

Stability Results for Perturbed Difference Equations of Volterra Type

Gabriela GALESCU and Pavel TALPALARU

Abstract. In this paper we study different kinds of stability for a perturbed difference equation of Volterra type under certain types of behaviour of the corresponding unperturbed difference equation.

Keywords: Volterra difference equation, h -stability, classical stability, difference inequality, variation of parameters formula.

1 Introduction

Difference equations occur in many branches of applied mathematics: numerical analysis, physics, control theory and optimization. In recent years, considerable attention has been paid to the development of qualitative theory for difference equations.

The study of stability and asymptotic behaviour of solutions of difference equations has made considerable use of the variation of parameters formula, both the classical linear version and the nonlinear version analogue to that of Alekseev obtained by Agarwal [1] and Bainov and Simeonov [2].

The nonlinear discrete of Alekseev's variation of parameters formula provides a most useful technique for investigating the effect of a perturbation on the solutions of nonlinear systems of difference equations.

In contrast to the usual Lyapunov function method, the use of this formula does not require that the unperturbed system satisfy very stringent hypotheses, and qualitative estimates are easily obtained.

Many authors have obtained results in qualitative behaviour of solutions of perturbed nonlinear systems using the nonlinear discrete variation of parameters formula. Among the contributions in this field we quote here those of Medina [6], Medina and Pinto [5], [7], Choi and Koo [3] and Zouyousefain and Leela [11].

In present paper we shall extend the study of different kinds of stability from [8] to the case of nonlinear Volterra type difference equations and we also shall establish relationships with the concept of h -stability introduced in [6] and widely developed in [7] and [3].

2 Notations, definitions and preliminary results

We adopt the following notations in the paper: \mathbb{N} is the set of nonnegative integers; $N(n_0) = \{n_0, n_0 + 1, \dots\}$, where $n_0 \in \mathbb{N}$; $N_m(n_0) = \{n_0, n_0 + 1, \dots, n_0 + m\}$, $m \in \mathbb{N}$; \mathbb{R}^q the q -dimensional real Euclidean space with norm $|x| = \sum_1^q |x_i|$, $x = (x_1, x_2, \dots, x_q) \in \mathbb{R}^q$. For a $q \times q$ matrix $A = (a_{ij})$, define its norm $|A|$ by $|A| = \max_j \sum_{i=1}^q |a_{ij}|$.

In the sequel we are interested in the relations between the solutions of the unperturbed system

$$x(n+1) = f(n, x(n)) \quad (P_0)$$

and the solutions of its perturbed Volterra type system

$$y(n+1) = f(n, y(n)) + \sum_{s=n_0}^{n-1} g(n, s, y(s)), \quad (P_1)$$

where $f : N(n_0) \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ and g is defined for $n \geq s \geq n_0$ and $|x| < H \leq \infty$ having its values in \mathbb{R}^q . We assume that $f_x = \frac{\partial f}{\partial x}$ exists and is continuous and invertible on $N(n_0) \times \mathbb{R}^q$ and, also, that $f(n, 0) = g(n, s, 0) = 0$.

Let $x(n) = x(n, n_0, x_0)$ be a solution of Eq. (P_0) with $x(n_0, n_0, x_0) = x_0$ and, respectively, $y(n) = y(n, n_0, y_0)$ a solution of Eq. (P_1) with $y(n_0, n_0, y_0) = y_0$.

Also, we consider the associated to $x(n, n_0, x_0)$ variational system

$$z(n+1) = f_x(n, x(n, n_0, x_0))z, \quad (V)$$

for which the fundamental matrix solution $\phi(n, n_0, x_0)$ is given by ([1], [4]) $\phi(n, n_0, x_0) = \partial x(n, n_0, x_0) / \partial x_0$, with $\phi(n_0, n_0, x_0) = I$ - the identity matrix.

If we assume that $x(n, n_0, x_0)$ and $x(n, n_0, y_0)$ are the solutions of (P_0) through (n_0, x_0) and (n_0, y_0) , respectively, which exist for $n \geq n_0$, and such that x_0 and y_0 belong to a convex subset of \mathbb{R}^q , then for $n \geq n_0$, we have ([4; Lemma 2.1])

$$x(n, n_0, x_0) - x(n, n_0, y_0) = \left[\int_0^1 \phi(n, n_0, x_0 + s(y_0 - x_0)) ds \right] (y_0 - x_0). \quad (2.1)$$

From here one obtains

$$x(n, n_0, x_0) = \left[\int_0^1 \phi(n, n_0, sx_0) ds \right] x_0. \quad (2.2)$$

If $x(n)$ and $y(n)$ are the solutions of (P_0) and (P_1) , respectively, and $x(n_0) = y(n_0)$, then the following analogue of Alekseev's formula holds ([1], [2], [4]).

$$y(x) = x(n) + \sum_{l=n_0}^{n-1} \left[\int_0^1 \phi(n, l+1, u(y(l), \tau)) d\tau \right] \sum_{s=n_0}^{l-1} g(l, s, y(s)), \quad (2.3)$$

where $u(y(n), \tau) = f(n, y(n)) + \tau \int_{s=n_0}^{n-1} g(n, s, y(s))$ and $\tau \in [0, 1]$. As is customary, in the

following, we use the convention $\sum_{i=s}^k u(i) = 0$ $\prod_{i=s}^k v(i) = 1$ when $s > k$.

We now recall some definitions of stability of the zero solution which will be used in the sequel.

