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A Unified Aproach to Some Results in Fixed Point Theory

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Abstract. Motivated by the existence of various extensions to multifunctions of the Brouwer fixed point theorem we have introduced in [1] the concept of fixed point property for pairs of classes of topological spaces. Continuing the study initiated in [1] we obtain in this framework, fixed point theorems, an intersection property for sets with $\mathcal{C}$-sections and quasi-equilibrium theorems.

Keywords: upper semicontinuous map, fixed point property, generalized convex space, quasi-equilibrium theorems.

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1 Preliminaries and notations

A *map* (or a *multifunction*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the values $T(x) \subset Y$. Given two maps $S : X \multimap Y$, $T : Y \multimap Z$ the *composite* $T \circ S : X \multimap Z$ is defined by $(T \circ S)x = T(Sx) = \cup\{Ty : y \in Sx\}$.

Let X and Y be topological spaces. A map $T : X \multimap Y$ is said to be *upper semicontinuous* if for each closed set $F \subset Y$ the *lower inverse* of F under T, that is $T^{-1}(F) = \{x \in X : T(x) \cap F \neq \emptyset\}$, is a closed subset of X or, equivalently, if for each open set $G \subset Y$ the *upper inverse* of G under T, that is $T^{+1}(G) = \{x \in X : T(x) \subset G\}$, is an open subset of X. Note that if Y is compact Hausdorff and $T(x)$ is closed for each $x \in X$, then T is upper semicontinuous if and only if the *graph* of T, that is $\{(x, y) \in X \times Y : y \in T(x)\}$, is closed in $X \times Y$. Recall also that the composite and the product of two (hence of a finite number of) upper semicontinuous maps are upper semicontinuous too. A map $T : X \multimap Y$ is said to be *closed* if its graph is closed in $X \times Y$.

For a class of sets $\mathcal{C}$ and a set X we shall denote by

$$\mathcal{C}(X) = \{C \in \mathcal{C} : C \subset X\} \text{ and } \mathcal{C}^*(X) = \{C \in \mathcal{C}(X) : C \neq \emptyset\}.$$

We say that a map $T : X \multimap Y$ has $\mathcal{C}$ (resp. $\mathcal{C}^*$) values if for each $x \in X$, $T(x) \in \mathcal{C}(X)$ (resp. $T(x) \in \mathcal{C}^*(X)$).

Motivated by the existence of various extensions to multifunctions of the well-known Brouwer's fixed point theorem we have introduced in [1] the concept of fixed point property for pair of classes of topological spaces defined as follows:

Definition. *We say that a pair* ($\mathcal{T}$, $\mathcal{C}$) *consisting of two classes of compact Hausdorff* *topological spaces has the fixed point property provided:*

(i) $X, Y \in \mathcal{T} \Rightarrow X \times Y \in \mathcal{T}$;

(ii) $C \in \mathcal{C}(X), D \in \mathcal{C}(Y) \Rightarrow C \times D \in \mathcal{C}(X \times Y)$, for each $X, Y \in \mathcal{T}$;

(iii) for each $X \in \mathcal{T}$, any upper semicontinuous map $T : X \multimap X$ with $\mathcal{C}^*$ values has a fixed point.

Four examples of pairs $(\mathcal{T}, \mathcal{C})$ having the fixed point property will be given in the sequel.

Example 1. Both $\mathcal{T}$ and $\mathcal{C}$ are the class of all compact convex subsets of all Hausdorff locally convex topological vector spaces. In this case condition (iii) in Definition is satisfied according to the Kakutani-Fan-Glicksberg fixed point theorem (see [3], [4], [6]).

Example 2. $\mathcal{T}$ is the class of all compact convex subsets of all Hausdorff locally convex topological vector spaces and for each $X \in \mathcal{T}$, $\mathcal{C}(X)$ consists of all compact acyclic subsets of X (recall that a topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish). The product of two acyclic sets is of course acyclic by the Kunneth formula (see [9]) and condition (iii) in Definition is satisfied according to Theorem 7 in [11].

In order to give two other examples of pairs $(\mathcal{T}, \mathcal{C})$ with the fixed point property we shall recollect some definitions introduced by Park and Kim (see [13] and [7]). For a set X we shall denote by $\langle X \rangle$ the set of all nonempty finite subsets of X.

