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The Darboux Problem for Third Order Hyperbolic Inclusions

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Abstract. In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form $u_{xyz} \in F(x,y,z,u)$. An existence theorem for a local solution of this problem is proved and some properties of the set of its solutions are established.

Keywords: multifunction, hyperbolic inclusion, upper-semicontinuity, initial values, absolutely continuous in Carathéodory's sense function.
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1 Introduction

In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form

$$\frac{\partial^3 u(x,y,z)}{\partial x \partial y \partial z} \in F(x,y,z,u), \quad (x,y,z) \in D = [0,a] \times [0,b] \times [0,c], \quad u \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

with initial values

$$\begin{cases} u(x,y,0) = \varphi(x,y), & (x,y) \in D_1 = [0,a] \times [0,b], \\ u(0,y,z) = \psi(y,z), & (y,z) \in D_2 = [0,b] \times [0,c], \\ u(x,0,z) = \chi(x,z), & (x,z) \in D_3 = [0,a] \times [0,c], \end{cases} \tag{1.2}$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense [1,§565 - §570], $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ and they satisfy the conditions

$$\begin{cases} u(x,0,0) = \varphi(x,0) = \chi(x,0) = v^1(x), & x \in [0,a], \\ u(0,y,0) = \varphi(0,y) = \psi(y,0) = v^2(y), & y \in [0,b], \\ u(0,0,z) = \psi(0,z) = \chi(0,z) = v^3(z), & z \in [0,c], \\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0, \end{cases} \tag{1.3}$$

where $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex and non-empty values, and $\Omega \subset \mathbb{R}^n$ is an open subset.

Under suitable assumptions, we prove an existence theorem for a local solution of the Darboux Problem (1.1)-(1.2) and that the set of its solutions is compact in Banach space $C(D_0;\mathbb{R}^n)$, $D_0 = [0,x_0] \times [0,y_0] \times [0,z_0] \subset D$; moreover, as a function of the initial values, this set defines an upper-semicontinuous multifunction.

This study was suggested by several papers which deal with the Darboux Problem for third order hyperbolic equations [4], [5], [8]-[11], [13]-[15], [17], [21], [23], [25]-[26].

2 Preliminaries

The definitions and Theorem 2.1. in this section are taken from [1], [6]-[8], [18]-[20], [22].

Definition 2.1. *Let X and Y be two non-empty sets. A* ***multifunction*** $\Phi : X \to 2^Y$ *is a function from X into the family of all non-empty subsets of Y.*

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup_{x\in X} \Phi(x)$ is the ***range*** of Φ.

Definition 2.2. *Let us consider $\Phi : X \to 2^Y$.*

a) *If $A \subset X$, the* ***image*** *of A by Φ is $\Phi(A) = \bigcup_{x\in A} \Phi(x)$;*

b) *If $B \subset Y$, the* ***counterimage*** *of B by Φ is*

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) *The graph of Φ, denoted* ***graph*** *Φ, is the set*

$$graph\ \Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. *Let us now take $\Phi : X \to 2^X$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a* ***fixed point*** *of the multifunction Φ.*

Definition 2.4. *A univalued function $\varphi : X \to Y$ is said to be a* ***selection*** *of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.*

Definition 2.5. *Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is* ***upper-semicontinuous*** *if, for any closed subset $B \subset Y$, $\Phi^-(B)$ is closed in X.*

Definition 2.6. *If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is* ***measurable*** *if $\Phi^-(B) \in \mathcal{F}$ for every closed subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.*

Theorem 2.1. [22]. *Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

(i) *the multifunction Φ is upper-semicontinuous;*

(ii) *the graph of Φ is a closed subset of $X \times Y$;*

(iii) *any would be the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$, $y_n \to y$, it follows $y \in \Phi(x)$.*

Definition 2.7. [6], [7]. *The function* $u : \Delta \to \mathbb{R}^n$, $\Delta \subset \mathbb{R}^2$, *is* ***absolutely continuous in Carathéodory's sense*** *[1, §565 - §570] iff* $u(x,y)$ *is continuous on* Δ, *absolutely continuous in* x *(for any* y*), absolutely continuous in* y *(for any* x*),* $u_x(x,y)$ *is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in* y *(for any* x*) and* u_{xy} *is Lebesgue-integrable on* Δ.