Definition 2.1. *The zero solution $x = 0$ of (P_0) is said to be:*

- a) *Stable (S)* if, for every $\varepsilon > 0$ and any $n_0 \in N(n_0) \equiv N(1)$, there exists a $\delta(\varepsilon, n_0) > 0$ such that, $|x_0| < \delta$ and $n \in N(n_0)$ imply $|x(n, n_0, x_0)| < \varepsilon$;
- b) *Uniformly stable (US)* if it is stable and the above δ is independent of n_0 ;
- c) *Asymptotically stable (AS)* if it is stable and, for any $n_0 \in N$, there exists $\delta_0(n_0) > 0$ such that $|x_0| < \delta_0$ implies $|x(n, n_0, x_0)| \rightarrow 0$ as $n \rightarrow \infty$ (i.e., it is attractive);
- d) *Uniformly asymptotically stable (UAS)* if it is (US) and, for any $n_0 \in N$ and $\varepsilon > 0$ there exist $\delta_0(n_0) > 0$ and $T(\varepsilon) \in N$ such that $|x_0| < \delta$ implies $|x(n, n_0, x_0)| < \varepsilon$ for all $n \in N(n_0 + T(\varepsilon))$.
- e) *Exponentially asymptotically stable (EAS)* if there exists $0 < p < 1$ and for every $\alpha > 0$ and $n_0 \in N$ there exists $\delta(n_0, \alpha) > 0$ such that $n \in N(n_0)$ and $|x_0| < \delta$ imply $|x(n, n_0, x_0)| < \alpha p^{n-n_0}$.

Definition 2.2. *The zero solution $x = 0$ of (P_0) is said to be h -stable (hS) if $c \geq 1$, $\delta > 0$ exist as well as a positive and bounded function $h : N(n_0) \rightarrow R$ such that $|x(n, n_0, x_0)| < c|x_0|h(n)h^{-1}(n_0)$, for $n \in N(n_0)$ and $|x_0| < \delta$ ($h^{-1}(n) = 1/h(n)$).*

Definition 2.3. *The zero solution $x = 0$ of (P_0) is said to be:*

- a) *Uniformly stable in variation (USV)* if there exists a constant $M \geq 1$ such that $|\phi(n, n_0, x_0)| \leq M$ for all $n \in N(n_0)$, whenever $|x_0| < H$;
- b) *Exponentially asymptotically stable in variation (EASV)* if there exist constants $M \geq 1$ and $0 < p < 1$ such that $|\phi(n, n_0, x_0)| \leq Mp^{n-n_0}$ for all $n \in N(n_0)$, whenever $|x_0| < H$.

Remark 2.1. The above definitions may be also formulated for the zero solution of (P_1) .

In view to develop different stability or growth relationships between solutions of (P_0) and (P_1) we need some discrete inequalities of Volterra type which are generalizations of Bellman–Bihari’s inequality.

Lemma 2.1. [9; Lemma 3.1.] *Let there exist functions $u(n)$, $v(n)$, $h(n, s)$ and $\omega(\rho)$ such that:*

- a) $u(n), v(n), h(n, s)$ are nonnegative for $n \geq s \geq n_0$;
- b) $\omega(\rho)$ is positive, continuous and non-decreasing for $\rho > 0$;
- c) for any $n \in N(n_0)$ we have the inequality

$$u(n) \leq c + \sum_{l=n_0}^{n-1} [v(l)\omega(u(l)) + \sum_{s=n_0}^l h(l, s)\omega(u(s))], \quad (2.4)$$

where $c \geq u(n_0)$ is a positive constant. Then, for any $n \in N(n_0)$, we have

$$\Omega(u(n)) \leq \Omega(c) + \sum_{l=n_0}^{n-1} [v(s) + \sum_{s=n_0}^l h(l, s)], \quad (2.5)$$

where $\Omega(z) = \int \frac{dz}{\omega(z)}$, $z > 0$.

Remark 2.2. When $h(l, s) = 0$, the inequality (2.5) reduces to a discrete version of a certain form of the Bellman–Gronwall inequality.

Remark 2.3. If $\omega(\rho) = \rho$, the inequality (2.5) becomes

$$u(n) \leq c \exp \left\{ \sum_{l=n_0}^{n-1} [v(s) + \sum_{s=n_0}^l h(l, s)] \right\}. \quad (2.6)$$

When $h(n, s) = 0$ from (2.6) one obtains the classical discrete Gronwall inequality.

Remark 2.4. Using b) and (2.5) it follows that there exists $z = \Omega^{-1}(w)$ and (2.5) may be written also

$$u(n) \leq \Omega^{-1}(\Omega(c) + \sum_{l=n_0}^{n-1} [v(s) + \sum_{s=n_0}^l h(l, s)]), \quad (2.7)$$

for $n_0 \leq n \leq T$, where

$$T = \sup\{n \in N(n_0) : \Omega(c) + \sum_{l=n_0}^{n-1} [v(s) + \sum_{s=n_0}^l h(l, s)] \in \text{dom } \Omega^{-1}\}.$$

In the particular case when $\omega(\rho) = \rho^m$, $0 < m < 1$, $\Omega : [0, \infty) \rightarrow [0, \infty)$, and

$$u(n) = [c^{1-m} + (1-m) \sum_{l=n_0}^{n-1} [v(s) + \sum_{s=n_0}^l h(l, s)]^{1/(1-m)}], \quad n \in N(n_0). \quad (2.8)$$

When $\omega(\rho) = \rho^m$, $m > 1$, $\Omega : (0, \infty) \rightarrow (-\infty, 0)$ and

$$u(n) = c[1 - (m-1) \sum_{l=n_0}^{n-1} (v(s) + \sum_{s=n_0}^l h(l, s))]^{-1/(m-1)}, \quad n \in N(n_0). \quad (2.9)$$

As follows from [6; Theorem 3], if $\omega(p) = \rho^m$, $m \geq 1$, for $c > 0$ small enough, there exists a constant $M \geq 1$ such that for all $\alpha > 0$,

$$\Omega^{-1}(\Omega(c) + \alpha) \leq Mc. \quad (2.10)$$

Lemma 2.2. Let $h(n)$ be a positive function defined for $n \in N(n_0)$. Suppose there exists a positive constant L such that for $n \in N(n_0)$, the inequality

$$\sum_{s=n_0}^n h(n)h^{-1}(s) \leq L \quad (2.11)$$

holds. Then, there exists a constant $\tilde{L} > 0$ such that the following inequalities are valid for $n \in N(n_0)$:

$$h(n) \leq \tilde{L} (1 - L^{-1})^{n-n_0} \leq \tilde{L} \exp[-L^{-1}(n - n_0)]. \quad (2.12)$$

