

On Solutions of Equations with Jumping Nonlinearities

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Abstract. We discuss the existence of solutions of the equation $Lu = \alpha u^+ - \beta u^- + g(x, u)$ where $u^\pm = \max(\pm u, 0)$ and g is a caratheodory function.

1 Introduction

In this work, we look for existence of solutions of the equation

$$Lu = N(x, u) = \alpha u^+ - \beta u^- + g(x, u) \text{ in } \Omega \tag{1}$$

where Ω is an open bounded set of \mathbb{R}^n . We suppose that $L : D(L) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is a selfadjoint operator with compact resolvent. Hence $D(L)$ is dense in $L^2(\Omega)$, L is closed and the injection of $(D(L), \| \cdot \|_{D(L)})$ in $(L^2(\Omega), \| \cdot \|)$ is compact. We denote by $\|u\|^2 = \int_{\Omega} u^2$, $\langle u, v \rangle = \int_{\Omega} uv$ and $\|u\|_{D(L)}^2 = \|u\|^2 + \|Lu\|^2$. Let $(\varphi_k)_k$ be the countable set of eigenfunctions of L such that $\|\varphi_k\| = 1$, this set is an orthonormal basis of $L^2(\Omega)$. We consider a simple eigenvalue λ and we denote by φ the corresponding eigenfunction and by $I =]\lambda_p, \lambda_g[$ where

$$\begin{aligned} \lambda_p &= \max \{ \tau \in \sigma(L) / \tau < \lambda \} \\ \lambda_g &= \min \{ \tau \in \sigma(L) / \tau > \lambda \} \end{aligned}$$

where $\sigma(L)$ is the spectrum of L . We suppose that $(\alpha, \beta) \in I \times I$ and that the function $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function (i.e continuous on \mathbb{R} and measurable on Ω). We denote by $V_1 = Vect \{ \dots, \lambda_p \}$, $V = \mathbb{R}\varphi$, $V_2 = Vect \{ \lambda_g, \dots \}$ and $G(x, t) = \int_0^t g(x, s) ds$.

When the function g doesn't depend on x , the equation $Lu = N(u)$ was studied by many authors and in particular N.P.Cac [1] and T.Gallouët and O.Kavian [3]. In [3], the authors suppose that N is differentiable at the origin and verifies the stronger condition $\lambda_p < \xi_p \leq [N(s) - N(t)] / [s - t] \leq \xi_g < \lambda_g^*$ for some real ξ_p, ξ_g . They show then (by using the Liapunov-Schmidt method) that a function u is a solution of (1) if and only if $\int u\varphi$ is a solution of a simple equation in \mathbb{R} . In the case of the Laplacian operator, N.P.Cac requires lower conditions (N may be not differentiable at the origin and doesn't verify the assumption *) and he shows that the equation $Lu = N(u)$ has at least one nontrivial solution.

In the third section of this paper, we consider a self adjoint operator with compact resolvent and we show that, under some hypotheses on g (which depend on the couple (α, β) position toward the Fucik spectrum), equation (1) has at least one solution.

First, we prove that if $C(\alpha, \beta) > 0$ (the mapping C was introduced by T.Gallouët and O.Kavian [3]) and :

- $\frac{\partial g}{\partial s}(x, 0)$ exists, $g(x, 0) = 0$.
- $\sup_{s \in \mathbb{R}} |g(x, s)| \in L^2(\Omega)$.
- There exists $\xi \in \mathbb{R}$ such that $(N(., s) - N(., t))/(s - t) \geq \xi > \lambda_p$ for each $(s, t) \in \mathbb{R}^2$.
- For each real s , we have $N(., s)/s \leq \lambda_g$ a.e. in Ω .
- There exists $k \in L^1(\Omega)$ such that for each real t , we have $G(x, t) \leq k(x)$ a.e. in Ω .
- $\lambda - \alpha \int (m + \varphi)^+ \varphi + \beta \int (m + \varphi)^- \varphi - \int \frac{\partial g}{\partial s}(., 0)(m + \varphi) \varphi < 0$ where m is the unique element of V_1 such that

$$Lm = P_{V_1} \left[\alpha(m + \varphi)^+ - \beta(m + \varphi)^- + \frac{\partial g}{\partial s}(., 0)(m + \varphi) \right]$$

then (1) has at least one nontrivial solution u with $\int u \varphi > 0$.

When (α, β) is not in the region between the Fučík curves, it is possible to allow greater freedom to the function g .

We show then that if $C(\alpha, \beta) * C(\beta, \alpha) > 0$ and if $|g(x, s)| \leq h^{1-\zeta}(x) |s|^\zeta + l(x)$ where $(h, l) \in L^2(\Omega) \times L^2(\Omega)$ and $0 \leq \zeta < 1$ then (1) has at least one solution.

After that, we give some remarks about the solutions of the equation $Lu = a(x)u^+ - b(x)u^-$ and we use them for proving that if $g(x, s) = h(x, s)s + l(x, s)$ and under the assumptions :

- $\sup_{s \in \mathbb{R}} |l(x, s)| \in L^2(\Omega)$.
- There exists $\eta > 0$ such that

$$\begin{aligned} \lambda_p + \eta &\leq \alpha + \inf_{s \geq 0} h(x, s) \leq \bar{\alpha} + \sup_{s \geq 0} h(x, s) \leq \alpha_0 \text{ a.e. in } \Omega \\ \lambda_p + \eta &\leq \beta + \inf_{s \leq 0} h(x, s) \leq \beta + \sup_{s \leq 0} h(x, s) \leq \beta_0 \text{ a.e. in } \Omega \end{aligned}$$

we have

a - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) = 0$ and $((meas(I_1) \neq 0$ or $meas(I_2) \neq 0)$ and $(meas(J_1) \neq 0$ or $meas(J_2) \neq 0)$), then (1) has at least one solution.

b - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) > 0$ and $(meas(I_1) \neq 0$ or $meas(I_2) \neq 0)$, then (1) has at least one solution.

c - If $C(\alpha_0, \beta_0) > 0$, $C(\beta_0, \alpha_0) = 0$ and $(meas(J_1) \neq 0$ or $meas(J_2) \neq 0)$, then (1) has at least one solution. Where

$$\begin{aligned} I_1 &= \left\{ x \in \Omega / w_1(x) > 0 \text{ and } \alpha + \sup_{s \geq 0} h(x, s) < \alpha_0 \right\} \\ I_2 &= \left\{ x \in \Omega / w_1(x) < 0 \text{ and } \beta + \sup_{s \leq 0} h(x, s) < \beta_0 \right\} \\ J_1 &= \left\{ x \in \Omega / w_2(x) < 0 \text{ and } \alpha + \sup_{s \geq 0} h(x, s) < \alpha_0 \right\} \end{aligned}$$

$$J_2 = \left\{ x \in \Omega / w_2(x) > 0 \text{ and } \beta + \sup_{s \leq 0} h(x, s) < \beta_0 \right\}$$

