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On a Class of Regular Hamiltonians

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Abstract. One considers the Hamiltonians $H = \varphi(K^2)$ for K^2 a regular Hamiltonian that is homogeneous of degree 2 in momenta p and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that $\varphi'(t) \neq 0$ and $\varphi' + 2t\varphi'' \neq 0$ for every $t \in \mathrm{Im}(K^2)$. One proves that H is regular and that its nonlinear connection coincides with that of K^2. Also, the h-covariant derivative of H coincides with that of K^2. Some consequences of these facts are derived.

Keywords: regular Hamiltonians, homogeneous Hamiltonians, nonlinear connections

MSC 200: 53B60

Introduction

A regular, autonomous Lagrangian $L : TM \to \mathbb{R}$, $(x, y) \to L(x, y)$ induces a semispray (second order differential equation) on the tangent manifold TM, for M a smooth, finite dimensional manifold, [2]. If $L = F^2$ is homogeneous of degree two in y, the said semispray reduces to a spray. P.L. Antonelli and D. Hrimiuc, [1], have considered a class of Lagrangians that are not homogeneous but generate sprays, too. They showed that the Lagrangians $L = \varphi(F^2)$ for $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with some properties, are regular Lagrangians that generate the same spray as F^2. The geometry of these Lagrangians called φ-Lagrangians is interesting in many respects. Some applications of it in Ecology were pointed out.

A regular Hamiltonian $H : T^*M \to R$, $(x, p) \to H(x, p)$ no longer defines a semispray, but if $K^2(x, p)$ is an homogeneous Hamiltonian of degree 2 in p, the Hamiltonians $H = \varphi(K^2)$ deserve a special attention. It is our purpose to study the geometry of these Hamiltonians, called φ-Hamiltonians, using the facts and methods as well as the terminology from the book [2]. First, we show that in the same conditions for φ as in Lagrangian case, the Hamiltonians $H = \varphi(K^2)$ are regular. Secondly, we prove that they define the same nonlinear connection as K^2. A nonlinear connection on T^*M is a regular distribution called horizontal that is supplementary to the vertical distribution defined by the projection $\tau : T^*M \to M$. It was proved by R. Miron, [2], that any regular Hamiltonian defines a nonlinear connection. The fact that all Hamiltonians $\varphi(K^2)$ define the same nonlinear connections implies that the "horizontal part" of their geometry is very close to the geometry of K^2 (the pair (M, K^2) was called in [2] a Cartan space). As we shall see below the "vertical part" of their geometry is drastically different from that of K^2. The manifold T^*M has a natural symplectic structure.

Thus the Hamiltonian fields X_H and X_{K^2} can be considered. It is easy to see these fields are linearly dependent on T^*M. Some comments on the duality between φ-Lagrangians and φ-Hamiltonians close the paper.

1 φ-Hamiltonians

Let $(U,(x^i))$ be a local chart on M. The indices $i,j,k,...$ will range from 1 to $n=\dim M$ and the Einstein convention on summation will be in use.

The local coordinates on $\tau^{-1}(U)$ will be (x^i,p_i) and a change of coordinates $(x^i,p_i)\to(\widetilde{x}^i,\widetilde{p}_i)$ has the form

$$\widetilde{x}^i=\widetilde{x}^i(x^1,...,x^n),\ \operatorname{rank}\left(\frac{\partial\widetilde{x}^i}{\partial x^j}\right)=n,\ \widetilde{p}_i=\frac{\partial x^j}{\partial\widetilde{x}^i}p_j. \tag{1}$$

A local basis in T^*U is $\left(\partial_i:=\dfrac{\partial}{\partial x^i},\dot{\partial}^i:=\dfrac{\partial}{\partial p_i}\right)$ and its dual is (dx^i,dp_i).

A *Cartan space* is a pair $C^n=(M,K)$ such that

1°K is a positive function on T^*M, smooth i.e. C^∞ on $T^*M\setminus\{(x,0)\}$ and continuous on the rest,

2°K is positively 1-homogeneous in the momenta (p_i), that is $K(x,\lambda p)=\lambda K(x,p)$ for every $\lambda>0$,

3°The matrix with the entries $g^{ij}(x,p)=\dfrac{1}{2}\dot{\partial}^i\dot{\partial}^jK^2$ is positive defined.