PROOF. Firstly we note that from (2.11) it follows that $L > 1$ and, from (2.12) we have $h(n) \rightarrow 0$ as $n \rightarrow \infty$. To prove our assertion we shall denote by $g(n) = \sum_{s=n_0}^n h^{-1}(s)$ and from (2.11) we obtain

$$h(n)g(n) = \sum_{s=n_0}^n h(n)h^{-1}(s) \leq L \quad \text{for } n \in N(n_0). \quad (2.13)$$

From (2.13) we get $h^{-1}(n) \geq L^{-1}g(n)$ and, with the definition of $g(n)$, $g(n) - g(n-1) = h^{-1}(n) \geq L^{-1}g(n)$, i.e., $g(n)(1 - L^{-1}) \geq g(n-1)$ for $n \in N(n_0 + 1)$. By a simple account one obtains $g(n) \geq g(n_0)L^{n-n_0}/(L-1)^{n-1}$ from where, in view of (2.13), we have

$$h(n) \leq Lg^{-1}(n) \leq L(L-1)^{n-n_0}/g(n_0)L^{n-n_0}.$$

From here and from the inequality $1 - u \leq e^{-u}$, $u \geq 0$, it follows (2.12) with $\tilde{L} \geq Lg^{-1}(n_0) = Lh^{-1}(n_0)$. ■

3 Stability, h -stability and uniform stability

In this section we shall examine the property of (hS) as well as those of (S) and (US) of the zero solution $y = 0$ of Eq. (P_1).

To this end, we assume the zero solution $z = 0$ of Eq. (V) is (hS) what, by Theorem 3.4 [7], implies that the zero solution $x = 0$ of Eq. (P_0) is (hS). According to Lemma 3.1 [7] this stability property is equivalent to the existence of a constant $c \geq 1$ and of a positive and bounded function h defined on $N(n_0)$ such that for every $x_0 \in \mathbb{R}^q$,

$$|\phi(n, n_0, x_0)| \leq ch(n)h^{-1}(n_0), \quad (3.1)$$

for $n \in N(n_0)$.

With respect to $g(n, s, y)$ we make the following hypothesis

(H1): $g(n, s, y) \leq b(n, s)\omega(|y|)$, $n \geq s \geq n_0$, $|y| < H$ where $b(n, s)$ is a positive function for $n \geq s \geq n_0$, and $\omega(\rho)$ is a continuous and non-decreasing function for $\rho > 0$ with $\Omega(0+) = -\infty$.

Moreover, we shall assume that there exists a positive function r , called multiplier function [7], such that

$$\omega(\alpha z) \leq r(\alpha)\omega(z), \quad \text{for } \alpha > 0, z \geq 0. \quad (3.2)$$

Theorem 3.1. *Assume that $z = 0$ of Eq. (V) is (hS) and hypothesis (H1) holds with $\omega(\rho)$ satisfying (3.2). If, in addition,*

$$h^{-1}(l+1) \sum_{s=n_0}^l r(h(s))b(l, s) \in l_1(n_0), \quad (3.3)$$

then, for $|y_0|$ small enough and $n_0 \in N$, $y(n) = y(n, n_0, y_0)$ satisfies $|y(n)| \leq h(n)\varphi(|y_0|)$, $n \in N(n_0)$ where $\varphi(|y_0|) \rightarrow 0$ as $|y_0| \rightarrow 0$. Moreover, $y = 0$ of Eq. (P_1) is stable.

PROOF. Using the discrete analogue of Alekseev's formula (2.3), (2.2), (3.1)–(3.3) we obtain

$$\begin{aligned} |y(n)| &\leq |x(n)| + \sum_{l=n_0}^{n-1} \left[\int_0^1 |\phi(n, l+1, y(y(l), \tau))| d\tau \right] \sum_{s=n_0}^{l-1} |g(l, s, y(s))| \leq \\ &\leq ch(n)h^{-1}(n_0)|y_0| + ch(n) \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} b(l, s)\omega(h(s)h^{-1}(s)|y(s)) \leq \\ &\leq h(n) \left[ch^{-1}(n_0)|y_0| + c \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^l b(l, s)r(h(s))\omega(h^{-1}(s)|y(s)) \right]. \end{aligned}$$

From here it follows that

$$h^{-1}(n)|y(n)| \leq ch^{-1}(n_0)|y_0| + c \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^l r(h(s))b(l, s)\omega(h^{-1}(s)|y(s)),$$

and applying Lemma 2.1 in particular case $v(n) \equiv 0$, we get

$$|y(n)| \leq h(n)\varphi(|y_0|) \quad \text{for } n \in N(n_0)$$

where

$$\begin{aligned} \varphi(|y_0|) &= \Omega^{-1}(\Omega(ch(n_0)|y_0|) + c \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^l r(h(s))b(l, s)) \leq \\ &\leq \Omega^{-1}(\Omega(ch(n_0)|y_0|) + c_1), \quad c_1 = c \sum_{l=n_0}^{\infty} h^{-1}(l+1) \sum_{s=n_0}^l r(h(s))b(l, s). \end{aligned}$$

From $\Omega(0+) = -\infty$, we may choose $|y_0|$ small enough such that

$$\Omega(ch(n_0)|y_0|) + c_1 \in \text{dom } \Omega^{-1} \equiv (-\infty, \Omega(\infty)).$$

Because $h(n)$ is a bounded function we have $h(n)\varphi(|y_0|) \leq M\varphi(|y_0|)$ and therefore, if we take $\delta(\varepsilon, n_0) = c^{-1}h(n_0)\Omega^{-1}(\Omega(\varepsilon/M) - c_1)$, it follows $|y(n)| = |y(n, n_0, y_0)| < \varepsilon$ for $|y_0| < \delta(\varepsilon, n_0)$, i.e., the stability of $y = 0$ of Eq. (P_1) . This completes the proof. ■

Remark 3.1. If $\omega(\rho) = \rho^m$, $m > 1$, because ([6; Theorem 3]) there exists a constant $K \geq 1$ such that $\varphi(|y_0|) \leq Kch^{-1}(n_0)|y_0|$, for $|y_0|$ small enough, it follows that $y = 0$ of Eq. (P_1) is (hS), under the assumptions of Theorem 3.1 with $r(h(s)) \equiv h^m(s)$ in (3.3).

