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Some Remarks on the Coupling of Dirichlet and Neumann Problems

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Abstract. In this note we are concerned by the coupling of Dirichlet and Neumann problems in the one-dimensional case. Indeed, we point out an absorption phenomena between the Dirichlet problem and the Neumann one through their sum form. We shall also show that the Robin problem is ill-posed in this particular case.

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1 Introduction

Let Ω be an open subset of $\mathbb{R}$. In the Hilbert space $\mathbf{H} = L^2(\Omega)$, consider the Dirichlet and the Neumann operators associated to the Dirichlet and the Neumann problems respectively, for the Laplace operator, that is, operators given respectively as, see, e.g., [3, 4]:

$$D(A_D) = \{u \in \mathbf{H^2}(\Omega) : \ u(x) = 0 \text{ on } \partial\Omega\}, \quad A_D u = -u'' \ \forall u \in D(A_D) \tag{1}$$

and,

$$D(A_N) = \{u \in \mathbf{H^2}(\Omega) : \ u'(x) = 0 \text{ on } \partial\Omega\}, \quad A_N u = -u'' \ \forall u \in D(A_N) \tag{2}$$

It is obvious to see that both A_D and A_N are unbounded self-adjoint operators in the Hilbert space $\mathbf{H} = L^2(\Omega)$. Let $0 < \theta < 1$, throughout this note we shall denote, $B_\theta = \theta A_N \oplus (1-\theta)A_D$, and $A_\theta = \theta A_N + (1-\theta)A_D$, the sum form, and the algebraic sum of θA_N with $(1-\theta)A_D$ respectively, where $\oplus$ denotes the sum form symbol. In the same way, it is obvious to see that fractional powers of both A_N with A_D are well-defined (both A_N with A_D are self-adjoint operators). In particular, we have, $D(A_D^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega)$, $D(A_N^{\frac{1}{2}}) = \mathbf{H}^1(\Omega)$, and $D(A_D^{\frac{1}{2}}) \cap D(A_N^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega)$. Thus, $D(A_\theta) = \mathbf{H}_0^2(\Omega)$, and $D(B_\theta^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega)$. In this note, we are concerned by the coupling problem of the Dirichlet and Neumann problems in the one-dimensional case. Such a coupling problenm is called the the Robin problem, see, e.g., [4]. Thus, we shall show that the operator, A_θ, given by the Robin problem is a nonnegative symmetric operator, and that it is not essentially-self-adjoint, see, e.g., [2]. In other words, the problem given by the Robin problem does not have a unique solution, in this particular case . Therefore, the Robin problem, in this particular case is ill-posed. We will also show an absoption phenomena through the sum form given by B_θ, that is, A_D absorbs A_N.

2 Coupling problem

As we stated in the introduction, we will focus on two points. The first one is to prove that the Robin problem is ill-posed. The second one is to point out an absorption phenomena, that is, "Dirichlet" absorbs "Neumann".

Proposition 2.1. *Let $0 < \theta < 1$, and consider the operator, $A_\theta = \theta A_N + (1-\theta)A_D$, given as above. Then A_θ is a nonnegative symmetric operator in $\mathbf{H} = L^2(\Omega)$. In addition, A_θ is not essentially self-adjoint.*

Proof. Let us first show that A_θ is a nonnegative symmetric operator. In fact,

$$A_\theta u = -u'' \ \forall u \in D(A_\theta) = \mathbf{H}_0^2(\Omega).$$

Let $u \in D(A_\theta)$, then

$$< A_\theta u, u >= \int_\Omega -u''(x)u(x)dx = \int_\Omega (u')^2(x)dx \geq 0.$$

In the same way, $\forall u, v \in D(A_\theta) = \mathbf{H}_0^2(\Omega)$, we have,

$$< A_\theta u, v >= \int_\Omega -u''(x)v(x)dx =< u, A_\theta v >$$

thus, A_θ is symmetric. Now, it is clear that A_θ admits at leat two distinct self-adjoint extensions, that is, both A_D, and A_N. Therefore A_θ is not essentially self-adjoint. The proof it complete. ■

Consequence 2.2. *The Robin problem, given by the operator A_θ does not have a unique solution. Therefore it is ill-posed.*

Proposition 2.3. *Let $0 < \theta < 1$, and consider the self-adjoint operator, B_θ as above. Then, B_θ does not depend on θ, and $B_\theta = A_D$.*

Proof. Let $0 < \theta < 1$. Thus, the sesqulinear form associated to B_θ is given as, see, e.g., [1, 2]:

$$\Phi_\theta(u, v) = \theta \int_\Omega u'(x)\overline{v'}(x)dx + (1-\theta)\int_\Omega u'(x)\overline{v'}(x)dx \quad \forall u, v \in \mathbf{H}_0^1(\Omega)$$

Hence,

$$\Phi_\theta(u, v) = \int_\Omega u'(x)\overline{v'}(x)dx, \quad u, v \in \mathbf{H}_0^1(\Omega)$$

Therefore,

$$\Phi_\theta(u, v) = \int_\Omega -u''(x)\overline{v}(x)dx, \quad u \in D(A_D), v \in \mathbf{H}_0^1(\Omega)$$

Since, $\Phi_\theta(u, v) =< A_\theta u, v > \ \ \forall u \in D(B_\theta), \ \ v \in D(\Phi_\theta) = D(B_\theta^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega)$. It follows that, B_θ does not depend on θ. Now, let Ψ be the sesquilinear form associated to A_D. Then, $\Psi(u, v) =< A_D u, v > \ \ \forall u \in D(A_D), \ \ v \in (\Psi) = D(A_D^{\frac{1}{2}}) = \mathbf{H}_0^1(\Omega)$. Therefore, $\Phi_\theta \equiv \Psi$. Thus, $B_\theta = A_D$. The proof is complete. ■

Remark 2.4. *As we described in the introduction, there is an absorption phenomena allowed by the process of the sum form. As input, we have, A_D, and A_N, and then we consider the sum form process given as, $B_\theta = \theta A_N \oplus (1-\theta)A_D$. As output, we get, $B_\theta = A_D$.*

The author is not aware about possible physical significations of such a phenomena. In fact we get a similar result, if we consider the operator, $B_{\alpha,\beta} = \alpha A \oplus \beta B$ *where,* $\alpha, \beta > 0$*, and* $\alpha + \beta = 1$.

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