The functions $(g^{ij}(x,p))$ behave like the local coefficients of a contravariant tensor field on M. One says that it defines a d-tensor field on T^*M. Here d is for "distinguished", see [2]. Let $(g_{jk}(x,p))$ be the inverse of the matrix $(g^{ij}(x,p))$. Then $(g_{jk}(x,y))$ define a a covariant d-tensor field. The functions $\overset{\circ}{C}{}^{ijk}=-\dfrac{1}{2}\dot{\partial}^kg^{ij}=-\dfrac{1}{4}\dot{\partial}^i\dot{\partial}^j\dot{\partial}^kK^2$, define a d-tensor field of type $(3,0)$. This is totally symmetric. If it vanishes then a Cartan space reduces to a Riemannian space. One puts $\overset{\circ}{p}{}^i=g^{ij}p_j$. The following formulae holds by virtue of the homogeneity of K

$$\begin{gathered}\overset{\circ}{p}{}^i=\frac{1}{2}\dot{\partial}^iK^2,\ K^2=g^{ij}p_ip_j=p_i\overset{\circ}{p}{}^i,\ g^{ij}=\dot{\partial}^j\overset{\circ}{p}{}^i,\\ \overset{\circ}{C}{}^{ijk}p_k=\overset{\circ}{C}{}^{ijk}p_j=\overset{\circ}{C}{}^{ijk}p_i=0.\end{gathered} \tag{2}$$

Let $\varphi:\mathbb{R}_+\to\mathbb{R}$ be a smooth function.

Theorem 1. *If the function φ has the property*

$$\varphi'(t)\neq0,\ \varphi'(t)+2t\varphi''(t)\neq0\ \forall t\in\operatorname{Im}(K^2), \tag{3}$$

then $H=\varphi(K^2)$ is a regular Hamiltonian.

Proof. The Hamiltonian H is regular if the matrix with the entries $a^{ij}(x,p)=\dfrac{1}{2}\dot{\partial}^i\dot{\partial}^jH$ is nondegenerate. We have $\dot{\partial}^jH=\varphi'\dot{\partial}^jK^2$ and then

$$a^{ij}(x,p) = \varphi' g^{ij}(x,p) + 2\varphi'' \overset{\circ}{p}{}^{i} \overset{\circ}{p}{}^{j}. \tag{4}$$

We look for the inverse of $(a^{ij}(x,p))$ in the form

$$g_{jk}(x,p) = \frac{1}{\varphi'}\ g_{jk} - cp_j p_k.$$

Using (2) in the condition $a^{ij}a_{jk} = \delta^i_k$ it comes out that c is uniquely determined as

$$c = \frac{2\varphi''}{\varphi'(\varphi' + 2K^2\varphi'')}. \tag{5}$$

Thus $H = \varphi(K^2)$ is regular. ■

If the quadratic form $a_{ij}(x,p)\xi^i\xi^j$, $(\xi) \in R^n$, is positive definite, the pair (M,H) is called a Hamilton space. If we put $p^i = a^{ik}p_k$, it follows

$$p^i = (\varphi' + 2\varphi'' K^2)\overset{\circ}{p}{}^{i}. \tag{6}$$

We note that if $\varphi' + 2\varphi'' K^2 = 0$, that is $\varphi'(t) = a \exp\left(-\int_0^t \frac{d\theta}{2\theta}\right)$, $a \in \mathbb{R}$, then $p^i = 0$ implies $a^{ik}p_k = 0$. Thus the matrix (a^{ik}) is singular.

The following examples of functions φ verifying (3) are taken from [1]

1.$\varphi(t) = at^m + b,\ t > 0,\ m \in \mathbb{R} \setminus \left\{0, \frac{1}{2}\right\}$

2.$\varphi(t) = ae^t + b,$

3.$\varphi(t) = a \ln t + b,\ t > 0,\ a, b \in \mathbb{R},\ a \neq 0.$

We note that the Hamiltonians K^2 and aK^2+b, $a, b \in \mathbb{R}$, $a \neq 0$, have the same geometry. Thus in the above examples we may take $a = 1$ and $b = 0$ with no lose of generality.

A converse of Theorem 1 is as follows.