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\Delta;\mathbb{R}^n)$ [6], [7].

Definition 2.8. [1], [8]. *The function* $u : D \subset \mathbb{R}^3 \to \mathbb{R}^n$, *is* ***absolutely continuous in Carathéodory's sense*** *[1,§565 - §570] if and only if* $u(x,y,z)$ *is continuous on* D, *absolutely continuous in each variable (for any pair of the other two variables), and - similarly -* $u_x(x,y,z)$, $u_y(x,y,z)$, $u_z(x,y,z)$, $u_{xy}(x,y,z)$, $u_{yz}(x,y,z)$, $u_{xz}(x,y,z)$, *and* u_{xyz} *is Lebesgue integrable.*

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(D;\mathbb{R}^n)$, [8].

3 Results

In a similar way as in [2], [24], we define the notion of a ***local solution*** for the Darboux Problem (1.1)+(1.2) and we prove an existence theorem for a local solution of this problem, together with some properties of the set of its solutions, namely that this is a compact subset in Banach space $C(D_0;\mathbb{R}^n)$ and, as a function of initial values, it defines an upper-semicontinuous multifunction.

Let the following hypotheses be satisfied:

(H_1) $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex, non-empty values in $\mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^n$ is an open subset.

(H_2) For any $(x,y,z) \in D$, the mapping $u \to F(x,y,z,u)$ is upper-semicontinuous on Ω.

(H_3) For any $u \in \Omega$ the mapping $(x,y,z) \to F(x,y,z,u)$ is Lebesgue-measurable on D.

(H_4) There exists a function $k : D \to \mathbb{R}_+$, $k \in \mathcal{L}^1(D;\mathbb{R}_+)$ such that

$$\|\zeta\| \le k(x,y,z), \quad \forall \zeta \in F(x,y,z,u), \quad \forall (x,y,z) \in D, \quad \forall u \in \Omega.$$

(H_5) The functions $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ are absolutely continuous in Carathéodory's sense and satisfy conditions (1.3);

Remark 1. The function $\alpha : D \to \mathbb{R}^n$ defined by

$$\begin{aligned} \alpha(x,y,z) &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) + \psi(0,0) = \\ &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0, \end{aligned} \tag{3.1}$$

is absolutely continuous in Carathéodory's sense on D, $\alpha \in C^*(D;\mathbb{R}^n)$ [1,§565 - §570].

Remark 2. Denote by $M \subset \Omega$ the convex compact set in which the function $\alpha : D \to \mathbb{R}^n$, defined by (3.1), takes its values for all $(x,y,z) \in D_0$.

Remark 3. Let $(x_0, y_0, z_0) \in]0, a] \times]0, b] \times]0, c]$ be a point such that

$$\int_0^{x_0} \int_0^{y_0} \int_0^{z_0} k(r, s, t)\, dr\, ds\, dt < d(M, C_\Omega),$$

where $d(M, C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$, an inequality immediately resulting from the integrability of function k.

Definition 3.1. *The* ***Darboux Problem*** *for the hyperbolic inclusion (1.1) means to determine a* ***solution*** *of this inclusion which satisfies the initial conditions (1.2).*

Definition 3.2. *A* ***local solution*** *of Darboux Problem (1.1)+(1.2) is defined as a function* $U : D_0 \to \Omega$, $U \in C^*(D_0; \mathbb{R}^n)$ *absolutely continuous in Carathéodory's sense [1], which satisfies (1.1) a.e. for* $(x, y, z) \in D_0$*, and also initial conditions (1.2) for all* $(x, y) \in [0, x_0] \times [0, y_0]$*, all* $(y, z) \in [0, y_0] \times [0, z_0]$*, all* $(x, z) \in [0, x_0] \times [0, z_0]$.

