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Approximate Fixed Point Theorems

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Abstract. Under some weakenings of the conditions of the well–known fixed point theorems of Brouwer, Kakutani and Banach the existence of approximate fixed points turns out to be still guaranteed. Also an approximate fixed point theorem is given for certain nonexpansive maps.

Keywords: fixed point, almost fixed point, contraction theorem, approximate fixed point.

1 Introduction

Fixed point theorems have proved to be a useful instrument in many applied areas such as:

(i) mathematical economics to prove the existence of Walras equilibria (Ref. 1,2),

(ii) non–cooperative game theory to prove the existence of Nash–equilibria (Ref. 3,4,5),

(iii) dynamic optimization and stochastic games to prove the existence of value functions in the case of a discounted payoff criterion (Ref. 6,7),

(iv) functional analysis, variational calculus, theory of integro–differential equations, etc. (Ref. 8).

However, for many practical situations, the conditions in the fixed point theorems are too strong, so there is then no guarantee that a fixed point exists. On the other hand, often in such situations one is content with approximate fixed points which guarantee then e.g. that supply and demand are approximatively equal, or that there exist approximate Nash equilibria (Ref. 9).

Here an approximate fixed point x of a function f has the property that $f(x)$ is 'near to' x in a sense to be specified.

In this note we discuss weakenings of the conditions in the fixed point theorems of Brouwer (Ref. 10), Kakutani (Ref. 11,12) and Banach (Ref. 13) which still guarantee the existence of approximate fixed points.

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2 Continuous functions and multifunctions on bounded convex sets and approximate fixed points

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^m$. Let C be a subset, let $f : C \to C$ be a function and let $F : C \twoheadrightarrow C$ be a multifunction, assigning to each $x \in C$ a subset $F(x)$ of C. A point $x^* \in C$ with the property that $f(x^*) = x^*$ is called a *fixed point of* f. Brouwer's famous fixed point theorem states that each continuous function from a compact and convex subset of $\mathbb{R}^m$ into itself possesses at least one fixed point. A point $x^* \in C$ with the property that $x^* \in F(x^*)$ is called a *fixed point of the multifunction* F. Kakutani's fixed point theorem states that each non–empty, compact and convex valued upper semicontinuous multifunction from a compact and convex subset of $\mathbb{R}^m$ into itself possesses at least one fixed point. Let ε be a positive real number. We will say that x^* is an ε*-fixed point* of $f : C \to C$ if $\|f(x^*) - x^*\| \leq \varepsilon$. Correspondingly we will say that x^* is an ε*-fixed point of* $F : C \twoheadrightarrow C$ if there is a $y^* \in F(x^*)$ such that $\|y^* - x^*\| \leq \varepsilon$).

We will say that a function $f : C \to C$ or a multifunction $F : C \twoheadrightarrow C$ has *the approximate fixed point property* if for each $\varepsilon > 0$ the function, respectively the multifunction, possesses at least one ε-fixed point.

The next two theorems can be seen as extensions of Brouwer's and Kakutani's fixed point theorems where the compactness condition for the region C is replaced by the boundedness condition and the condition that the image of a point of a multifunction is compact is also replaced by a boundedness condition. These weakenings can destroy the existence of fixed points, but we prove that still the approximate fixed point property holds.

Theorem 2.1. *Let $C \subset \mathbb{R}^m$ be a non–empty bounded and convex subset of $\mathbb{R}^m$, and let $f : C \to C$ be a continuous function. Then f has the approximate fixed point property.*

Proof. Suppose without loss of generality that $0 \in C$. Take $\varepsilon > 0$ and let $\alpha := \sup\{\|x\| \mid x \in C\} < \infty$. Take $\delta \in (0, 1)$ such that $\delta\alpha < \varepsilon$. Let D be the compact and convex subset of C, defined by $D = (1 - \delta)\overline{C}$, where $\overline{C}$ is the closure of C. Define the continuous function $g : D \to D$ by

$$g(x) = (1 - \delta)f(x) \quad \text{for each } x \in D.$$

By Brouwer's fixed point theorem, there is an $x^* \in D$ such that $g(x^*) = x^*$. This implies: $(1 - \delta)f(x^*) = x^*$, $\|f(x^*) - x^*\| = \|\delta f(x^*)\| \leq \delta\alpha < \varepsilon$. So x^* is an ε-fixed point of f. ■

Theorem 2.2. *Let $C \subset \mathbb{R}^m$ be a non–empty bounded and convex subset of $\mathbb{R}^m$, and let $F : C \twoheadrightarrow C$ be an upper semicontinuous multifunction, such that $F(x)$ is a non–empty, bounded and convex subset of C for each $x \in C$. Then F has the approximate fixed point property.*

Proof. The proof will run along the same lines as the proof of Theorem 2.1. Let $\varepsilon > 0$ and without loss of generality assume that $0 \in C$. Define, as in the proof of Theorem 2.1, α, δ and the compact and convex set D. Define the upper semicontinuous multifunction $G : D \twoheadrightarrow D$, which is non–empty, compact and convex valued, by

$$G(x) = (1 - \delta)\overline{F(x)} \quad \text{for each } x \in D,$$

where $\overline{F(x)}$ is the closure of $F(x)$.