Theorem 1'. *Let H be a regular Hamiltonian and $\psi : R \to R_+$ a smooth function such that $\psi'(t) \neq 0$ and $\psi'' \neq 0$ for every $t \in \mathrm{Im}(H)$. If the function $K(x,p) = \psi(H(x,p))$ is positively homogeneous of degree 1 in p, then K^2 is a regular Hamiltonian, that is the pair (M,K) is a Cartan space.*

PROOF. We have $\dot\partial^i K = \psi' \dot\partial^i H$. Contracting by p_i we get

$$p_i \dot\partial^i H = \frac{\psi}{\psi'} \tag{$*$}$$

using the Euler theorem on homogeneous function. Then we take $\dot\partial^j \dot\partial^i K = \psi'' \dot\partial^j H \dot\partial^i H + \psi' \dot\partial^j \dot\partial^i H$ and contract again by p_i. Since $\dot\partial^j K$ is homogeneous of degree 0 one gets

$$\frac{\psi''\psi}{\psi'}\dot\partial^j H + 2\psi' a^{ji}p_i = 0, \text{ where } a^{ij} = \frac{1}{2}\dot\partial^i \dot\partial^j H.$$

If we put $z_i = \dfrac{p_i}{\psi}$, it follows

$$a^{ji}z_i = -\frac{\psi''}{2\psi'^2}\dot\partial^j H,\ a^{ji}z_j z_i = -\frac{\psi''}{2\psi'^3}. \tag{$**$}$$

We compute $g^{ij} = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j K^2$ and we find $g^{ij} = 2\psi\psi' a^{ij} + (\psi''\psi + {\psi'}^2)\dot{\partial}^i H \dot{\partial}^j H$. We look for its inverse in the form $g_{jk} = \frac{1}{2\psi\psi'}\, a_{jk} - dz_j z_k$, for a function d to be determined.

The condition $g^{ij} g_k = \delta^i_k$ gives, using $(*)$ and $(**)$ $d = -\frac{\psi''\psi + {\psi'}^2}{\psi\psi''}$ and so the proof is complete. ■

2 Nonlinear connection of a φ-Hamiltonian

Let $D_u\tau : T_u T^*M \to T_{\tau(u)}M$, $u \in T^*M$, be the differential of the projection τ. Then $V : u \to \operatorname{Ker} D_u\tau := V_u$ is a regular distribution on T^*M called vertical distribution. It is locally spanned by $(\dot{\partial}^i)$, hence it is integrable. A *nonlinear connection* is a regular distribution $N : u \to N_u$ called horizontal such that

$$T_u T^*M = N_u \oplus V_u \text{ (direct sum) for every } u \in T^*M. \tag{7}$$

It is usual to take N as being locally spanned by the local d-vector fields

$$\delta_i = \partial_i + N_{ik}\dot{\partial}^k, \tag{8}$$

where the functions $(N_{ik}(x,p))$ are called the local coefficients of the given nonlinear connection. Conversely, $(N_{ik}(x,p))$ with a convenient behavior by a change of local coordinates on T^*M determines the nonlinear connection N. Now it is well known due to R. Miron that any regular Hamiltonian determines a nonlinear connection. He found ([2]) the coefficients $(N_{ij}(x,p))$ in the form

$$4N_{ij} = \{a_{ij}, H\} - (a_{ik}\dot{\partial}^k(\delta_j H) + a_{jk}\dot{\partial}^k(\partial_i H)), \tag{9}$$

where $\{\ ,\ \}$ denotes the usual Poisson bracket. Note that $N_{ij} = N_{ji}$.

Let be $H = \varphi(K^2)$ a φ-Hamiltonian. If we denote by $(\overset{\circ}{N}_{ij})$ the coefficients of the nonlinear connection derived from K^2, obviously we have

$$4\overset{\circ}{N}_{ij} = \{g_{ij}, K^2\} - (g_{ik}\dot{\partial}^k(\partial_j K^2) + g_{jk}\dot{\partial}^k(\partial_i K^2)). \tag{10}$$

Theorem 2. *The Hamiltonians K^2 and $\varphi(K^2)$ induce the same nonlinear connection on T^*M, that is,*

$$N_{ij}(x,y) = \overset{\circ}{N}_{ij}(x,p), \text{ for every } \varphi \text{ satisfying (3).} \tag{11}$$

Proof. By a calculation as follows.