Theorem 3.1. *Let the hypotheses* (H_1)*-*(H_5) *be satisfied. Then:*

(i) *there exists at least a local solution* U *of the Darboux Problem (1.1)+(1.2);*

(ii) *the set* S_α *of the local solutions* U *is compact in the Banach space* $C(D_0; \mathbb{R}^n)$*;*

(iii) *the multifunction* $\alpha \to S_\alpha$ *is upper-semicontinuous on* $C^*(D_0; \mathbb{R}^n)$*, taking values in* $C(D_0; \mathbb{R}^n)$.

Proof. (i) Let $C^*(D_0; \mathbb{R}^n)$ be the set of absolutely continuous functions in Carathéodory's sense defined on D_0, with values in $\mathbb{R}^n$ [1]. We denote by $\mathcal{U}_M$ the set of functions $U : D_0 \to \mathbb{R}^n$, $U \in C^*(D_0; \mathbb{R}^n)$, which satisfy the inequality

$$\left\| \frac{\partial^3 U(x, y, z)}{\partial x \partial y \partial z} \right\| \leq k(x, y, z), \quad \text{a.e. for } (x, y, z) \in D_0, \tag{3.2}$$

and also conditions (1.2). The notation $\mathcal{U}_M$ is suitable because, by Remark 2, $\alpha(x, y, z) \in M$ for $(x, y, z) \in D_0$. We remark that the absolute continuity in Carathéodory's sense of the function U assures the existence of derivative $\dfrac{\partial^3 U(x, y, z)}{\partial x \partial y \partial z}$ a.e. for $(x, y, z) \in D_0$ [1, §565 - §570].

We have $\mathcal{U}_M \subset C^*(D_0; \mathbb{R}^n)$. Then, by Remark 3 and inequality (3.2), for any $U \in \mathcal{U}_M$, it follows that $U(x, y, z) \in \Omega$.

Indeed, integrating $\dfrac{\partial^3 U(x, y, z)}{\partial x \partial y \partial z}$ on D_{xyz}, where

$$D_{xyz} = \{(r, s, t) \mid 0 \leq r \leq x,\ 0 \leq s \leq y,\ 0 \leq t \leq z\}, \quad (x, y, z) \in D_0,$$

and using conditions (1.2), we obtain

$$\begin{aligned} U(x, y, z) &= U(x, y, 0) + U(x, 0, z) - U(x, 0, 0) + U(0, y, z) - U(0, y, 0) - \\ &\quad - U(0, 0, z) + U(0, 0, 0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 U(r, s, t)}{\partial r \partial s \partial t}\, dr\, ds\, dt = \end{aligned}$$

$$= \varphi(x,y)+\psi(y,z)+\chi(x,z)-\varphi(x,0)-\varphi(0,y)-\psi(0,z)+$$
$$+\psi(0,0)+\int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt=$$
$$= \varphi(x,y)+\psi(y,z)+\chi(x,z)-v^1(x)-v^2(y)-v^3(z)+$$
$$+v^0+\int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt=$$
$$= \alpha(x,y,z)+\int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt. \quad (3.3)$$

From the Remark 3 it results that

$$\|U(x,y,z)-\alpha(x,y,z)\| = \left\|\int_0^x\int_0^y\int_0^z \frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\,dr\,ds\,dt\right\| \le$$
$$\le \int_0^x\int_0^y\int_0^z \left\|\frac{\partial^3 U(r,s,t)}{\partial r\partial s\partial t}\right\|dr\,ds\,dt \le \int_0^x\int_0^y\int_0^z k(r,s,t)\,dr\,ds\,dt \le$$
$$\le \int_0^{x_0}\int_0^{y_0}\int_0^{z_0} k(r,s,t)\,dr\,ds\,dt < d(M,C_\Omega). \quad (3.4)$$