By Kakutani's fixed point theorem, there is an $x^* \in D$ such that $x^* \in G(x^*)$. This implies that $x^* \in G(x^*) = (1-\delta)\overline{F(x)}$. So there is a $z \in \overline{F(x^*)}$ such that $x^* = (1-\delta)z$, $\|z - z^*\| = \|\delta z\| < \varepsilon$. Then there is also a $z' \in F(x^*)$ with $\|z' - x^*\| < \varepsilon$, so x^* is an ε–fixed point of F. ■

For other approximate fixed point theorems for multifunctions in a topological vector space we refer to Ref. 14 and Ref. 15.

3 Contractions and approximate fixed points

Let $< V, d >$ be a metric space. A function $f : V \to V$ is called a *contraction* with contraction factor $\beta \in [0,1)$ if for all $v, w \in V$ we have: $d(f(v), f(w)) \leq \beta d(v,w)$. The contraction theorem of Banach (Ref. 13) states that each contraction from a complete metric space into itself possesses a unique fixed point. In the next theorem we remove the completeness condition in Banach's theorem and obtain still the existence of ε–fixed points for each $\varepsilon > 0$, together with the property that the ε–fixed points are concentrated in a set with the diameter going to zero if $\varepsilon \to 0$.

Theorem 3.1. *Let $< V, d >$ be a metric space and let $f : V \to V$ be a contraction with contraction factor $\beta \in [0,1)$. Then for each $\varepsilon > 0$: $FIX_\varepsilon(f) = \{x \in V \mid d(f(x), x) \leq \varepsilon\} \neq \emptyset$ and the diameter of $FIX_\varepsilon(f)$ is not larger that $(1-\beta)^{-1}2\varepsilon$.*

PROOF. Take $\varepsilon > 0$ and $x \in V$. Consider the infinite sequence $x_0, x_1, x_2, x_3, ...$, where $x_0 = x$, $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, and so on. Then, by the contraction property $d(x_{n+1}, x_n) \leq \beta^n d(x_1, x_0)$. For n large: $d(x_{n+1}, x_n) = \beta^n d(x_1, x_0) < \varepsilon$ implying that x_n is an ε–fixed point of f. This proves that $FIX_\varepsilon(f) \neq \emptyset$. For the assertion about the diameter of $FIX_\varepsilon(f)$, take $x, y \in FIX_\varepsilon(f)$ and note that by the triangle inequality: $d(x,y) \leq d(x, f(x)) + d(y, f(y)) + d(f(x), f(y)) \leq 2\varepsilon + \beta d(x,y)$, so $d(x,y) \leq (1-\beta)^{-1}2\varepsilon$. ■

Corollary 3.1. *Suppose that the contraction $f : V \to V$ in Theorem 3.1 possesses a fixed point x^*. Then*

(i)x^* is the unique fixed point of f,

(ii)for each sequence $x_1, x_2, x_3, \ldots$ with the property that for each $n \in N$ the point x_n is an n^{-1} fixed point we have $\lim_{n\to\infty} x_n = x^*$.

PROOF. Assertion (i) is obvious and (ii) follows from the fact that $x^* \in FIX_\varepsilon(f)$ for each $\varepsilon > 0$, so by Theorem 3.1 $d(x_n, x^*) \leq \text{diam}(FIX_{n^{-1}}(f)) \leq (1-\beta)^{-1}2n^{-1}$. Hence, $\lim_{n\to\infty} d(x_n, x^*) = 0$. (Here diam$(V)$ denotes the diameter of V.) ■

4 Nonexpansive maps and approximate fixed points

In Ref 8 (Theorem 9.5, p.202) it is proved that approximate fixed points (called quasi–fixed points in the reference) exist for a function$f : C \to C$ on a non–empty, closed and convex bounded subset C of a Banach space, if f is nonexpansive i.e. if for all $a, b \in C$: $d(f(a), f(b)) \leq d(a, b)$.

We generalize this result in the next theorem.

Theorem 4.1. *Let V be a normed linear space with the norm $\|\cdot\|$. Let W be a non–empty bounded convex subset of V. Let $f : W \to W$ be a nonexpansive map. Then f has approximate fixed points.*

Proof. Suppose without loss of generality that $0 \in W$. Let $R = \sup\{\|w\| \mid w \in W\} < \infty$. Take $\varepsilon > 0$ and a $\beta \in (0, 1)$ with $(1 - \beta)R \leq \varepsilon/2$. Then $\beta f : W \to W$ is a contraction map. So, by Theorem 3.1, there is a $z \in W$ with $\|\beta f(z) - z\| < \varepsilon/2$. Then $\|f(z) - z\| \leq \|(1 - \beta)f(z)\| + \|\beta f(z) - z\| \leq (1 - \beta)R + \varepsilon/2 \leq \varepsilon$. So z is an ε–fixed point of f. ∎

For another approximate fixed point theorem for nonexpansive mappings see Ref. 16.

5 Concluding remark

It might be interesting to consider other well–known fixed point theorems and deduce from them approximate fixed point theorems by weakening the conditions in the original theorem. Also it might be interesting to look for more applications for the developed approximate fixed point theorems.

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