A *generalized convex space* (or a G-convex space) (X, Γ) consists of a topological space X and a map $\Gamma : \langle X \rangle \multimap X$ such that:

(a) $A, B \in \langle X \rangle$, $A \subset B \Rightarrow \Gamma_A = \Gamma(A) \subset \Gamma_B$; and

(b) for each $A \in \langle X \rangle$ with $|A| = n + 1$ there exists a continuous function $\Phi_A : \Delta_n \to \Gamma_A$ such that $J \in \langle A \rangle$ implies $\Phi_A(\Delta_J) \subset \Gamma_J$ (here Δ_n denotes the standard n-simplex and Δ_J denotes the face of Δ_n coresponding to $J \in \langle A \rangle$).

For an (X, Γ) a subset Y of X is said to be *G-convex* if $A \in \langle Y \rangle$ implies $\Gamma_A \subset Y$.

A G-convex space (X, Γ) is called:

(a) *locally G-convex uniform space* if it satisfies the following conditions:

  (a_1) X is a Hausdorff uniform space with the basis $\mathcal{V}$; and

  (a_2) for each $V \in \mathcal{V}$ and $x \in X$ the set $\{y \in X : (x, y) \in V\}$ is G-convex.

(b) *of type II* if it is separated and satisfies the following conditions:

  (b_1) for each $x \in X$, $\{x\}$ is G-convex; and

  (b_2) for any compact G-convex subset Y of X and each open neighborhood V of Y there exists an open neighborhood U of Y such that $\cap\{Z : U \subset Z \subset X$ and Z is G-convex$\} \subset U$.

Example 3. $\mathcal{T}$ is the class of all compact locally G-convex uniform spaces and for each $X \in \mathcal{T}$, $\mathcal{C}(X)$ consists of all compact G-convex subsets of X. In this case condition (iii) in Definition is satisfied according to Lemma 4 and Theorem 4 in [12].

Example 4. $\mathcal{T}$ is the class of all compact G-convex spaces of type II and for each $X \in \mathcal{T}$, $\mathcal{C}(X)$ consists of all compact G-convex subsets of X. In this case condition (iii) in Definition is satisfied according to Theorem 2 in [7].

In the sequel let us fix a pair $(\mathcal{T}, \mathcal{C})$ of classes of compact Hausdorff topological spaces having the fixed point property. In this framework we shall obtain two fixed point theorems and as applications intersection results and quassi-equilibrium theorems.

2 Fixed points, quasi-equilibrium theorems, intersection properties

Let $\{X_i\}_{1 \le i \le n}$ be a finite family of sets $(n \ge 2)$. Let

$$X = \prod_{i=1}^{n} X_i \text{ and } X^i = \prod_{\substack{j=1 \\ j \ne i}}^{n} X_j.$$

Any $x = (x_1, x_2, \ldots, x_n) \in X$ can be expresed as $x = (x^i, x_i)$ for any $i \in \{1, 2, \ldots, n\}$, where x^i denotes the canonical projections of x on X^i.

Let us beginning with the following collectively fixed point theorem:

Theorem 1. *Let $X_i \in \mathcal{T}(1 \le i \le n)$ and for each $i \in \{1, 2, \ldots, n\}$ let $T_i : X \multimap X_i$ be an upper semicontinuous map with $\mathcal{C}^*$ values. Then there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in X$ for each $i \in \{1, 2, \ldots, n\}$.*

Proof. Define $T : X \multimap X$ by $T(x) = \prod_{i=1}^{n} T_i(x)$ for each $x \in X$. Then T is upper semicontinuous by virtue of Fan [3, Lemma 7] with $\mathcal{C}^*$ values. Consequently T has a fixed point $\widehat{x} \in X$; that is $\widehat{x} \in T(\widehat{x})$ whence $\widehat{x}_i \in T_i(\widehat{x})$ for each $i \in \{1, 2, \ldots, n\}$ ■

When the pair of topological spaces $(\mathcal{T}, \mathcal{C})$ is that which appears in Example 4, then Theorem 1 reduces to Theorem 3 in [7].