$$\begin{aligned}
&\{a_{ij}, H\} := (\dot{\partial}^k a_{ij})(\partial_k H) - (\dot{\partial}^k H)(\partial_k a_{ij}) \overset{(2,4)}{=\!=\!=} \\
&= \dot{\partial}^k\left(\frac{1}{\varphi'}\, g_{ij} - c p_i p_j\right)\varphi'(\partial_k K^2) - 2\varphi' \overset{\circ}{p}{}^k \partial_k \left(\frac{1}{\varphi'}\, g_{ij} - c p_i p_j\right) = \\
&= (\dot{\partial}^k g_{ij})(\partial_k K^2) - (\dot{\partial}^k K^2)(\partial_k g_{ij}) - c\varphi'(p_j \partial_i K^2 + p_i \partial_j K^2),
\end{aligned}$$

since the other four terms cancel each other. Thus we get

$$\{a_{ij}, H\} = \{g_{ij}, K^2\} - c\varphi'(p_j \partial_i K^2 + p_i \partial_j K^2). \tag{12}$$

Then we have

$$a_{ik}\dot{\partial}^k(\partial_j H) = a_{ik}(\dot{\partial}^k(\varphi'\partial_j K^2)) = a_{ik}(2\varphi''\overset{\circ}{p}{}^k\partial_j K^2 + \varphi'\dot{\partial}^k\partial_j K^2) =$$
$$= \left(\frac{1}{\varphi'}\, g_{ik} - cp_ip_k\right)(2\varphi''\overset{\circ}{p}{}^k\partial_j K^2 + \varphi'\dot{\partial}^k\partial_j K^2) =$$
$$= g_{ik}\dot{\partial}^k(\partial_j K^2) + 2\,\frac{\varphi''}{\varphi'}\, p_i\partial_j K^2 - 2c\varphi'' K^2 p_i\partial_j K^2 - 2c\varphi' p_i\partial_j K^2.$$

The last term takes the given form because of Euler Theorem applied to the 2–homogeneous function in p, $\partial_j K^2$. It results

$$a_{ik}\dot{\partial}^k(\partial_j H) = g_{ik}\dot{\partial}^k(\partial_j K^2) - \varphi' cp_i\partial_j K^2.$$

Thus the second term in (N_{ij}) from (9) is

$$\begin{aligned} a_{ik}\dot{\partial}^k(\partial_j H) + a_{jk}\dot{\partial}^k(\partial_i H) = g_{ik}\dot{\partial}^k(\partial_j K^2) + g_{jk}\dot{\partial}^k(\partial_i K^2) - \\ - c\varphi'(p_i\partial_j K^2 + p_j\partial_i K^2). \end{aligned} \tag{13}$$

Using (12) and (13) in (9) one gets $N_{ij} = \overset{\circ}{N}{}_{ij}$. ■

3 Canonical metrical N-linear connection of a φ-Hamiltonian

A linear connection D on T^*M is called an N-linear connection if: i) the distributions N and V are preserved by parallel translations defined by D and ii) the natural symplectic structure $\theta = dp_i \wedge dx^i$ is absolute parallel with respect to D, that is the skew–symmetric tensor field $\widetilde{\theta}$ defining θ satisfies $D\widetilde{\theta} = 0$.

The decomposition (7) produces decompositions of vector fields of tensor fields and of an N–linear connection. For vector fields we shall write $X = X^H + X^V$ and

$$D^H_X = D^{X^H}, \quad D^V_X = D_{X^V}, \tag{14}$$

are two operators in algebra of d-tensor fields on T^*M called h- and v-covariant derivations.

In the basis $(\delta_i, \dot{\partial}^i)$ that is adapted to the decomposition (7) an N-linear connection has the form

$$\begin{aligned} D_{\delta_j}\delta_i = H^k_{ij}\delta_k, \quad D_{\delta_j}\dot{\partial}^i = -H^i_{kj}\dot{\partial}^k, \\ D_{\dot{\partial}^j}\delta_i = C_i^{kh}\delta_h, \quad D_{\dot{\partial}^j}\dot{\partial}^i = -C_h^{ij}\dot{\partial}^h, \end{aligned} \tag{15}$$

where the functions (C_i^{kh}) define a d-tensor field and (H^k_{ij}) behave like the coefficients of a linear connection on M by a change of local coordinates on T^*M.