Theorem 3.1 and the result from Remark 3.1 extend Theorems 2, 3 [6] to difference equations of Volterra type.

Remark 3.2. If $\omega(\rho) = \rho$ then (3.3) becomes

$$h^{-1}(l+1) \sum_{l=n_0}^l h(s)b(l, s) \in l_1(n_0) \tag{3.3'}$$

and the application of Gronwall's inequality implies

$$\begin{aligned} |y(n)| &\leq ch(n)h^{-1}(n_0)|y_0| \exp \left(c \sum_{l=n_0}^{n-1} \sum_{s=n_0}^l h^{-1}(l+1)h(s)b(l,s) \right) \leq \\ &\leq c_1h(n)h^{-1}(n_0)|y_0|, \quad n \in N(n_0), \end{aligned}$$

where $c_1 = c \sum_{l=n_0}^{\infty} h^{-1}(l+1) \sum_{s=n_0}^l h(s)b(l,s) < \infty$. Therefore, in this case $y = 0$ of Eq. (P_1) is stable.

The condition (3.3) from Theorem 3.1 may be improved if we shall use in the proof a result of Fubini type [3; Lemma 2], [11; Lemma 2.1]

$$\sum_{l=n_0}^{n-1} a(n, l+1) \sum_{s=n_0}^{l-1} b(l, s) = \sum_{l=n_0}^{n-1} \sum_{s=l+1}^{n-1} a(n, s+1)b(s, l). \tag{3.4}$$

Corollary 3.1. *Assume that $z = 0$ of Eq. (V) is (hS) and $(H1)$ holds with $\omega(\rho)$ satisfying (3.2). In addition, suppose that*

$$\sup \left\{ \sum_{s=l}^{n-1} h^{-1}(s+1)b(s, l) : n_0 \leq l \leq n-1 \right\} \leq K, \tag{3.5}$$

for some constant $K > 0$, and $r(h(n)) \in l_1(n_0)$. Then the conclusions of Theorem 3.1 occur.

PROOF. According to (3.5), as above we obtain

$$\begin{aligned} h^{-1}(n)|y(n)| &\leq ch^{-1}(n_0)|y_0| + c \sum_{l=n_0}^{n-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} r(h(s))b(l, s)\omega(h^{-1}(s)|y(s)|) \\ &\leq ch^{-1}(n_0)|y_0| + c \sum_{l=n_0}^{n-1} r(h(l)) \left[\sum_{s=l}^{n-1} h^{-1}(s+1)b(s, l) \right] \omega(h^{-1}(l)|y(l)|) \leq \\ &\leq ch^{-1}(n_0)|y_0| + cK \sum_{l=n_0}^{n-1} r(h(l))\omega(h^{-1}(l)). \end{aligned}$$

From here, by (2.5) with $h(l, s) \equiv 0$, we get again $|y(n)| \leq h(n)\varphi(|y_0|)$ where c_1 in $\varphi(|y_0|)$ is given by $c_1 = \sum_{l=n_0}^{\infty} r(h(l)) < \infty$. ■

Corollary 3.2. *Assume that $z = 0$ of Eq. (V) is (hS) and $(H1)$ holds with $\omega(\rho) = \rho$. In addition, suppose that*

$$\sum_{l=n_0}^{\infty} h(l) \sum_{s=l}^{\infty} h^{-1}(s+1)b(s, l) \leq K < c^{-1}. \tag{3.6}$$

Then $y = 0$ of Eq. (P_1) is (hS) .

PROOF. By the same process in the proof of Corollary 3.1, we obtain for $v(n) = h^{-1}(n)|y(n)|$,

$$v(n) \leq cv(n_0) + \sup\{v(l) : n_0 \leq l \leq n\}.$$

Thus we have

$$\sup\{v(l) : n_0 \leq l \leq n\} \leq cv(n_0) + cK \sup\{v(l) : n_0 \leq l \leq n\},$$

and since $cK < 1$, we get

$$\sup\{v(l) : n_0 \leq l \leq n\} < c(1 - cK)^{-1}v(n_0) = c_1v(n_0),$$

where $c_1 = (1 - cK)^{-1} < 1$. Thus $|y(n)| \leq c_1|y_0|h(n)h^{-1}(n_0)$, for $|y_0| \leq \delta$, i.e., the (hS) of $y = 0$ of Eq. (P_1) . ■

A result concerning the stability of $y = 0$ of Eq. (P_1) is given by the next theorem where it is not asked the (hS) of $z = 0$ of Eq. (V) but a growth property for its fundamental matrix $\Phi(n, s, x)$.

Theorem 3.2. *Assume that (H1) holds and that there exist a positive function $a(n)$, $n \in N(n_0)$, and a positive constant L such that*

$$|\Phi(n, s, x)| \leq La(s), \quad n \geq s \geq n_0, \quad (3.7)$$

and $a(l+1) \sum_{s=n_0}^l b(l, s) \in l_1(n_0)$. Then $y = 0$ of Eq. (P_1) is stable.