2 Preliminaries

Theorem 1. [3] *Let $q(x, s) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous on \mathbb{R} and measurable on Ω such that $q(x, 0) = 0$. Suppose in addition that*

$$\exists (\alpha, \beta) \in I \times I, \forall (s, t) \in \mathbb{R} \times \mathbb{R}, \alpha \leq \frac{q(x, s) - q(x, t)}{s - t} \leq \beta$$

then for each $s \in \mathbb{R}$ and $f \in L^2(\Omega)$ such that $\int f \varphi = 0$, the equation $Lv = P_{\mathbb{R}\varphi^\perp}(q(\cdot, v + s\varphi)) + f$ admits a unique solution v in $D(L)$ with $\int v \varphi = 0$. ($P_{\mathbb{R}\varphi^\perp}$: orthogonal projection on $(\mathbb{R}\varphi)^\perp$).

Theorem 2. [3] *For each $(\alpha, \beta) \in I \times I$, there exists a unique $(u, C(\alpha, \beta)) \in D(L) \times \mathbb{R}$ such that*

$$\begin{cases} Lu &= \alpha u^+ - \beta u^- + C(\alpha, \beta)\varphi \\ \int u \varphi &= 1 \end{cases}$$

the function $C : I \times I \rightarrow \mathbb{R}$ has the following properties :

- 1- $C(\alpha, \alpha) = \lambda - \alpha$.
- 2- If $\varphi^- \equiv 0$, then $C(\alpha, \beta) = \lambda - \alpha$.
- 3- If $\|\varphi^-\| \neq 0$ and $\|\varphi^+\| \neq 0$, then C is decreasing in each variable.

It was shown in [3] that the Fucik spectrum Σ (i.e. the set of $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that the equation $Lu = \alpha u^+ - \beta u^-$ has nontrivial solution in $D(L)$) consists in $I \times I$ of two nonincreasing curves passing through the point (λ, λ) and a couple $(\alpha, \beta) \in I \times I$ belongs to Σ if and only if $C(\alpha, \beta) * C(\beta, \alpha) = 0$.

3 Existence Results

Theorem 3. *Suppose that*

A1 : $\frac{\partial g}{\partial s}(x, 0)$ exists, $g(x, 0) = 0$.

A2 : $\sup_{s \in \mathbb{R}} |g(x, s)| \in L^2(\Omega)$.

A3 : There exists $\xi \in \mathbb{R}$ such that $(N(\cdot, s) - N(\cdot, t))/(s - t) \geq \xi > \lambda_p$ for each $(s, t) \in \mathbb{R}^2$.

A4 : For each real s , we have $N(\cdot, s)/s \leq \lambda_g$ a.e. in Ω .

A5 : There exists $k \in L^1(\Omega)$ such that for each real t , we have $G(x, t) \leq k(x)$ a.e. in Ω .

A6 : $\lambda - \alpha \int (m + \varphi)^+ \varphi + \beta \int (m + \varphi)^- \varphi - \int \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \varphi < 0$ where m is the unique element of V_1 such that

$$Lm = P_{V_1} \left[\alpha(m + \varphi)^+ - \beta(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right]$$

If in addition $C(\alpha, \beta) > 0$, then (1) has at least one solution u with $\int u \varphi > 0$.

Theorem 4. *Suppose that A1, A2, A3, A4, A5 are verified. If in addition A7 : $\lambda - \beta \int (m + \varphi)^+ \varphi + \alpha \int (m + \varphi)^- \varphi - \int \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \varphi < 0$ where m is the unique element of V_1 such that*

$$Lm = P_{V_1} \left[\beta(m + \varphi)^+ - \alpha(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right]$$

and $C(\beta, \alpha) > 0$, then (1) has at least one solution u with $\int u \varphi < 0$.

The proof of theorem 3 needs the following results. Consider the mapping $J : D(L) \rightarrow \mathbb{R}$ defined by $J(u) = \frac{1}{2} \langle Lu, u \rangle - \frac{\alpha}{2} \int (u^+)^2 - \frac{\beta}{2} \int (u^-)^2 - \int_{\Omega} G(\cdot, u)$, J is differentiable on $D(L)$ and [6]

$$(DJ(u), h) = \langle Lu, h \rangle - \alpha \int u^+ h + \beta \int u^- h - \int g(\cdot, u) h$$

Let $K : V_1 \times (V + V_2) \rightarrow \mathbb{R}$ be such that $K(v_1, w) = J(v_1 + w)$.

Proposition 5. *Under the assumptions of theorem 3, for each $w \in V + V_2$, there exists a unique $v_1^0 \in V_1$ such that $K(v_1^0, w) = \max_{v_1 \in V_1} K(v_1, w)$.*

PROOF. Let $w \in V + V_2$.

a - The function $K(\cdot, w)$ is strictly concave on V_1 . Let $v_1^0 \in V_1$ and $v_1^1 \in V_1$ such that $v_1^0 \neq v_1^1$, we have :

$$\begin{aligned} \left\langle \frac{\partial K}{\partial v_1}(v_1^0, w) - \frac{\partial K}{\partial v_1}(v_1^1, w), v_1^0 - v_1^1 \right\rangle &= \langle L(v_1^0 - v_1^1), v_1^0 - v_1^1 \rangle \\ &\quad - \alpha \int [(v_1^0 + w)^+ - (v_1^1 + w)^+] (v_1^0 - v_1^1) \\ &\quad + \beta \int [(v_1^0 + w)^- - (v_1^1 + w)^-] (v_1^0 - v_1^1) \\ &\quad - \int [g(\cdot, v_1^0 + w) - g(\cdot, v_1^1 + w)] (v_1^0 - v_1^1) \end{aligned}$$