From Theorem 1 we get the following fixed point theorem for composites of upper semicontinuous maps:

Theorem 2. *Let $X_i \in \mathcal{T}(1 \le i \le n)$ and $S_i : X_i \multimap X_{i+1}(1 \le i \le n-1)$, $S_n : X_n \multimap X_1$ be upper semicontinuous maps with $\mathcal{C}^*$ values. Then the composite $S_n \circ \cdots \circ S_1$ has a fixed point.*

Proof. Let $T_i : X \multimap X_i(1 \le i \le n)$ be the maps defined by

$$T_1(x_1, x_2, \ldots, x_n) = S_n(x_n)$$

$$T_i(x_1, x_2, \ldots, x_n) = S_{i-1}(x_{i-1}) \text{ for } 2 \le i \le n.$$

By Theorem 1 there exist $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) \in X$ such that

$$\widehat{x}_1 \in T_1(\widehat{x}) = S_n(\widehat{x}_n),\ \widehat{x}_2 \in T_2(\widehat{x}) = S_1(\widehat{x}_1), \ldots,\ \widehat{x}_n \in T_n(\widehat{x}) = S_{n-1}(\widehat{x}_{n-1}),$$

whence $\widehat{x}_1 \in (S_n \circ \cdots \circ S_1)(\widehat{x}_1)$. ■

Theorem 2 generalizes results of Granas and Liu [5], Park [11], Balaj [1, 2].

As a consequence of Theorem 1 we obtain the folowing generalization of von Neumann intersection theorem [14] for sets with $\mathcal{C}^*$ sections.

Theorem 3. *For each* $i \in \{1,2,\ldots,n\}$ *let* $X_i \in \mathcal{T}$ *be a closed subset of* X *such that* $M_i(x^i) = \{x_i \in X_i : (x^i, x_i) \in M_i\} \in \mathcal{C}(X_i)$ *for each* $x^i \in X^i$. *Then* $\cap_{i=1}^n M_i = \emptyset$.

Proof. We use Theorem 1 with $T_i : X \multimap X_i$ defined by

$$T_i(x) = M_i(x^i),\ x \in X.$$

Then

$$\begin{aligned} X_i \times M_i &= \{(y_i, x) \in X_i \times X : x \in M_i\} = \\ &= \{(y_i, x^i, x_i) \in X_i \times X : x \in M_i(x^i)\} = \\ &= \{(y_i, x^i, x_i) \in X_i \times X : x_i \in T_i(y_i, x^i)\} \end{aligned}$$

which implies that T_i is a closed map with $\mathcal{C}^*$ values. Since X is compact, T_i is upper semicontinuous. Therefore, by Theorem 1, there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for all $i \in \{1,2,\ldots,n\}$. So we have $\widehat{x} = (\widehat{x}^i, \widehat{x}_i) \in M_i$, for all $i \in \{1,2,\ldots,n\}$. This completes the proof. ■

Theorem 1 can be reformulated to the form of a quasi-equilibrium theorem as follows:

Theorem 4. *For each* $i \in \{1,2,\ldots,n\}$ *let* $X_i \in \mathcal{T}$, $S_i : X \multimap X_i$ *closed map and* $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ *upper semicontinuous functions. Suppose that for each* $i \in \{1,2,\ldots,n\}$ *the following conditions are satisfied:*

(a) $g_i(x) \le f_i(x)$ *for each* $x \in X$;

(b) *the function* M_i *defined on* X *by*

$$M_i(x) = \max_{y \in S_i(x)} g_i(x^i, y)$$

is lower semicontinuous;

(c) *for each* $x \in X$

$$\{y \in S_i(x) : f_i(x^i, y) \ge M_i(x)\} \in \mathcal{C}(X).$$

There exist an $\widehat{x} \in X$ *such that for each* $i \in \{1,2,\ldots,n\}$

$$\widehat{x}_i \in S_i(\widehat{x}) \text{ and } f_i(\widehat{x}^i, \widehat{x}_i) \ge M_i(\widehat{x}).$$