PROOF. Indeed, this time we have

$$|y(n)| \leq La(n_0)|y_0| + L \sum_{l=n_0}^{n-1} a(l+1) \sum_{s=n_0}^l b(l, s)\omega(|y(s)|),$$

from where, according to (2.7),

$$|y(n)| \leq \Omega^{-1}(La(n_0)|y_0|) + c_1, \quad c_1 = L \sum_{l=n_0}^{\infty} a(l+1) \sum_{s=n_0}^l b(l, s) < \infty.$$

Reasoning as in the proof of Theorem 3.1 one obtains the desired result with $\delta(\varepsilon, n_0) = L^{-1}a^{-1}(n_0)\Omega^{-1}(\Omega(\varepsilon) - c_1)$. ■

Remark 3.3. If $f(n, x) = A(n)x$, where $A(n)$ is a $q \times q$ matrix, Theorem 3.2 reduces to Theorem 3.1 [8].

Remark 3.4. We note that, since $h(n)$ in Definition 2.2 is a bounded function, the (hS) of $z = 0$ of Eq. (V) implies (3.7) with $h^{-1}(s) = a(s)$.

A result concerning the (US) of $y = 0$ of Eq. (P_1) one can obtain under more restrictive condition than (3.7).

Theorem 3.3. *Assume that $x = 0$ of Eq. (P_0) is (USV) and (H1) holds. If, in addition, $\sum_{s=n_0}^l b(l, s) \in l_1(n_0)$, then $y = 0$ of Eq. (P_1) is (US).*

PROOF. For any $0 < \varepsilon < H$ let $\delta(\varepsilon) = M^{-1}\Omega^{-1}(\Omega(\varepsilon) - MK)$ and $|y_0| < \delta(\varepsilon)$, where, according to Definition 2.3, $M \geq 1$, $|\Phi(n, n_0, x_0)| \leq M$, $n \in N(n_0)$, and $K = \sum_{l=n_0}^{\infty} \sum_{s=n_0}^l b(l, s) < \infty$.

By similar arguments made above we have $|y(n)| \leq \Omega^{-1}(\Omega(M|y_0|) + MK) < \varepsilon$, $n \in N(n_0)$.
 ■

Remark 3.5. If $f(n, x) = A(n)x$, where $A(n)$ is a $q \times q$ matrix, Theorem 3.3 reduces to Theorem 4.1 [8]. In this case the (USV) is equivalent to the (US) of $x = 0$ of Eq. (P_0).

4 Asymptotic stability, attractivity and h -stability

In this section we study firstly the asymptotic stability of the zero solution of Eq. (P_1) and then its connection with the (hS) of the zero solution of Eq. (V). To this end, we shall assume that there exist two positive functions $a(n)$ and $c(s)$ defined for $n \geq s \geq n_0$, such that

$$|\Phi(n, s, x)| \leq a(n)c(s), \quad n \geq s \geq n_0, \quad (4.1)$$

with $c(s) \in l_1(n_0)$ and $a(n) \rightarrow 0$ as $n \rightarrow \infty$. Let us note by $L = \sum_{s=n_0}^{\infty} c(s) < \infty$ and let us observe that there exists a positive constant T such that $a(n) \leq T$ for all $n \in N(n_0)$. With regard to $g(n, s, y)$ we shall assume that hypothesis (H1) holds with $\omega(\rho) = \rho$, i.e.,

$$|g(n, s, y)| \leq b(n, s)|y|, \quad n \geq s \geq n_0, \quad |y| < H. \quad (4.2)$$

Theorem 4.1. *Assume that the conditions (4.1) and (4.2) hold and, in addition,*

$$\sup \left\{ \sum_{s=n_0}^n b(n, s) : n \in N(n_0) \right\} < (LT)^{-1}, \quad (4.3)$$

$$\lim_{m \rightarrow \infty} \sum_{s=n_0}^m b(m, s) = 0 \quad \text{for all } n \in N(n_0). \quad (4.4)$$

Then $y = 0$ of Eq. (P_1) is (AS).

PROOF. Firstly we show the stability of $y = 0$. From (4.3) there exists a positive constant γ such that

$$0 < \gamma < (LT)^{-1} \quad \text{and} \quad \sup \left\{ \sum_{s=n_0}^n b(n, s) : n \in N(n_0) \right\} \leq \gamma. \quad (4.5)$$

For any $0 < \varepsilon < H$ and $n_0 \in N$, we choose

$$\delta = \delta(n_0, \varepsilon) < \min\{(1 - \gamma LT)\varepsilon/Tc(n_0), \varepsilon\}.$$

Assume that there exists $n_1 \in N(n_0)$ such that $|y(n_1)| = \varepsilon$ and $|y(n)| < \varepsilon$ for $n \in \{n_0, n_0 + 1, \dots, n_1 - 1\}$. For all $n \in \{n_0, n_0 + 1, \dots, n_1\}$, for the solution $y(n)$ of Eq. (P_1) with $|y_0| \leq \delta$, we have

$$\begin{aligned} |y(n)| &\leq a(n)c(n_0)|y_0| + \sum_{l=n_0}^{n-1} a(n)c(l+1) \sum_{s=n_0}^{l-1} b(l, s)|y(s)| \\ &< Tc(n_0)|y_0| + \varepsilon \sum_{l=n_0}^{n-1} a(n)c(l+1) \sum_{s=n_0}^{l-1} b(l, s) < (1 - \gamma LT)\varepsilon + \gamma LT\varepsilon = \varepsilon. \end{aligned}$$

Thus, $y = 0$ of Eq. (P_1) is stable. Next we will show that is also attractive. From the stability, taking $\varepsilon = 1$, it follows the existence of $\delta_0 = \delta(1, n_0) < 1$ such that $n_0 \in \mathbb{N}$ and $|y_0| < \delta_0$ imply

$$|y(n)| = |y(n, n_0, y_0)| < \min(H, 1) \quad \text{for all } n \in N(n_0). \quad (4.6)$$