so

$$\begin{aligned} &\left\langle \frac{\partial K}{\partial v_1}(v_1^0, w) - \frac{\partial K}{\partial v_1}(v_1^1, w), v_1^0 - v_1^1 \right\rangle \\ &= \langle L(v_1^0 - v_1^1), v_1^0 - v_1^1 \rangle - \int [N(\cdot, v_1^0 + w) - N(\cdot, v_1^1 + w)] (v_1^0 - v_1^1) \\ &\leq \int [\lambda_p - (N(\cdot, v_1^0 + w) - N(\cdot, v_1^1 + w)) / (v_1^0 - v_1^1)] (v_1^0 - v_1^1)^2 \end{aligned}$$

we conclude by A3 that

$$\left\langle \frac{\partial K}{\partial v_1}(v_1^0, w) - \frac{\partial K}{\partial v_1}(v_1^1, w), v_1^0 - v_1^1 \right\rangle \leq (\lambda_p - \xi) \int (v_1^0 - v_1^1)^2 < 0$$

b - Let $B_{1n} = \{v_1 \in V_1 / \|v_1\|_{D(L)} \leq n\}$. Since $K(\cdot, w)$ is concave, there exists $v_{1n} \in B_{1n}$ such that $K(v_{1n}, w) = \max_{v_1 \in B_{1n}} K(v_1, w)$. So

$$K(v_{1n}, w) \geq K(0, w)$$

which is equivalent to

$$\begin{aligned} & \langle Lv_{1n}, v_{1n} \rangle + \langle Lw, w \rangle - \alpha \int ((v_{1n} + w)^+)^2 - \beta \int ((v_{1n} + w)^-)^2 \\ & - 2 \int G(\cdot, v_{1n} + w) \geq \langle Lw, w \rangle - \alpha \int (w^+)^2 - \beta \int (w^-)^2 - 2 \int G(\cdot, w) \end{aligned}$$

so

$$\begin{aligned} & \lambda_p \|v_{1n}\|^2 - \min(\alpha, \beta) \int (v_{1n} + w)^2 + \max(\alpha, \beta) \int w^2 \\ & \geq 2 \int (G(\cdot, v_{1n} + w) - G(\cdot, w)) \\ & \geq -2 \int \sup_{s \in \mathbb{R}} |g(x, s)| |v_{1n}| \\ & \geq - \int \left[\frac{1}{\varepsilon^2} \sup_{s \in \mathbb{R}}^2 |g(x, s)| + \varepsilon^2 v_{1n}^2 \right] \end{aligned}$$

for some $\varepsilon > 0$ very small. Thus

$$(\min(\alpha, \beta) - \lambda_p - \varepsilon^2) \|v_{1n}\|^2 \leq (\max(\alpha, \beta) - \min(\alpha, \beta)) \int w^2 + \frac{1}{\varepsilon^2} \int \sup_{s \in \mathbb{R}}^2 |g(x, s)|$$

the sequence (v_{1n}) is then bounded in $L^2(\Omega)$.

c - In the other hand, v_{1n} verifies

$$\left(\frac{\partial K}{\partial v_1}(v_{1n}, w), v_1 - v_{1n} \right) \leq 0 \text{ for each } v_1 \in B_{1n}$$

Let $a_i = \langle v_{1n}, \varphi_i \rangle$ where $\varphi_i \in V_1$ is an eigenfunction of L and consider the element $v_1 = v_{1n} - a_i \varphi_i$ which belongs to B_{1n} . So we have :

$$\left(\frac{\partial K}{\partial v_1}(v_{1n}, w), -a_i \varphi_i \right) \leq 0$$

which is equivalent to

$$\langle Lv_{1n}, a_i \varphi_i \rangle - \alpha \int (v_{1n} + w)^+ a_i \varphi_i + \beta \int (v_{1n} + w)^- a_i \varphi_i - \int g(\cdot, v_{1n} + w) a_i \varphi_i \geq 0$$

hence

$$\begin{aligned} & a_i^2 \lambda_i - \alpha \int (v_{1n} + w)^+ a_i \varphi_i + \beta \int (v_{1n} + w)^- a_i \varphi_i \\ & - \int g(\cdot, v_{1n} + w) a_i \varphi_i \geq 0 \end{aligned} \tag{2}$$

multiplying (2) by $(\lambda_i - \lambda)$ which is negative, we obtain :

$$\begin{aligned} & a_i^2 \lambda_i (\lambda_i - \lambda) - \alpha \int (v_{1n} + w)^+ a_i (\lambda_i - \lambda) \varphi_i + \beta \int (v_{1n} + w)^- a_i (\lambda_i - \lambda) \varphi_i \\ & - \int g(\cdot, v_{1n} + w) a_i (\lambda_i - \lambda) \varphi_i \leq 0 \end{aligned}$$

thus

$$\begin{aligned} a_i^2(\lambda_i - \lambda)^2 - \alpha \int (v_{1n} + w)^+ a_i(\lambda_i - \lambda) \varphi_i + \beta \int (v_{1n} + w)^- a_i(\lambda_i - \lambda) \varphi_i \\ - \int g(\cdot, v_{1n} + w) a_i(\lambda_i - \lambda) \varphi_i + a_i^2 \lambda (\lambda_i - \lambda) \leq 0 \end{aligned} \quad (3)$$

By adding the precedent equation over i such that $\lambda_i < \lambda$, we obtain :

$$\|Lv_{1n} - \lambda v_{1n}\|^2 - \langle N(\cdot, v_{1n} + w) - \lambda v_{1n}, Lv_{1n} - \lambda v_{1n} \rangle \leq 0 \quad (4)$$

we deduce from (4) and from the fact that (v_{1n}) is bounded in $L^2(\Omega)$ that (Lv_{1n}) is also bounded in $L^2(\Omega)$. We deduce also that there exists a constant $c > 0$ not depending on w neither nor n such that $\|v_{1n}\|_{D(L)} \leq c(1 + \|w\|)$. Since the sequence (v_{1n}) is bounded in $D(L)$, we can extract from it a subsequence (v_{1n_k}) which converges weakly in $D(L)$ to v_1^0 . Let $v_1 \in V_1$, there exists $N > 0$ such that for each $n \geq N$, we have $v_1 \in B_{1n_k}$, so $K(v_{1n_k}, w) \geq K(v_1, w)$ and when n goes to ∞ , we deduce that $K(v_1^0, w) \geq K(v_1, w)$. Hence $K(v_1^0, w) = \max_{v_1 \in V_1} K(v_1, w)$. ■

Under the assumptions of theorem 3, we define the mapping $\theta_1 : V + V_2 \rightarrow V_1$ such that

$$K(\theta_1(w), w) = \max_{v_1 \in V_1} K(v_1, w)$$

so $\theta_1(w)$ verifies

$$\begin{aligned} \langle L\theta_1(w), v_1 \rangle = \alpha \int (\theta_1(w) + w)^+ v_1 - \beta \int (\theta_1(w) + w)^- v_1 \\ + \int g(\cdot, \theta_1(w) + w) v_1 \end{aligned} \quad (5)$$

for each $v_1 \in V_1$. The equation (5) is equivalent to

$$L\theta_1(w) = P_{V_1} [\alpha(\theta_1(w) + w)^+ - \beta(\theta_1(w) + w)^- + g(\cdot, \theta_1(w) + w)]$$

It is not difficult to verify that :

- $\theta_1(0) = 0$.
- There exists a constant $c > 0$ not depending on w such that

$$\|\theta_1(w)\|_{D(L)} \leq c(1 + \|w\|)$$

- If (w_n) tends to w weakly in $D(L)$, then $(\theta_1(w_n))$ tends to $\theta_1(w)$ in $D(L)$.