Hence, it follows

$$d(U(x,y,z),\alpha(x,y,z)) = \|U(x,y,z)-\alpha(x,y,z)\| < d(M,C_\Omega), \quad (3.5)$$

and, from the Remark 2 stating that $\alpha(x,y,z)\in M$ for $(x,y,z)\in D_0$, we conclude that $U(x,y,z)\in\Omega$.

The set of functions $\mathcal{U}_M$ is ***convex*** and ***compact*** in $C(D_0;\mathbb{R}^n)$. The convexity of $\mathcal{U}_M$ results by the definition of this set, and its compactness from the Arzelà-Ascoli theorem, using hypothesis (H_5), and Remarks 1-3.

We denote by $\mathcal{G}$ the set of the triples $(\alpha,U,V)\in C^*(D_0;\mathbb{R}^n)\times\mathcal{U}_M\times\mathcal{U}_M$ with the property that U and V satisfy the membership relation

$$\frac{\partial^3 V(x,y,z)}{\partial x\partial y\partial z}\in F(x,y,z,U(x,y,z)) \text{ a.e. for } (x,y,z)\in D_0. \quad (3.6)$$

We prove that, for each $\alpha\in C^*(D_0;\mathbb{R}^n)$ with $\alpha(x,y,z)\in M$ for $(x,y,z)\in D_0$, the set of those pairs (U,V) such that $(\alpha,U,V)\in\mathcal{G}$ is non-empty and the set $\mathcal{G}$ is closed.

Indeed, let us take $U\in\mathcal{U}_M$. By Theorem 1 [2], there exists a μ-measurable (under the μ-Lebesgue measure) multifunction $\Gamma : D_0\to 2^{\mathbb{R}^n}$ with compact, non-empty values in $\mathbb{R}^n$ such that

$$\Gamma(x,y,z)\subset F(x,y,z,U(x,y,z)),\ \ \forall(x,y,z)\in D_0. \quad (3.7)$$

Then, by Theorem 2 or Theorem 3 [3], there exists a measurable selection β of Γ, i.e. a measurable univalued function $\beta : D_0\to\mathbb{R}^n$ with $\beta(x,y,z)\in\Gamma(x,y,z)$ for $(x,y,z)\in D_0$.

Let the function $V : D_0\to\mathbb{R}^n$ be defined by

$$V(x,y,z)=\alpha(x,y,z)+\int_0^x\int_0^y\int_0^z \beta(r,s,t)\,dr\,ds\,dt. \quad (3.8)$$

Then, the set of those pairs (U, V) such that $(\alpha, U, V) \in \mathcal{G}$ is ***non-empty*** because

$$\beta(x,y,z) \in \Gamma(x,y,z) \subset F(x,y,z,U(x,y,z)), \text{ a.e. for } (x,y,z) \in D_0, \tag{3.9}$$

$$\frac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} = \beta(x,y,z) \in \Gamma(x,y,z) \subset F(x,y,z,U(x,y,z)), \text{ a.e. } (x,y,z) \in D_0, \tag{3.10}$$

$$\left\| \frac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} \right\| = \|\beta(x,y,z)\| \leq k(x,y,z), \quad (x,y,z) \in D_0, \tag{3.11}$$

by hypothesis (H_4) for $\zeta = \beta(x,y,z)$, and

$$\begin{cases} V(x,y,0) = \varphi(x,y), & (x,y) \in [0,x_0] \times [0,y_0], \\ V(0,y,z) = \psi(y,z), & (y,z) \in [0,y_0] \times [0,z_0], \\ V(x,0,z) = \chi(x,z), & (x,z) \in [0,x_0] \times [0,z_0], \end{cases} \tag{3.12}$$