In the sequel we consider that solution with $|y_0| < \delta_0$ and assume there exists y_0 and $y(n) = y(n, n_0, y_0)$ such that

$$\limsup_{n \rightarrow \infty} |y(n)| = \mu > 0. \quad (4.7)$$

Since $\gamma LT < 1$, by (4.5), there exists a constant θ such that $\gamma LT < \theta < 1$. By (4.7), there exists $n_1 \in N(n_0)$ such that

$$|y(n)| < \mu\theta^{-1} \quad \text{for all } n \in N(n_1). \quad (4.8)$$

From (4.4), there exists $m_1 > n_1$ such that

$$\sum_{s=n_0}^{n_1} b(m, s) < (\theta - \gamma LT)\mu/2\theta LT \quad \text{for all } m \in N(n_1). \quad (4.9)$$

Then we have

$$\begin{aligned} |y(n)| \leq & a(n)c(n_0)\delta_0 + a(n) \sum_{l=n_0}^{m_1-1} c(l+1) \sum_{s=n_0}^{l-1} b(l, s)|y(s)| + \\ & + \sum_{l=m_1}^{n-1} a(n)c(s+1) \sum_{s=n_0}^{n_1-1} b(l, s)|y(s)| + \sum_{l=m_1}^{n-1} a(n)c(s+1) \sum_{s=n_1}^{l-1} b(l, s)|y(s)|. \end{aligned}$$

From (4.1), (4.5) and (4.9) we have

$$\sum_{l=m_1}^{n-1} a(n)c(s+1) \sum_{s=n_0}^{n_1-1} b(l, s)|y(s)| \leq LT(\theta - \gamma LT)\mu/2\theta LT.$$

Thus,

$$|y(n)| \leq a(n) \left[c(n_0)\delta_0 + \sum_{l=n_0}^{m_1-1} c(l+1) \sum_{s=n_0}^{l-1} b(l, s)|y(s)| \right] + (\theta + \gamma LT)\mu/2\theta. \quad (4.10)$$

Since $a(n) \rightarrow 0$ as $n \rightarrow \infty$, from (4.10), we get $\mu \leq (\theta + \gamma LT)\mu/\theta$ and thus $\mu < \mu$ which is impossible. Therefore, $y = 0$ of Eq. (P_1) is attractive and the proof is now complete. ■

Corollary 4.1. *Assume that (4.1) holds and $|g(n, s, y)| \leq d(n-s)|y|$ for $n \geq s \geq n_0$ and $|y| < H$, where $\sum_{k=0}^{\infty} d(k) < (LT)^{-1}$. Then $y = 0$ of Eq. (P_1) is (AS).*

PROOF. Since $\sup_{n \geq n_0} \sum_{s=n_0}^n d(n-s) = \sup_{n \geq n_0} \sum_{k=0}^{n-n_0} d(k) = \sum_{k=0}^{\infty} d(k) < (LT)^{-1}$, it follows that (4.3) holds. Then, for any $n \in N(n_0)$, we have

$$\lim_{m \rightarrow \infty} \sum_{s=n_0}^n d(m-s) = \lim_{m \rightarrow \infty} \sum_{k=m-n}^{m-n_0} d(k) = 0,$$

i.e., (4.4) is satisfied. Thus, by Theorem 4.1, $y = 0$ of Eq. (P_1) is (AS). ■

Corollary 4.2. *Assume that (4.1) holds and $|g(n, s, y)| \leq f(n)g(s)|y|$ for $n \geq s \geq n_0$ and $|y| < H$, where $f(n) \geq 0$, $g(s) \geq 0$, $\sup_{n \geq n_0} f(n) \sum_{s=n_0}^n g(s) < (LT)^{-1}$ and $\sum_{s=n_0}^{\infty} g(s) = \infty$. Then $y = 0$ of Eq. (P_1) is (AS).*

PROOF. Since $\sum_{s=n_0}^n g(s)$ is an increasing function which tends to infinity and

$\sup_{n \geq n_0} f(n) \sum_{s=n_0}^n g(s) < \infty$, then $f(n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, for any $n \in N(n_0)$, we have $\lim_{m \rightarrow \infty} f(m) \sum_{s=n_0}^m g(s) = 0$, i.e. (4.4) holds and thus, by Theorem 3.1, $y = 0$ of Eq. (P_1) is (AS). ■

Remark 4.1. Condition (4.1) under the assumptions $a(n) \rightarrow 0$, as $n \rightarrow \infty$, and $c(s) \in l_1(n_0)$ cannot be replaced by that resulting from the (hS) of $z = 0$ of Eq. (V) , i.e., by taking $c(s) = \lambda a^{-1}(s)$, $\lambda \in [1, \infty)$. However, it is possible to formulate a result giving a connection between the (hS) of $z = 0$ of Eq. (V) and the (AS) of $y = 0$ of Eq. (P_1) .

Corollary 4.3. *Assume that $z = 0$ of Eq. (V) is (hS) and that (4.2) and (4.4) hold. With regard to the function $h(n)$ from the definition of (hS) the condition*

$$\sum_{s=n_0}^n h(n)h^{-1}(s) \leq L < \infty \quad \text{for } n \in N(n_0), \tag{4.11}$$

occurs. Moreover, we assume that

$$\sup \left\{ \sum_{s=n_0}^n b(n, s) : n \in N(n_0) \right\} < (cL)^{-1}. \tag{4.12}$$

Then $y = 0$ of Eq. (P_1) is (AS).

PROOF (an outline). We shall emphasize only the differences which arise in the proof of Theorem 4.1. According to Lemma 2.2, from (4.11) it follows that $h(n) \rightarrow 0$ as $n \rightarrow \infty$ and therefore there exist the positive constant T . The number γ from (4.5) will satisfy $0 < \gamma < (cL)^{-1}$ and $\delta = \delta(n_0, \varepsilon) < \min\{(1 - c\gamma L)h(n_0)\varepsilon/cT, \varepsilon\}$. The relation (4.12) yields

$$\sum_{s=n_0}^{n_1} b(m, s) < (\theta - c\gamma L)\mu/2\theta cL \quad \text{for } m \geq m_1 > n_1$$

where $0 < c\gamma L < \theta < 1$. Further the proof continues taking $h^{-1}(s)$ instead of $c(s)$ and taking into account the relation (4.11). Finally, one obtains

$$|y(n)| \leq ch(n) \left[h^{-1}(n_0)\delta_0 + \sum_{l=n_0}^{m_1-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} b(l, s)|y(s)| \right] + (\theta + c\gamma L)\mu/2\theta,$$

from where the desired result follows. ■

Remark 4.2. Condition (4.11) is obviously satisfied in the case of the (EAS) of $z = 0$ of Eq. (V).