Proposition 6. *Under the assumptions of theorem 3, we have :*

a - When τ goes to 0^+ , the function $\theta_1(\tau\varphi)/\tau$ converges weakly in $D(L)$ to $m \in V_1$ where m is the unique solution of the equation

$$Lm = P_{V_1} \left[\alpha(m + \varphi)^+ - \beta(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right]$$

b - When τ goes to 0^- , the function $\theta_1(\tau\varphi)/\tau$ converges weakly in $D(L)$ to $m \in V_1$ where m is the unique solution of the equation

$$Lm = P_{V_1} \left[\beta(m + \varphi)^+ - \alpha(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right]$$

PROOF. Let (τ_n) be a sequence of positive real converging to 0. We claim that the sequence $(v_n = \theta_1(\tau_n \varphi)/\tau_n)$ is bounded in $D(L)$ and we will prove it by contradiction. Suppose that $\forall k > 0, \exists n_k > 0$ such that $\alpha_{n_k} = \|v_{n_k}\|_{D(L)} \geq k$ and

$$Lv_{n_k} = P_{V_1} \left[\alpha(v_{n_k} + \varphi)^+ - \beta(v_{n_k} + \varphi)^- + \frac{1}{\tau_{n_k}} g(\cdot, \tau_{n_k}(v_{n_k} + \varphi)) \right]$$

Put $z_{n_k} = v_{n_k}/\alpha_{n_k}$, that function verifies $\|z_{n_k}\|_{D(L)} = 1$ and

$$\begin{aligned} Lz_{n_k} &= P_{V_1} \left[\alpha(z_{n_k} + \varphi/\alpha_{n_k})^+ - \beta(z_{n_k} + \varphi/\alpha_{n_k})^- \right. \\ &\quad \left. + \frac{g(\cdot, \tau_{n_k}(v_{n_k} + \varphi))}{\tau_{n_k}(v_{n_k} + \varphi)} (z_{n_k} + \varphi/\alpha_{n_k}) \right] \end{aligned} \quad (6)$$

We conclude from A3 and A4 and by using Lebesgue theorem that $\frac{g(\cdot, \tau_{n_k}(v_{n_k} + \varphi))}{\tau_{n_k}(v_{n_k} + \varphi)}$ converges to $\frac{\partial g}{\partial s}(\cdot, 0)$ in $L^2(\Omega)$ (and weakly star in $L^\infty(\Omega)$). Since the sequence (z_{n_k}) is bounded in $D(L)$, we can extract from it a subsequence which converges weakly in $D(L)$ to z such that

$$Lz = P_{V_1} \left[\alpha z^+ - \beta z^- + \frac{\partial g}{\partial s}(\cdot, 0)z \right] \quad (7)$$

but assumption A3 implies that (7) has a unique solution which is $z = 0$. So the right side of (6) converges to 0 in $L^2(\Omega)$. This implies that (z_{n_k}) converges to 0 in $D(L)$. In contradiction with $\|z_{n_k}\|_{D(L)} = 1$.

Since the sequence (v_n) is bounded in $D(L)$, we can extract from it a subsequence denoted (v_{n_k}) which converges weakly in $D(L)$ to m . The function m is a solution of the equation

$$Lm = P_{V_1} \left[\alpha(m + \varphi)^+ - \beta(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right] \quad (8)$$

but assumption A3 implies also that (8) has a unique solution, so the sequence (v_n) converges weakly in $D(L)$ to m . The proof of the case b is similar to that of the case a. ■

PROOF OF THEOREM 3. First, we remark that $u = 0$ is a solution of (1) and we have to show that (1) has at least one nontrivial solution. Let the function $I : V + V_2 \rightarrow \mathbb{R}$ be such that $I(w) = J(\theta_1(w) + w)$. It is well known [1] that $(DI(w), h) = (DJ(\theta_1(w) + w), h)$ for each $h \in V + V_2$ and a function $u = v_1 + w$ is a solution of (1) if and only if $DI(w) = 0$.

a - We claim that there exists $w_0 \in \mathbb{R}^+ \varphi + V_2 = W^+$ such that $I(w_0) = \min_{W^+} I(w)$. Let $B_n = \left\{ w \in V + V_2 / \|w\|_{D(L)} \leq n \right\}$. Since the mapping I is lower semi-continuous, there exists $w_n \in W^+ \cap B_n$ such that $I(w_n) = \min_{w \in W^+ \cap B_n} I(w)$. So

$$I(w_n) = J(\theta_1(w_n) + w_n) \leq I(0) = 0$$

if we write w_n as $w_n = s_n \varphi + w_{2n}$ with $s_n \geq 0$, we have :

$$\max_{v_1 \in V_1} J(v_1 + s_n \varphi + w_{2n}) \leq 0$$

which is equivalent to

$$\max_{v_1 \in V_1} \left[J_0(v_1 + s_n \varphi + w_{2n}) - \int G(\cdot, v_1 + s_n \varphi + w_{2n}) \right] \leq 0$$

where $J_0(u) = \frac{1}{2} \langle Lu, u \rangle - \frac{\alpha}{2} \int (u^+)^2 - \frac{\beta}{2} \int (u^-)^2$. Since $G(x, t) \leq k(x)$, we deduce that

$$\max_{v_1 \in V_1} J_0(v_1 + s_n \varphi + w_{2n}) \leq \int k(x) \quad (9)$$

if $s_n > 0$, then

$$s_n^2 \max_{v_1 \in V_1} J_0(v_1 + \varphi + \frac{1}{s_n} w_{2n}) \leq \int k(x)$$

