(a) Let $\{X_1, ...X_n\}$ be a finite system of (nonempty) closed parts of E. If all these are finite then $X = X_1 \cup ... \cup X_n$ is finite too; hence closed. So, without loss, one may assume

(5H1) $J = \{j \in \{1, ..., n\};\ X_j = \text{infinite}\} \neq \emptyset$ (hence $X = \text{infinite}$).

Let G be an infinite super chain of E with $G \subseteq X$. By (5H1) above,

$$G_i = G \cap X_i \quad \text{is infinite, for at least one } i \in J.$$

This, along with the definition of the upper bound, yields

$$\text{ubd}(G) \subseteq \text{ubd}(G_i) \subseteq X_i \subseteq X; \quad \text{and therefore, } X \text{ is closed.}$$

(b) Let $\{X_i;\ i \in I\}$ be an arbitrary family of (nonempty) closed parts of E. If $X = \cap\{X_i;\ i \in i\}$ is finite, we are done. So, without loss, one may assume

(5H2) $X =$ infinite (hence $X_i =$ infinite, for all $i \in I$).

Let G be an infinite super chain of E with $G \subseteq X$. We have

$$G \subseteq X_i \text{ (wherefrom } \text{ubd}(G) \subseteq X_i\text{), for all } i \in I;$$

and so, $\text{ubd}(G) \subseteq X$. Hence, X is closed.

It remains now to verify that the introduced topology has all properties we need. This will be done in a number of steps.

(i) Firstly, it is clear, via (5D1), that

$$\mathcal{T} \text{ is Frechet } (= T_1) \text{ separable (i.e.: } (\Theta_1) \text{ holdsover } (E, \mathcal{T})). \tag{5.5}$$

(ii) Secondly, we claim that

$$(\leq) \text{ is semi-closed (i.e.: } (\Delta_1) \text{ is retainable over } (E, \leq, \mathcal{T})). \tag{5.6}$$

Indeed, let $x \in E$ be arbitrary fixed; we have to show that $E(x, \leq)$ is closed. This is clearly true when the set in question is finite; so, without loss,one may accept that $E(x, \leq)$ is infinite. Let G be some infinite super chain with $G \subseteq E(x, \leq)$. By the very definition of this last set, one has $\text{ubd}(G) \subseteq E(x, \leq)$; and this proves our claim.

(iii) Finally, we assert that

$$\text{any super chain has a limit (i.e.: } (C_3^1) \text{ holds over } (E, \leq, \mathcal{T})). \tag{5.7}$$

For, let X be such an object. If X is finite,then $z = \text{lst}(X) \in \lim(X)$, because $\{z\} \in \mathcal{B}_X$ (cf.the convention in Section 3). So, we may assume that X is infinite. We claim that, in such a circumstance,

$$\text{ubd}(X) \subseteq \lim(X); \tag{5.8}$$

and this, by the working assumption about $(E, \leq)$, proves our claim. Suppose by contradiction that $z \in \text{ubd}(X)$ is not a limit point of X. By definition, there must an open neighborhood U of z with

$$Y = X \cap U^c \quad \text{is cofinal in } X. \tag{5.9}$$

This, by the working assumption about X, yields $Y =$ infinite; hence U^c is infinite too. Moreover,the same property gives $\text{ubd}(X) = \text{ubd}(Y)$. But then, the closdness of U^c implies

$$\text{ubd}(X) \subseteq U^c \text{ [because } \text{ubd}(Y) \subseteq U^c]; \quad \text{wherefrom } z \in U^c.$$

This, however, is in contradiction with the choice of U; and the assertion follows.

Summing up, the topological ordered structure $(E, \leq, \mathcal{T})$ fulfils all the premises of the maximality principle $\mathcal{M}_g^1(C_3^1\Delta_1; \Omega_2, \Theta_2)$. So, by its conclusion, $\max(E, \leq) \neq \emptyset$; and the proof is complete. ■

The obtained result may be viewed as a refinement of the one due to Brunner [4]; see also Manka [10]. A natural open question to be posed here is of to what extent can the conclusion of Theorem 4 be valid for the remaining maximality principles of (5.1); i.e., of whether or not are implications like

$$\mathcal{M}_g^1(C_i^r\Delta_j;\Omega_2,\Theta_2) \Longrightarrow \text{ (AC), in } \{ZF\}_0 \tag{5.10}$$

retainable, for some triple (i, j, r) in $\{1,2,3\} \times \{1,2,3\} \times \{0,1\}$ distinct from the ones in (5.3). At a first glance, the answer must be positive if one takes into account a related statement in this area due to Borwein [2]. Some further developments of these facts will be performed in a future paper.

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