Remark 4.3. In the linear case, $f(n, s) = A(n, x)$, we have $x(n, n_0, x_0) = X(n)X^{-1}(n_0)x_0$, $\Phi(n, n_0, x_0) = X(n)X^{-1}(n_0)$ and Eq. (V) coincides with Eq. (P₀). Here $X(n)$ is a fundamental matrix for the linear system (P₀). Now we have $|\Phi(n, n_0, x_0)| \leq |X(n)||X^{-1}(n_0)|$, hence (4.1) occurs with $a(n) = |X(n)|$, $c(s) = |X^{-1}(s)|$ and $\sum_{s=n_0}^n |X(n)X^{-1}(s)| \leq L$ which assures [10; Lemma] that $X(n) \rightarrow \infty$ as $n \rightarrow \infty$. Thus Theorem 4.1 reduces to Theorem 5.1 [8].

The next theorem establishes the (AS) of $y = 0$ of Eq. (P₁) under more general hypotheses than those from Theorem 4.1 or Corollary 4.3.

Theorem 4.2. Assume that hypothesis (H1) and condition (4.1) hold with $a(n) \rightarrow 0$ as $n \rightarrow \infty$ and, in addition,

$$\sum_{l=n_0}^{\infty} c(l+1) \sum_{s=n_0}^l b(l, s) = K < \infty, \tag{4.13}$$

for some constant $K > 0$. Then $y = 0$ of Eq. (P₁) is (AS).

PROOF. Reasoning as in the previous theorems one obtains

$$|y(n)| \leq Tc(n_0)|y_0| + T \sum_{l=n_0}^{n-1} \sum_{s=n_0}^l c(l+1)b(l, s)\omega(|y(s)|),$$

from where, by Lemma 2.1,

$$|y(n)| \leq \Omega^{-1}(\Omega(Tc(n_0)|y_0|) + KT), \tag{4.14}$$

for $n \in N(n_0)$, and $a(n) \leq T$ for $n \in N(n_0)$. From (4.14), by the same argument from the proof of Theorem 3.1, it follows the stability of $y = 0$ of Eq. (P₁). To prove the attractivity of $y = 0$, we firstly observe that from its stability, there exists $\delta(n_0) > 0$ such that $|x_0| < \delta(n_0)$ implies $|y(n)| < \varepsilon = 1$, $n \in N(n_0)$. Thus one obtains

$$|y(n)| \leq a(n) \left[c(n_0)|y_0| + \sum_{l=n_0}^{n_1-1} \sum_{s=n_0}^{l-1} c(l+1)b(l, s)\omega(1) \right] + a(n) \sum_{l=n_1}^{n-1} \sum_{s=n_0}^{l-1} c(l+1)b(l, s)\omega(1), \tag{4.15}$$

where $n_1 \in N(n_0)$, $n_0 < n_1 < n$, is large enough such that $\sum_{l=n_1}^{\infty} \sum_{s=n_0}^l c(l+1)b(l, s) < \varepsilon T^{-1}$.

Thus, using (4.15), we get $\lim |y(n)| = 0$ if $|y_0| < \delta(n_0)$ because $a(n) \rightarrow 0$ as $n \rightarrow \infty$ and $\varepsilon > 0$ is an arbitrary number. This completes the proof. ■

Corollary 4.4. Assume that $z = 0$ of Eq. (V) is (hS), h satisfying (4.11), the hypothesis (H1) holds and (4.13) is satisfied for $c(s) = h^{-1}(s)$. Then $y = 0$ of Eq. (V) is (AS).

PROOF. Indeed, in this case we have $a(n) = h(n)$, $c(s) = h^{-1}(s)$ and (4.11) implies $h(n) \rightarrow 0$ as $n \rightarrow \infty$ (Lemma 2.2). The proof continues as above. ■

Remark 4.4. If we shall use in the proof of Theorem 4.2 (respectively, Corollary 4.4) the mentioned formula (3.4), under the same conditions excepting (4.13) which will become

$$\sup \left\{ \sum_{s=l}^{n-1} c(s+1)b(s,l) : n_0 \leq l \leq n-1 \right\} \leq K \quad (4.13')$$

(respectively, with $c(s) = h^{-1}(s)$), then the conclusions are also true.

In what follows, instead of (H1), we shall make the hypothesis

(H2): $|g(n, s, y)| \leq [a(s) + b(n, s)]\omega(|y|)$, $n \geq s \geq n_0$, $|y| < H$, where $a(s)$, $b(n, s)$ are positive functions for $n \geq s \geq n_0$ and $\omega(\rho)$ defined in (H1) is such that $\omega(\rho) \leq \rho$, $\rho > 0$.

Theorem 4.3. Assume that (H2) holds and $z = 0$ of Eq. (V) is (hS) , where h satisfies (4.1). Moreover, assume that

$$\sum_{s=n_0}^l [a(s) + b(l, s)] \in l_1(n_0). \quad (4.16)$$

Then, if $L < (cA)^{-1}$, where $A = \sup \left\{ \sum_{s=n_0}^l a(s) : n_0 \leq l \leq n \right\}$, the solution $y = 0$ of Eq. (P_1) is (AS).