or $0 < C(\alpha, \beta) = 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} J_0(v_1 + \varphi + v_2)$ [3], hence

$$\frac{1}{2} s_n^2 C(\alpha, \beta) \leq \int k(x)$$

we conclude then that (s_n) is bounded in \mathbb{R} . In the other hand, we deduce from (9) that

$$J_0(s_n \varphi + w_{2n}) \leq \int k(x)$$

which is equivalent to

$$s_n^2 \lambda + \langle Lw_{2n}, w_{2n} \rangle - \alpha \int ((s_n \varphi + w_{2n})^+)^2 - \beta \int ((s_n \varphi + w_{2n})^-)^2 \leq 2 \int k(x)$$

hence

$$s_n^2 (\lambda - \max(\alpha, \beta)) + (\lambda_g - \max(\alpha, \beta)) \|w_{2n}\|^2 \leq 2 \int k(x)$$

so (w_{2n}) is bounded in $L^2(\Omega)$. By proceeding as the step c of the proof of proposition 4, we show that (Lw_{2n}) is also bounded in $L^2(\Omega)$. Since the sequence (w_n) is bounded in $D(L)$, we can extract from it a subsequence which converges weakly in $D(L)$ to $w_0 \in W^+$ and it is easy to check that w_0 verifies $I(w_0) = \min_{w \in W^+} I(w)$.

b - We have to prove now that $w_0 \in \mathbb{R}_*^+ \varphi + V_2$. Let $w \in V_2$, we have :

$$\begin{aligned} I(w) &= J(\theta_1(w) + w) \\ &\geq J(w) = \frac{1}{2} \langle Lw, w \rangle - \int_{\Omega} \int_0^w N(., s) ds \\ &\geq \int_{\Omega} \int_0^w [\lambda_g - N(., s)/s] s ds \\ &\geq 0 \end{aligned}$$

(by assumption A4). Consider now the mapping $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $\psi(t) = I(t\varphi)$. We have $\psi(0) = 0$, $\frac{d\psi}{dt}(0) = 0$. The term $\frac{d^2\psi}{dt^2}(0^+) < 0$ because

$$\begin{aligned} \frac{d^2\psi}{dt^2}(0^+) &= \lim_{t \rightarrow 0^+} \frac{1}{t} \left[\frac{d\psi}{dt}(t) - \frac{d\psi}{dt}(0) \right] \\ &= \lim_{t \rightarrow 0^+} \left[\left\langle L \left(\frac{1}{t} \theta(t\varphi) + \varphi \right), \varphi \right\rangle - \alpha \int \left(\frac{1}{t} \theta(t\varphi) + \varphi \right)^+ \varphi \right. \\ &\quad \left. + \beta \int \left(\frac{1}{t} \theta(t\varphi) + \varphi \right)^- \varphi - \int \frac{1}{t} g(., \theta(t\varphi) + t\varphi) \varphi \right] \\ &= \lambda - \alpha \int (m + \varphi)^+ \varphi + \beta \int (m + \varphi)^- \varphi - \int \frac{\partial g}{\partial s}(., 0) (m + \varphi) \varphi \\ &< 0 \end{aligned}$$

(by assumption A6). So there exists $t_0 > 0$ such that $I(t_0\varphi) < 0$. Then $w_0 \in \mathbb{R}_*^+\varphi + V_2$, which implies that $DI(w_0) = 0$. Hence, (1) has at least one nontrivial solution u with $\int u\varphi > 0$. ■

The proof of theorem 4 is similar to that of theorem 3.

Remark 1. Suppose that A1, A2 are verified. If in addition

A8 : There exists $\xi \in \mathbb{R}$ such that $(N(\cdot, s) - N(\cdot, t))/(s - t) \leq \xi < \lambda_g$ for each $(s, t) \in \mathbb{R}^2$.

A9 : For each real s , we have $N(\cdot, s)/s \geq \lambda_p$ a.e. in Ω .

A10 : There exists $k \in L^1(\Omega)$ such that for each real t , we have $G(x, t) \geq k(x)$ a.e. in Ω .

A11 : $\lambda - \alpha \int (m + \varphi)^+\varphi + \beta \int (m + \varphi)^-\varphi - \int \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi)\varphi > 0$ where m is the unique element of V_2 such that

$$Lm = P_{V_2} \left[\alpha(m + \varphi)^+ - \beta(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right]$$

and $C(\alpha, \beta) < 0$, then (1) has at least one solution u with $\int u\varphi > 0$. If $C(\beta, \alpha) < 0$ and instead of A11, we have :

A12 : $\lambda - \beta \int (m + \varphi)^+\varphi + \alpha \int (m + \varphi)^-\varphi - \int \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi)\varphi > 0$ where m is the unique element of V_2 such that

$$Lm = P_{V_2} \left[\beta(m + \varphi)^+ - \alpha(m + \varphi)^- + \frac{\partial g}{\partial s}(\cdot, 0)(m + \varphi) \right]$$

then (1) has at least one solution u with $\int u\varphi < 0$.

Remark 2. If $\frac{\partial g}{\partial s}(\cdot, 0) > \lambda - \min(\alpha, \beta)$ a.e. in Ω , then the assumptions A6 and A7 are satisfied. If $\frac{\partial g}{\partial s}(\cdot, 0) < \lambda - \max(\alpha, \beta)$ a.e. in Ω , then the assumptions A11 and A12 are satisfied.

When (α, β) is not in the region between the Fucik curves, it is possible to allow greater freedom for the function g as we will see it.

Theorem 7. If $C(\alpha, \beta) * C(\beta, \alpha) > 0$ and $|g(x, s)| \leq h^{1-\zeta}(x)|s|^\zeta + l(x)$ where $(h, l) \in L^2(\Omega) \times L^2(\Omega)$ and $0 \leq \zeta < 1$, then (1) has at least one solution.