The h-covariant derivative denoted by $_{|k}$ is as follows

$$\begin{aligned} X^i_{|k} = \delta_k X^i + H^i_{sk}X^s,\ \omega_{j|k} = \delta_k\omega_j - H^s_{jk}\omega_s, \\ g_{ij|k} = \delta_k g_{ij} - H^s_{ik}g_{sj} - H^s_{jk}g_{is} \text{ and so on.} \end{aligned} \tag{16}$$

The v-covariant derivative denoted by $\big|^k$ has the form

$$\begin{aligned} X^i\big|^k = \dot{\partial}^k X^i + C_h^{ik}X^h,\ \omega_j\big|^k = \dot{\partial}^k\omega_j - C_j^{sk}\omega_s, \\ g^{ij}\big|^k = \dot{\partial}^k g^{ij} + C_s^{ik}g^{sj} + C_s^{jk}g^{is} \text{ and so on.} \end{aligned} \tag{17}$$

Let H be any Hamiltonian, N its nonlinear connection and $a^{ij} = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j H$. An N-linear connection is called metrical if $a^{ij}{}_{|k} = 0$ and $a^{ij}\big|^k = 0$. One proves that there exists an unique N-linear connection i.e. $D\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ that is metrical and symmetrical i.e. $H^i_{jk} = H^i_{kj}$, $C^{jk}_i = C^{kj}_i$. It is given by

$$\begin{aligned} H^i_{jk} &= \frac{1}{2}(\delta_j a_{hk} + \delta_k a_{jh} - \delta_h a_{jk}), \\ C^{jk}_i &= -\frac{1}{2}\, a_{ih}(\dot{\partial}^j a^{hk} + \dot{\partial}^k a^{jh} - \dot{\partial}^h a^{jk}) = -\frac{1}{2} a_{ih}\dot{\partial}^j a^{hk}. \end{aligned} \tag{18}$$

Let be now $H = \varphi(K^2)$ a φ-Hamiltonian. We have N and $D\Gamma(N)$ with H^i_{jk} and C^{jk}_i given by (18).

Similarly, for the Hamiltonian K^2 we have the nonlinear connection $\overset{\circ}{N} = N$ by Theorem 2 and the N-linear connection $D\overset{\circ}{\Gamma}(N) = (\overset{\circ}{H}{}^i_{jk}, \overset{\circ}{C}{}^{jk}_i)$, where these coefficients are obtained from (18) replacing (a^{ij}) by (g^{ij}) and (a_{ij}) by (g_{ij}). We are interested to express $D\Gamma(N)$ using $D\overset{\circ}{\Gamma}(N)$. We prove

Theorem 3. *The canonical metrical N-linear connection $D\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ of a φ-Hamiltonian $H = \varphi(K^2)$ is given as follows*

$$H^i_{jk} \;=\; \overset{\circ}{H}{}^i_{jk}, \tag{19}$$

$$\begin{aligned} C^{jk}_i \;=\; & \overset{\circ}{C}{}^{jk}_i - \frac{\varphi''}{\varphi'}(\overset{\circ}{p}{}^j\delta^k_i + \overset{\circ}{p}{}^k\delta^j_i) - \frac{\varphi''}{\varphi' + 2K^2\varphi''}\, p_i g^{jk} - \\ & -2\,\frac{\varphi''\varphi' - 2\varphi''^2}{\varphi'(\varphi' + 2K^2\varphi'')}\, p_i\overset{\circ}{p}{}^j\overset{\circ}{p}{}^k. \end{aligned} \tag{20}$$

Proof. We recall from [2, Prop.4.3, Ch.6] the following properties of the pair (M, K^2).

$$\begin{aligned} K^2_{\overset{\circ}{|}j} = 0,\ K^2\overset{\circ}{\big|}{}^j = 2p^i,\ p_{i\overset{\circ}{|}j} = 0,\ p_i\overset{\circ}{\big|}{}^j = \delta^j_i, \\ \overset{\circ}{p}{}^i{}_{\overset{\circ}{|}j} = 0,\ \overset{\circ}{p}{}^i\overset{\circ}{\big|}{}^j = g^{ij}, \end{aligned} \tag{21}$$

where $\overset{\circ}{|}k$ and $\overset{\circ}{\big|}{}^k$ denotes the h- and v-derivatives with respect to $D\overset{\circ}{\Gamma}(N)$. From $p_{i\overset{\circ}{|}j} = 0$ it results $\delta_j p_i = H^k_{ij}p_k$ and $K^2_{\overset{\circ}{|}j} = 0$ implies $(f(K^2))_{\overset{\circ}{|}j} = 0$ for any smooth function f. Then we have

$$\delta_j a_{hk} = \frac{1}{\varphi'}\ \delta_j g_{hk} - cp_s(H^s_{hj}p_k + H^s_{kj}p_h). \tag{22}$$

Using (22) the second factor in H^i_{jk}, denoted by G_{hjk} takes the form

$$G_{jkh} = \frac{1}{\varphi'}\ \overset{\circ}{G}{}_{jkh} - 2cp_h p_s H^s_{jk}, \tag{23}$$

where $\overset{\circ}{G}{}_{jhk}$ is the similar factor from $\overset{\circ}{H}{}^i_{jk}$.