For the proof that $\mathcal{G}$ is closed, we consider a sequence $\{(\alpha_n, U_n, V_n)\}_{n \in \mathbb{N}}$ of elements in $\mathcal{G}$, convergent to (α, U, V) in the space $C^*(D_0;\mathbb{R}^n) \times C(D_0;\mathbb{R}^n) \times L^1(D_0;\mathbb{R}^n)$. We must check that $(\alpha, U, V) \in \mathcal{G}$, what implies, by the definition of set $\mathcal{G}$, that conditions (1.2) and (3.10) are satisfied by U and V.

The set $\left\{ \frac{\partial^3 V_n(x,y,z)}{\partial x \partial y \partial z} \right\}_{n \in \mathbb{N}}$ is ***relatively weakly compact*** in $L^1(D_0;\mathbb{R}^n)$ by the Dunford-Pettis Criterion [12]. It follows that $\left\{ \frac{\partial^3 V_n(x,y,z)}{\partial x \partial y \partial z} \right\}_{n \in \mathbb{N}}$ is weakly convergent to a function $W \in L^1(D_0;\mathbb{R}^n)$. For each $(x,y,z) \in D_0$, we have

$$\begin{aligned} V(x,y,z) &= w - \lim_{n\to\infty} V_n(x,y,z) = \\ &= w - \lim_{n\to\infty} \left[\alpha_n(x,y,z) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 V_n(r,s,t)}{\partial r \partial s \partial t}\, dr\, ds\, dt \right] = \\ &= \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z W(r,s,t)\, dr\, ds\, dt. \end{aligned} \tag{3.13}$$

From the weak convergence $\frac{\partial^3 V_n(x,y,z)}{\partial x \partial y \partial z} \rightharpoonup W(x,y,z)$, $(x,y,z) \in D_0$, using the Corollary of Mazur's Theorem [16], it follows that there exists a sequence of convex combinations $\{X_n\}_{n \in \mathbb{N}}$ of the set $\left\{ \frac{\partial^3 U_n}{\partial x \partial y \partial z}, \frac{\partial^3 U_{n+1}}{\partial x \partial y \partial z}, \dots \right\}$, strongly convergent to W in $L^1(D_0;\mathbb{R}^n)$. Then, we can extract a subsequence from the sequence $\{X_n\}_{n \in \mathbb{N}}$, which converges a.e. to $W : X_{n_i} \to W$ a.e. for $(x,y,z) \in D_0$.

Since $F(x,y,z,U)$ is convex and compact for all $(x,y,z) \in D$ and for all $U \in \Omega$, we obtain from the previous results and from Lemma 2 [2] that

$$\begin{aligned} W(x,y,z) &\in \bigcap_{l=1}^{\infty} \overline{conv\left(\bigcup_{n=l}^{\infty} \frac{\partial^3 V_n(x,y,z)}{\partial x \partial y \partial z} \right)} \subset \bigcap_{l=1}^{\infty} \overline{conv\left(\bigcup_{n=l}^{\infty} F\left(x,y,z,U_n(x,y,z)\right) \right)} \subset \\ &\subset F\left(x,y,z,U(x,y,z)\right), \text{ a.e. for} (x,y,z) \in D_0, \end{aligned} \tag{3.14}$$

from which it follows that $\mathcal{G}$ ***is closed***.

Indeed, (3.14) shows that $W(x, y, z) \in F(x, y, z, U(x, y, z))$ a.e. for $(x, y, z) \in D_0$, and we obtain $\dfrac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} = W(x, y, z)$ from (3.13); then, using (3.3) and (3.14) we have

$$W(x, y, z) = \frac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} \in F(x, y, z, U(x, y, z)) \text{ a.e. for } (x, y, z) \in D_0, \tag{3.15}$$

and also (3.12), hence V satisfies the initial conditions (1.2) for $(x, y, z) \in D_0$.