PROOF. Suppose that $C(\alpha, \beta) > 0$ and $C(\beta, \alpha) > 0$. Let $\rho \in]\lambda_p, \min(\alpha, \beta)[$. For each $s \in [0, 1]$, we define the equation

$$Lu = s [\alpha u^+ - \beta u^- + g(\cdot, u)] + (1 - s)\rho u \quad (3.1)$$

A function u is a solution of (10) if and only if $u = T(s, u)$ where $T : [0, 1] \times L^2(\Omega) \rightarrow L^2(\Omega)$ is the completely continuous operator defined by

$$T(s, u) = (L - \gamma I)^{-1} [s(\alpha u^+ - \beta u^- + g(\cdot, u)) + (1 - s)\rho u - \gamma u] \quad (11)$$

where γ is a scalar such that $(L - \gamma I)^{-1}$ is compact. Under the assumptions of theorem 7, we prove by contradiction that there exists $R > 0$ such that for $s \in [0, 1]$ and u a solution of (10), we have $\|u\|_{D(L)} < R$. Suppose that $\forall n > 0, \exists s_n \in [0, 1], \exists u_n \in D(L)$ such that $\alpha_n = \|u_n\|_{D(L)} \geq n$ and

$$Lu_n = s_n [\alpha u_n^+ - \beta u_n^- + g(\cdot, u_n)] + (1 - s_n)\rho u_n$$

We can extract from $(v_n = u_n/\alpha_n)$ and (s_n) subsequences denoted in the same manner such that (v_n) tends weakly to v in $D(L)$ and (s_n) tends to s in \mathbb{R} . Since $\|g(\cdot, u_n)\| \leq \|h\|^{1-\zeta} \|u_n\|^\zeta + \|l\|$, the function v verifies

$$Lv = (s\alpha + (1-s)\rho)v^+ - (s\beta + (1-s)\rho)v^-$$

and $\|v\|_{D(L)} = 1$. But $C(s\alpha + (1-s)\rho, s\beta + (1-s)\rho) \geq C(\alpha, \beta) > 0$ and $C(s\beta + (1-s)\rho, s\alpha + (1-s)\rho) \geq C(\beta, \alpha) > 0$, so $v = 0$. We have then a contradiction. Hence $D_{LS}(I - T(1, \cdot), B(R), 0) = D_{LS}(I - T(0, \cdot), B(R), 0) = \pm 1$ where D_{LS} means the Leray-Schauder degree. Then (1) has at least one solution. ■

Let $(\alpha_0, \beta_0) \in I \times I$. If $C(\alpha_0, \beta_0) = 0$, we denote by w_1 the unique element of $D(L)$ such that $Lw_1 = \alpha_0 w_1^+ - \beta_0 w_1^-$ and $\int w_1 \varphi = 1$ and if $C(\beta_0, \alpha_0) = 0$, we denote by w_2 the unique element of $D(L)$ such that $Lw_2 = \beta_0 w_2^+ - \alpha_0 w_2^-$ and $\int w_2 \varphi = 1$.

We give now an existence result of solution of (1) when $g(x, s) = h(x, s)s + l(x, s)$.

Theorem 8. *Suppose that*

B1 - $g(x, s) = h(x, s)s + l(x, s)$.

B2 - $\sup_{s \in \mathbb{R}} |l(x, s)| \in L^2(\Omega)$.

B3 - *There exists $\eta > 0$ such that*

$$\lambda_p + \eta \leq \alpha + \inf_{s \geq 0} h(x, s) \leq \alpha + \sup_{s \geq 0} h(x, s) \leq \alpha_0 \text{ a.e. in } \Omega$$

$$\lambda_p + \eta \leq \beta + \inf_{s \leq 0} h(x, s) \leq \beta + \sup_{s \leq 0} h(x, s) \leq \beta_0 \text{ a.e. in } \Omega$$

a - *If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) = 0$ and $((\text{meas}(I_1) \neq 0$ or $\text{meas}(I_2) \neq 0)$ and $(\text{meas}(J_1) \neq 0$ or $\text{meas}(J_2) \neq 0))$, then (1) has at least one solution.*

b - *If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) > 0$ and $(\text{meas}(I_1) \neq 0$ or $\text{meas}(I_2) \neq 0)$, then (1) has at least one solution.*

c - *If $C(\alpha_0, \beta_0) > 0$, $C(\beta_0, \alpha_0) = 0$ and $(\text{meas}(J_1) \neq 0$ or $\text{meas}(J_2) \neq 0)$, then (1) has at least one solution. Where*

$$I_1 = \left\{ x \in \Omega / w_1(x) > 0 \text{ and } \alpha + \sup_{s \geq 0} h(x, s) < \alpha_0 \right\}$$

$$I_2 = \left\{ x \in \Omega / w_1(x) < 0 \text{ and } \beta + \sup_{s \leq 0} h(x, s) < \beta_0 \right\}$$

$$J_1 = \left\{ x \in \Omega / w_2(x) < 0 \text{ and } \alpha + \sup_{s \geq 0} h(x, s) < \alpha_0 \right\}$$

$$J_2 = \left\{ x \in \Omega / w_2(x) > 0 \text{ and } \beta + \sup_{s \leq 0} h(x, s) < \beta_0 \right\}$$

Remark 3. *Suppose that B1, B2 are verified and there exists $\eta > 0$ such that*

$$\alpha_0 \leq \alpha + \inf_{s \geq 0} h(x, s) \leq \alpha + \sup_{s \geq 0} h(x, s) \leq \lambda_g - \eta \text{ a.e. in } \Omega$$

$$\beta_0 \leq \beta + \inf_{s \leq 0} h(x, s) \leq \beta + \sup_{s \leq 0} h(x, s) \leq \lambda_g - \eta \text{ a.e. in } \Omega$$

a - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) = 0$ and $((\text{meas}(M_1) \neq 0$ or $\text{meas}(M_2) \neq 0)$ and $(\text{meas}(N_1) \neq 0$ or $\text{meas}(N_2) \neq 0))$, then (1) has at least one solution.

b - If $C(\alpha_0, \beta_0) < 0$, $C(\beta_0, \alpha_0) = 0$ and $(\text{meas}(N_1) \neq 0$ or $\text{meas}(N_2) \neq 0)$, then (1) has at least one solution.

c - $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) < 0$ and $(\text{meas}(M_1) \neq 0$ or $\text{meas}(M_2) \neq 0)$, then (1) has at least one solution. Where

$$\begin{aligned} M_1 &= \left\{ x \in \Omega / w_1(x) > 0 \text{ and } \alpha_0 < \alpha + \inf_{s \geq 0} h(x, s) \right\} \\ M_2 &= \left\{ x \in \Omega / w_1(x) < 0 \text{ and } \beta_0 < \beta + \inf_{s \leq 0} h(x, s) \right\} \\ N_1 &= \left\{ x \in \Omega / w_2(x) < 0 \text{ and } \alpha_0 < \alpha + \inf_{s \geq 0} h(x, s) \right\} \\ N_2 &= \left\{ x \in \Omega / w_2(x) > 0 \text{ and } \beta_0 < \beta + \inf_{s \leq 0} h(x, s) \right\} \end{aligned}$$