We contract (23) by a^{ih} given by (4). After some calculation we find $H^i_{jk} = \overset{\circ}{H}{}^i_{jk} + \phi \overset{\circ}{p}{}^i p_s \overset{\circ}{H}{}^s_{jk}$, where ϕ is a smooth function of K^2. By some algebra it results $\phi = 0$, that is (19) holds.

In order to prove (20) we show first that

$$\dot{\partial}^j a^{hk} = \varphi' \dot{\partial}^j g^{hk} + 2\varphi''(\overset{\circ}{p}{}^j g^{hk} + \overset{\circ}{p}{}^k g^{hj} + g^{jk} \overset{\circ}{p}{}^h) + 4\varphi''' \overset{\circ}{p}{}^k \overset{\circ}{p}{}^j \overset{\circ}{p}{}^h.$$

Then contracting this with a_{ih}, after some calculations one finds (20). ■

Corollary 1. *The h-covariant derivative of* $D\Gamma(N)$ *coincides with the h-covariant derivative of* $D\overset{\circ}{\Gamma}(N)$.

Indeed, this follows from $N = \overset{\circ}{N}$ and $H^i_{jk} = \overset{\circ}{H}{}^i_{jk}$. The same coincidences lead to

$$H_{|j} = 0, \quad p_{i|j} = 0.$$

4 Some final remarks

As T^*M is a symplectic manifold, we may associate to the functions H and K^2 the Hamiltonian vector fields

$$X_H = (\dot{\partial}^i H)\partial_i - (\partial_i H)\dot{\partial}^i, \ X_{K^2} = (\dot{\partial}^i K^2)\partial_i - (\partial_i K^2)\dot{\partial}^i.$$

By $\dot{\partial}^i H = \varphi'(\dot{\partial} K^2)$ and $\partial_i H = \varphi'(\partial_i K^2)$ it results

$$X_H(x,p) = H(x,p) X_{K^2}, \ (x,p) \in T^*M. \tag{24}$$

As H is a first integral of the dynamical system defined by X_H, that is, $X_H(H) = 0$, it follows

$$[X_H, X_{K^2}] = 0. \tag{25}$$

Thus, if $\{\psi^H_t\}$ the 1-parameter groups defined by H and K^2, respectively, we have

$$\psi^H_t \circ \psi^{K^2}_t = \psi^{K^2}_t \circ \psi^H_t, \text{ for } t \text{ in an open subset of } \mathbb{R}. \tag{26}$$

We notice that all functions $H = \varphi(K^2)$ are first integrals for X_{K^2}, that is $X_{K^2} H = 0$. This follows also from

$$\{K^2, H = \varphi\left(K^2\right)\} = 0. \tag{27}$$

The regular Hamiltonian K^2 defines a Legendre transformation $\mathcal{L}_{K^2} : D^* \to D$, for D^* an open set in T^*M and D an open set in TM, given by

$$(x^i, p_i) \to \left(x^i, \frac{1}{2}\, \dot{\partial}^i K^2\right), \tag{28}$$

as well as a regular Lagrangian $F^2(x,y) = K^2(x,p)$, for (x,p) given by $\mathcal{L}^{-1}_{K^2}(x,y)$, $(x,y) \in D$.

Of course, given a regular Lagrangian F^2, for F a fundamental Finsler function, we have a Legendre map $\mathcal{L}_{F^2} : D \to D^*$ given by

$$(x^i, y^i) \to \left(x^i, p_i = \frac{1}{2}\, \dot{\partial}_i F^2\right), \ \left(\dot{\partial}_i = \frac{\partial}{\partial y^i}\right) \tag{29}$$

and a regular hamiltonian $K^2(x,p) = F^2(x,y)$ with (x,y) given by $\mathcal{L}^{-1}_{F^2}(x,p)$, is obtained.

The Hamiltonian $H = \varphi(K^2)$ induces a regular Lagrangian $L = H \circ \mathcal{L}_{K^2}^{-1}$ but generally we have not $L = \varphi(F^2)$. Thus the techniques from Ch. 7 in [2] involving the Legendre transformation can not be used in the study of φ - Hamiltonians.

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