PROOF. As we have seen, (4.11) yields to the existence of a positive constant L_1 such that $h(n) \leq L_1$, $n \in N(n_0)$ (Lemma 2.2). Formula (2.3) implies for $v(n) = \max\{|y(s)| : s = n_0, n_0 + 1, \dots, n\}$ the inequality

$$v(n) \leq L_1 ch^{-1}(n_0)|y_0| + cL \sum_{l=n_0}^{n-1} \left[\left(\sum_{s=n_0}^l a(s) \right) \omega(r(l)) + \sum_{s=n_0}^l b(l, s) \omega(r(s)) \right]$$

from where, using Lemma 2.1, we get

$$|y(n)| \leq v(n) \leq \Omega^{-1}(\Omega(L_1 ch^{-1}(n_0)|y_0|) + cLK),$$

$K = \sum_{l=n_0}^{\infty} \sum_{s=n_0}^l [a(s) + b(l, s)]$. From here, as in the proof of Theorem 3.1, it follows the stability of $y = 0$.

Therefore, for $n_0 \in N$, there exists $\delta(n_0) > 0$ such that any solution $y(n)$ of (P_1) with $|y_0| < \delta$ satisfies $|y(n)| = |y(n, n_0, y_0)| < 1$ for all $n \in N(n_0)$. To complete the proof we shall show that, if $|y_0| < \delta(n_0)$, we have $y(n) \rightarrow 0$ as $n \rightarrow \infty$. To this end, define $M = \limsup_{n \rightarrow \infty} |y(n)|$. Obviously, $0 \leq M \leq 1$. Choose d such that $0 < McAL < dM < 1$.

If we assume that $M > 0$, then there exists $n_1 \in N(n_0)$ such that $|y(n)| < Md^{-1}$ for all $n \in N(n_1)$. From the formula (2.3) and the hypotheses of our theorem we get

$$\begin{aligned} |y(n)| \leq & ch(n) \left[h^{-1}(n_0)|y_0| + \sum_{l=n_0}^{n_1-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} |g(l, s, y(s))| \right] + \\ & + cAL\omega(Md^{-1}) + cL\omega(Md^{-1}) \sum_{l=n_1}^{n-1} \sum_{s=n_0}^{l-1} b(l, s). \end{aligned}$$

By virtue of (4.16) and $\omega(Md^{-1}) \leq Md^{-1}$, taking, if necessary, a larger n_1 and keeping it fixed, we obtain

$$|y(n)| \leq ch(n) \left[h^{-1}(n_0)|y_0| + \sum_{l=n_0}^{n_1-1} h^{-1}(l+1) \sum_{s=n_0}^{l-1} |g(l, s, y(s))| \right] + cALMd^{-1} + \varepsilon.$$

By the same reasoning from the final part of Theorem 4.2 it follows the attractivity of $y = 0$.
 ■

Remark 4.5. The given proof is valid only if $a(s) > 0$. When $a(s) = 0$, a similar result is obtained from Corollary 4.4.

Remark 4.6. The condition $\omega(\rho) \leq \rho$ occurs, at least, for $\omega(\rho) = \rho^m$, $m \in [1, \infty)$.

In what follows we shall give conditions under which the property of the (UAS) of $z = 0$ of Eq. (V) will be transferred to $y = 0$ of Eq. (P₁). We note that the (UAS) of $z = 0$ of Eq. (V) is equivalent [8; Lemma 2.1] to the existence of two constants $M \geq 1$ and $0 < p < 1$, such that

$$|\Phi(n, s, x)| \leq Mp^{n-s}, \quad n \geq s \geq n_0, \quad (4.17)$$

and, on the other hand, this means the (EASV) of $x = 0$ of (P₀). We also note that the (UAS) is a particular case of the (hS) for $h(n) = p^n$.

Corollary 4.5. Assume that $z = 0$ of Eq. (V) is (UAS), (H2) holds with $\omega(\rho) = \rho$ and, in addition, the following conditions: $\sum_{s=n_0}^{\infty} a(s) = A < \infty$, $\sum_{l=n_0}^{\infty} \sum_{s=n_0}^l b(l, s) = B < \infty$, $MAp^{-1} + \ln p < 0$ are fulfilled. Then $y = 0$ of Eq. (P₁) is (UAS).

PROOF. Setting $v(n) = \max\{|y(s)| : s = n_0, n_0 + 1, \dots, n\}$, by the variation parameters formula, we get

$$|y(n)| \leq v(n) \leq Mp^{n-n_0}|y_0| + M \sum_{l=n_0}^{n-1} \sum_{s=n_0}^{l-1} [a(s) + b(l, s)]p^{n-s-1}v(l).$$

From this, multiplying both sides by $p^{-(n-n_0)}$ and using Gronwall's inequality one obtains

$$|y(n)| \leq M|y_0| \exp(n - n_0)[MAp^{-1} + \ln p + MBp^{-1}(n - n_0)^{-1}], \quad n \in N(n_0),$$

which implies

$$|y(n)| \leq M|y_0| \exp(MBp^{-1}).$$

Let $q > 0$ be such that $K = MAp^{-1} + \ln p + q < 0$. Then, for all $n \in N(n_0)$ such that $n - n_0 \in N(n_1)$, where n_1 is a sufficiently large positive integer independent of n_0 with $MBp^{-1}(n - n_0)^{-1} \leq MBp^{-1}n_1^{-1} < q$, we have

$$|y(n)| \leq M|y_0| \exp M(n - n_0), \quad n \in N(n_0),$$

from which it follows the (UAS) of $y = 0$ of Eq. (P₁). ■

Remark 4.7. In the case of linear system (P₀), $f(n, x) = A(n)x$, condition (4.17) becomes $|X(n)X(s)| \leq Mp^{n-1}$, $n \geq s \geq n_0$, where $X(n)$ is a fundamental matrix of the linear system (P₀) and Corollary 4.5 reduces to Theorem 6.1 [8].

Remark 4.8. In this case, $h(n) = p^n$, condition (4.11) is fulfilled with $L = (1 - p)^{-1} > 1$ and from $MAp^{-1} + \ln p < 0$ it follows $cAL = MA(1 - p)^{-1} < -p(1 - p)^{-1} \ln p < 1$.

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