Before the proof of theorem 8, we give some remarks about the solution of the equation

$$Lu = a(x)u^+ - b(x)u^- \quad (12)$$

Theorem 9. *Suppose that there exists $\eta > 0$ such that $\lambda_p + \eta \leq a(x) \leq \alpha_0$ and $\lambda_p + \eta \leq b(x) \leq \beta_0$ a.e. in Ω .*

a - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) = 0$ and

$$C1 : \int (\alpha_0 - a(x))(w_1^+)^2 + \int (\beta_0 - b(x))(w_1^-)^2 > 0$$

$$C2 : \int (\alpha_0 - a(x))(w_2^-)^2 + \int (\beta_0 - b(x))(w_2^+)^2 > 0$$

then (12) has only the trivial solution.

b - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) > 0$ and C1 is verified, then (12) has only the trivial solution.

c - If $C(\alpha_0, \beta_0) > 0$, $C(\beta_0, \alpha_0) = 0$ and C2 is verified, then (12) has only the trivial solution.

d - If $C(\alpha_0, \beta_0) < 0$, $C(\beta_0, \alpha_0) = 0$ and C2 is verified, then (12) has no solution such that $\int u\varphi < 0$.

e - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) < 0$ and C1 is verified, then (12) has no solution such that $\int u\varphi > 0$.

PROOF. Case a : If (12) has a nontrivial solution u , then $\int u\varphi \neq 0$ because if not, we deduce by theorem 1 that $u = 0$. Suppose that $\int u\varphi > 0$ and let $v = u / \int u\varphi$, the function v verifies $Lv = a(x)v^+ - b(x)v^-$ and $\int v\varphi = 1$. Let the mapping $K(a, b) : V_1 \times V_2 \rightarrow \mathbb{R}$ be such that

$$\begin{aligned} K(a, b)(v_1, v_2) &= \frac{1}{2} \langle L(v_1 + \varphi + v_2), v_1 + \varphi + v_2 \rangle - \frac{1}{2} \int a(x)((v_1 + \varphi + v_2)^+)^2 \\ &\quad - \frac{1}{2} \int b(x)((v_1 + \varphi + v_2)^-)^2 \end{aligned}$$

By proceeding as for the proof of proposition 5, we show that for each $v_2 \in V_2$, there exists a unique $\theta(a, b)v_2 \in V_1$ such that

$$K(a, b)(\theta(a, b)v_2, v_2) = \max_{v_1 \in V_1} K(a, b)(v_1, v_2)$$

Put $v = v_1^0 + \varphi + v_2^0$ and $w_1 = w_{11}^0 + \varphi + w_{12}^0$. We deduce from the equation of v that $K(a, b)(v_1^0, v_2^0) = \max_{v_1 \in V_1} K(a, b)(v_1, v_2^0)$, so

$$0 = K(a, b)(v_1^0, v_2^0) \geq \max_{v_1 \in V_1} K(\alpha_0, \beta_0)(v_1, v_2^0) = K(\alpha_0, \beta_0)(\theta(\alpha_0, \beta_0)v_2^0, v_2^0)$$

Since $0 = C(\alpha_0, \beta_0) = 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} K(\alpha_0, \beta_0)(v_1, v_2)$, we deduce that $K(\alpha_0, \beta_0)(\theta(\alpha_0, \beta_0)v_2^0, v_2^0) = 0$. Then

$$\begin{aligned} \min_{v_2 \in V_2} \max_{v_1 \in V_1} K(\alpha_0, \beta_0)(v_1, v_2) &= \max_{v_1 \in V_1} \min_{v_2 \in V_2} K(\alpha_0, \beta_0)(v_1, v_2) \\ &= K(\alpha_0, \beta_0)(\theta(\alpha_0, \beta_0)v_2^0, v_2^0) \\ &= 0 \end{aligned}$$

The point $(\theta(\alpha_0, \beta_0)v_2^0, v_2^0)$ is then a saddle point of $K(\alpha_0, \beta_0)$ as like the point (w_{11}, w_{12}) [3]. Since the mapping $K(\alpha_0, \beta_0)$ is strictly concave on V_1 and strictly convex on V_2 , we conclude that $\theta(\alpha_0, \beta_0)v_2^0 = w_{11}$ and that $v_2^0 = w_{12}$. We claim that $\theta(\alpha_0, \beta_0)v_2^0 \neq v_1^0$ because if not, we have $v = w_1$, so

$$Lw_1 = \alpha_0 w_1^+ - \beta_0 w_1^- = a(x)w_1^+ - b(x)w_1^-$$

thus

$$(\alpha_0 - a(x))w_1^+ - (\beta_0 - b(x))w_1^- = 0$$

which implies that

$$\int (\alpha_0 - a(x))(w_1^+)^2 + \int (\beta_0 - b(x))(w_1^-)^2 = 0$$

in contradiction with the assumption C1. Since $\theta(\alpha_0, \beta_0)v_2^0 \neq v_1^0$, we have :

$$0 = K(a, b)(v_1^0, v_2^0) > K(a, b)(\theta(\alpha_0, \beta_0)v_2^0, v_2^0) \geq K(\alpha_0, \beta_0)(\theta(\alpha_0, \beta_0)v_2^0, v_2^0) = 0$$

which is impossible. When $\int u\varphi < 0$, we proceed as for the case when $\int u\varphi > 0$.

Case b : The case $\int u\varphi > 0$ gives a contradiction (we proceed as for the case a). Suppose now that $\int u\varphi < 0$, then the function $v = u / \int u\varphi$ verifies $Lv = b(x)v^+ - a(x)v^-$ and $\int v\varphi = 1$. Put $v = v_1^0 + \varphi + v_2^0$. We deduce from the equation of v that

$$0 = K(b, a)(v_1^0, v_2^0) = \max_{v_1 \in V_1} K(b, a)(v_1, v_2^0)$$

which is in contradiction with

$$0 < C(\beta_0, \alpha_0) = 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} K(\beta_0, \alpha_0)(v_1, v_2) \leq 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} K(b, a)(v_1, v_2)$$

■

We prove the other results by using the same method.