Let us take $\alpha \in C^*(D_0; \mathbb{R}^n)$ with $\alpha(x, y, z) \in M$ for $(x, y, z) \in D_0$. To each $U \in \mathcal{U}_M$ we associate the set $\Phi(U) \subset \mathcal{U}_M$ as follows:

$$V \in \Phi(U) \Leftrightarrow V \in \mathcal{U}_M,\ \frac{\partial^3 V(x,y,z)}{\partial x \partial y \partial z} \in F(x, y, z, U(x, y, z)), \tag{3.16}$$

a.e. for $(x, y, z) \in D_0$. We thus define a multifunction $\Phi : \mathcal{U}_M \to 2^{\mathcal{U}_M}$. The set $\Phi(U)$ is ***convex***, ***compact*** and ***non-empty***. It can be seen that $\Phi(U)$ is convex since $F(x, y, z, U(x, y, z))$ is convex by hypothesis (H_1). We have $\Phi(U) \subset \mathcal{U}_M$ but $\mathcal{U}_M$ is compact. The multifunction Φ has a closed graph because *graph* Φ is the set $\mathcal{G}$ for each fixed α and $\mathcal{G}$ is closed. It follows that $\Phi(U)$ is compact in $C(D_0; \mathbb{R}^n)$ as a closed subset of the compact set $\mathcal{U}_M$. The set $\Phi(U)$ is non-empty since there exists V, defined by (3.8), with the property $V \in \Phi(U)$.

The multifunction $\Phi : \mathcal{U}_M \to 2^{\mathcal{U}_M}$, having a closed graph, is upper-semicontinuous by Theorem 2.1. Taking into account all the properties of Φ, the Kakutani-Ky Fan fixed point Theorem [12], [22] can be applied. Indeed, $\Phi : \mathcal{U}_M \to 2^{\mathcal{U}_M}$ is defined on $\mathcal{U}_M$ which is a convex, compact and non-empty set; it is also upper-semicontinuous and its set-values $\Phi(U)$ are convex, closed and non-empty in $\mathcal{U}_M$. From Kakutani-Ky Fan fixed point Theorem it follows that the multifunction Φ has at least a fixed point, i.e. there exists at least an element $\mathcal{U} \in \mathcal{U}_M$ such that $U \in \Phi(U)$, hence $U = V$; but V is of the form (3.8), therefore this fixed point U is a solution of Darboux Problem (1.1)+(1.2).

(ii) We denote by S_α the set of solutions to problem (1.1)+(1.2), a notation showing that any solution U depends on the function α defined by (3.1). The set S_α contains at least an element. The set S_α is ***compact***, ***non-empty*** in the Banach space $C(D_0; \mathbb{R}^n)$, being the set of the fixed points of multifunction Φ.

iii) The graph $\mathcal{H}$ of the multifunction $\alpha \to S_\alpha$, defined on $C^*(D_0; \mathbb{R}^n)$ with values in $2^{\mathcal{U}_M}$, $S_\alpha \subset \Phi(\mathcal{U}_M) \subset 2^{\mathcal{U}_M}$, is closed in $C^*(D_0; \mathbb{R}^n) \times \mathcal{U}_M$ since $\mathcal{H}$ is the image of the compact set $\mathcal{H}_1$ of the triples $(\alpha, U, V) \in \mathcal{G}$ with $U = V$ through the projection mapping $(\alpha, U, V) \to (\alpha, U)$. The mapping S_α is $-$ in general $-$ a multifunction because several solutions of the problem (1.1)+(1.2) can exist, which are fixed points of mapping Φ corresponding to the same function α. Because the mapping $\alpha \to S_\alpha$ has a closed graph $\mathcal{H}$ by Theorem 2.1, it follows that $\alpha \to S_\alpha$ is upper-semicontinuous on $C^*(D_0; \mathbb{R}^n)$, what completes the proof. ■

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