Proof. For each $i \in \{1,2,\ldots,n\}$ define a map $T_i : X \multimap X_i$ by

$$T_i(x) = \{y \in S_i(x) : f_i(x^i, y) \ge M_i(x)\} \text{ for } x \in X.$$

Note that each T_i is nonempty by (i), since $S_i(x)$ is compact and $g_i(x^i, \cdot)$ is upper semicontinuous on $S_i(x)$. We show that graph T_i is closed in $X \times X_i$. In view of this let $(x_\alpha, y_\alpha) \in$ graph T_i such that $(x_\alpha, y_\alpha) \to (x, y)$. Then

$$\begin{aligned} f_i(x^i, y) &\ge \limsup_\alpha f_i(x^i_\alpha, y_\alpha) \ge \limsup_\alpha M_i(x_\alpha) \\ &\ge \liminf_\alpha M_i(x_\alpha) \ge M_i(x) \end{aligned}$$

and, since graph S_i is closed in $X \times X_i$, $y_\alpha \in S_i(x_\alpha)$ implies $y \in S_i(x)$. Hence $(x, y) \in$ graphT_i. Since X_i is compact, T_i is upper semicontinuous. By Theorem 1, there exists an $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for each $i \in \{1,2,\ldots,n\}$; that is $\widehat{x}_i \in S_i(\widehat{x})$ and $f_i(\widehat{x}^i, \widehat{x}_i) \ge$

$M_i(\widehat{x})$. This completes the proof. ■

The origin of Theorem 4 goes back to Nash equilibrium theorem [10]. From Theorem 4 we have the following quasi-equilibrium result:

Corollary 5. *Let* $X \in \mathcal{T}$, $S : X \multimap X$ *be a closed map and* $f, g : X \times X \to \mathbb{R}$ *be upper semicontinuous functions. Suppose that:*

(a) $g(x) \leq f(x)$ *for each* $x \in X$ *;*

(b) *the function* M *defined on* X *by*

$$M(x) = \max_{y \in S(x)} g(x, y) \text{ for } x \in X$$

is lower semicontinuous;

(c) *for each* $x \in X$*, the set*

$$\{y \in S(x) : f(x, y) \geq M(x)\} \in \mathcal{C}(X).$$

There exists $\widehat{x} \in X$ *such that*

$$\widehat{x} \in S(\widehat{x}) \text{ and } f(\widehat{x}, \widehat{x}) \geq M(\widehat{x}).$$

REFERENCES

[1] M. Balaj, *Pairs of classes of topological spaces with the fixed point property*, Seminar on Fixed Point Theory, "Babes-Bolyai" University, Cluj-Napoca, 3 (2002), 197–202.

[2] M. Balaj, *Fixed point theorems, section properties and minimax inequalities on K-G-convex spaces*, J. Korean Math. Soc., 39 (2002), 387–395.

[3] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. USA, 38 (1952), 121–126.

[4] I. L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. 3 (1952), 170–174.

[5] A. Granas an F.C. Liu, *Coincidences for set-valued maps and minimax inequalities*, J. Math. Pures. Appl., 65 (1986), 119–148.

[6] S. Kakutani, *A generalization of Brouwer's fixed-point theorem*, Duke Math. J. 8 (1941), 457–459.

[7] H. Kim, *Fixed point theorems on generalized convex spaces*, J. Korean. Math. Soc. 35 (1998), 491–502.

[8] M. Lassonde, *Fixed point for Kakutani factorizable multifunctions*, J. Math. Anal. Appl. 152 (1990), 46–60.

[9] W. S. Massey, *Singular Homology Theory*, Springer-Verlag, New York, 1980.

[10] J. Nash, *Non-cooperative games*, Ann. Math., 54 (1951), 286–295.

[11] S. Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM*

theory, Fixed Point Theory and Applications (K.-K. Tan, ed.), World Sci. Publ. River Edge, NJ, 1992, pp. 248–277.

[12] S. Park, *Remarks on fixed point theorems for generalized convex spaces*, Proc. Internat. Conf. on Math. Anal. Appl., (Chinju, 1998) 1-A, 1999, pp. 95–104.

[13] S. Park and H. Kim, *Admissible classes of multifunctions on generalized convex spaces*, Proc. Coll. Natur. Sci. SNU, 18 (1993), 1–21.

[14] J. von Neumann, *Uber ein okonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpuncktstzes*, Ergebn.eines Math. Kolloqn., 8 (1937), 73–83.