Remark 4. Suppose that there exists $\eta > 0$ such that $\alpha_0 \leq a(x) \leq \lambda_g - \eta$ and $\beta_0 \leq b(x) \leq \lambda_g - \eta$ a.e. in Ω .

a - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) = 0$ and

$$C3 : \int (a(x) - \alpha_0)(w_1^+)^2 + \int (b(x) - \beta_0)(w_1^-)^2 > 0$$

$$C4 : \int (a(x) - \alpha_0)(w_2^-)^2 + \int (b(x) - \beta_0)(w_2^+)^2 > 0$$

then (12) has only the trivial solution.

b - If $C(\alpha_0, \beta_0) < 0$, $C(\beta_0, \alpha_0) = 0$ and C4 is verified, then (12) has only the trivial solution.

c - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) < 0$ and C3 is verified, then (12) has only the trivial solution.

d - If $C(\alpha_0, \beta_0) = 0$, $C(\beta_0, \alpha_0) > 0$ and C3 is verified, then (12) has no solution such that $\int u\varphi > 0$.

e - If $C(\alpha_0, \beta_0) > 0$, $C(\beta_0, \alpha_0) = 0$ and C4 is verified, then (12) has no solution such that $\int u\varphi < 0$.

PROOF OF THEOREM 8. Let $\rho \in]\lambda_p, \min(\alpha_0, \beta_0)[$. We claim that there exists $R > 0$ such that for $s \in [0, 1]$ and u a solution of (10), we have $\|u\|_{D(L)} < R$. We will prove that result by contradiction. Suppose that $\forall n > 0$, $\exists s_n \in [0, 1]$, $\exists u_n \in D(L)$ such that $\alpha_n = \|u_n\|_{D(L)} \geq n$ and

$$\begin{aligned} Lu_n &= s_n (\alpha u_n^+ - \beta u_n^- + u_n h(\cdot, u_n) + l(\cdot, u_n)) + (1 - s_n) \rho u_n \\ &= [s_n (\alpha + h(\cdot, u_n^+)) + (1 - s_n) \rho] u_n^+ - [s_n (\beta + h(\cdot, -u_n^-)) + (1 - s_n) \rho] u_n^- \\ &\quad + s_n l(\cdot, u_n) \end{aligned}$$

We can extract from $(v_n = u_n/\alpha_n)$, (s_n) , $(h(\cdot, u_n^+))$ and $(h(\cdot, -u_n^-))$ subsequences such that (v_n) tends to v in $L^2(\Omega)$, (Lv_n) tends to Lv weakly in $L^2(\Omega)$, (s_n) tends to s in \mathbb{R} , $(h(\cdot, u_n^+))$ tends to h_+^* weakly star in $L^\infty(\Omega)$ (and weakly in $L^2(\Omega)$) and $(h(\cdot, -u_n^-))$ tends to h_-^* . The function v is a solution of the equation

$$Lv = (s(\alpha + h_+^*) + (1 - s)\rho)v^+ - (s(\beta + h_-^*) + (1 - s)\rho)v^-$$

First we have $\min(\lambda_p + \eta, \rho) \leq s(\alpha + h_+^*) + (1 - s)\rho \leq \alpha_0$ and $\min(\lambda_p + \eta, \rho) \leq s(\beta + h_-^*) + (1 - s)\rho \leq \beta_0$. Suppose per example that $meas(I_1) \neq 0$ and $meas(J_1) \neq 0$, then

$$\begin{aligned} &\int_{\Omega} (\alpha_0 - s(\alpha + h_+^*) - (1 - s)\rho) (w_1^+)^2 \\ &\quad + \int_{\Omega} (\beta_0 - s(\beta + h_-^*) - (1 - s)\rho) (w_1^-)^2 \\ &\geq \int_{I_1} \left(\alpha_0 - s(\alpha + \sup_{s \geq 0} h(x, s)) - (1 - s)\rho \right) (w_1^+)^2 \\ &\quad + \int_{I_2} \left(\beta_0 - s(\beta + \sup_{s \leq 0} h(x, s)) - (1 - s)\rho \right) (w_1^-)^2 \\ &> 0 \end{aligned}$$

because $[\alpha_0 - s(\alpha + \sup_{s \geq 0} h(x, s)) - (1 - s)\rho] (w_1^+)^2 > 0$ on I_1 . We have also :

$$\begin{aligned} &\int_{\Omega} (\alpha_0 - s(\alpha + h_+^*) - (1 - s)\rho) (w_2^-)^2 \\ &\quad + \int_{\Omega} (\beta_0 - s(\beta + h_-^*) - (1 - s)\rho) (w_2^+)^2 \\ &\geq \int_{J_1} \left(\alpha_0 - s(\alpha + \sup_{s \geq 0} h(x, s)) - (1 - s)\rho \right) (w_2^-)^2 \\ &\quad + \int_{J_2} \left(\beta_0 - s(\beta + \sup_{s \leq 0} h(x, s)) - (1 - s)\rho \right) (w_2^+)^2 \\ &> 0 \end{aligned}$$

we deduce then by theorem 9 that $v = 0$. So (v_n) converges to 0 in $D(L)$. In contradiction with $\|v\|_{D(L)} = 1$. By the Invariance of Leray-Schauder degree, we prove that (1) has at

least one solution. ■

References

- [1] N. P. Cac, *On nontrivial solutions of an asymptotically linear Dirichlet problem*, Journal of Differential Equations, pp.103-117, 1988.
- [2] N. P. Cac, *On nontrivial solutions of a Dirichlet problem whose jumping nonlinearities crosses a multiple eigenvalues*, Journal of Differential Equations 80, pp.379-404, 1989.
- [3] T. Gallouët and O. Kavian, *Résultats d'existence et de non-existence pour certains problèmes demi-linéaires à l'infini*, Annales Faculté Des Sciences Toulouse, vol III, 1981, pp.201-246.
- [4] T. Gallouët and O. Kavian, *Resonance for jumping nonlinearities*, Comm. in partial differential equations, 7(3), pp.323-342, 1982.
- [5] O. Kavian, *Quelques remarques sur le spectre demi-linéaire de certains opérateurs auto-adjoints*, Publication du laboratoire d'analyse numérique, Université Pierre et Marie Curie (Paris IV), 1985.
- [6] O. Kavian, *Introduction à la théorie des points critiques et application aux problèmes elliptiques*, Springer-Verlag, France, Paris 1993.
- [7] A. Ambrosetti and G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura i Applicata, 1972, pp.231-246.
- [8] M. Schechter, *The Fucik spectrum*, Indiana university mathematics journal, vol.43, No.4 (1994), pp.1136-1157.
- [9] M. Schechter, *Type (II) regions between curves of the Fucik Spectrum*, Nonlinear differential equations and applications NoDEA. 4 (1997), pp.459-476.