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Stable Range in Unitary Modules

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Abstract. The concept of stable range in commutative rings with identity is naturally extended to unitary R-modules. Let A be a unitary R-module and $J(A)$ the Jacobson radical of A. It is shown that any unimodular sequence of elements in A that meets $J(A)$ is stable. For any submodule $B \subseteq J(A)$ of a finitely generated R-module A, A is n-stable if and only if A/B is n-stable. If A is a finitely generated n-stable R-module, then $1 \leq \text{rank}(A) \leq n$. A torsion-free cyclic R-module is strongly n-stable (resp., n-stable) if and only if R is a strongly n-stable (resp., an n-stable) ring. The homomorphic image of a strongly n-stable R-module is strongly n-stable. The homomorphic image of a cyclic n-stable R-module is also n-stable.

1 Preliminaries

All rings considered (unless otherwise indicated) are commutative rings with identity and all modules are unitary modules. For any R-module A, $J(A)$ the Jacobson radical of A is defined to be the intersection of all maximal submodules of A. If A has no maximal submodules, then we set $J(A) = A$. An element u of an R-module A is said to be a unit provided that u is not contained in any maximal submodule of A. A minimal generating set of an R-module A is a subset X of A such that $\langle X \rangle = A$ and no proper subset of X spans A. For a finitely generated R-module A, we say that A is of rank m (m a positive integer) if A has a minimal generating set of m elements and does not have a minimal generating set of fewer than m elements. For any integer $n \geq 1$, a sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ of elements of an R-module A is said to be a unimodular sequence whenever the submodule $\langle a_1, a_2, \ldots, a_n, a_{n+1} \rangle = A$. A sequence $a_1, a_2, \ldots, a_n, a_{n+1}$ of elements of A is said to be stable whenever $\langle a_1, a_2, \ldots, a_n, a_{n+1} \rangle = \langle a_1 + r_1 a_{n+1}, a_2 + r_2 a_{n+1}, \ldots, a_n + r_n a_{n+1} \rangle$ for some $r_1, r_2, \ldots, r_n \in R$. For any fixed integer $n \geq 1$, A is said to be strongly n-stable (resp., n-stable) provided that any (resp., unimodular) sequence of elements in A of size larger than n is stable. For convenience, a strongly n-stable (resp., n-stable) module is called strongly stable (resp., stable) whenever $n = 1$. It follows that if A is a strongly n-stable (resp., an n-stable) module, then also A is strongly m-stable (resp., m-stable) for any fixed integer $m \geq n$. For a detailed study of stable range in commutative rings, see [1], [4], [5], [7].

The following first results are obtained from [3] and we state here for the sake of reference.

Result 1.1. *In a finitely generated R-module A, every proper submodule of A is contained in a maximal submodule of A.*

Proof. See Theorem 1.2 in [3]. The result is easily obtained simply by using an argument quite similar to the standard proof given for maximal ideals in a ring with identity. □

Result 1.2. *Let A be a finitely generated R-module. Then $J(A) = A$ if and only if $A = \langle 0 \rangle$.*

Proof. See Corollary 1.3 in [3]. □

Result 1.3. Let $B \subseteq J(A)$ be a submodule of a finitely generated R-module A. If $A/B = \langle \{a_i + B\}_{i \in I} \rangle$ for an arbitrary index set I, then $A = \langle \{a_i\}_{i \in I} \rangle$.

Proof. See Lemma 2.2 in [3]. □

Result 1.4. *In a finitely generated R-module A, $u \in A$ is a unit if and only if $\langle u \rangle = A$.*

Proof. See Theorem 1.4 in [3]. □

Result 1.5. *Let A be an R-module (not necessarily finitely generated) such that A has a unit. Then $x \in J(A)$ if and only if $u - rx$ is a unit in A for any element $r \in R$ and any unit u in A.*

Proof. See Theorem 1.6 in [3]. □

Theorem 1.1. *In a finitely generated R-module A, any unimodular sequence $a_1, a_2, \ldots, a_n, a_{n+1} \in A$ is stable whenever $\{a_1, a_2, \ldots, a_n, a_{n+1}\} \cap J(A) \neq \emptyset$. More precisely if $a_i \in J(A)$ for some $1 \leq i \leq n$, then $\langle a_1, \ldots, a_{i-1}, a_i + a_{n+1}, a_{i+1}, \ldots, a_n \rangle = A$ and for the case $i = n+1$,*

$$\langle a_1 + a_{n+1}, a_2, \ldots, a_n \rangle = \langle a_1, a_2 + a_{n+1}, \ldots, a_n \rangle = \cdots = \langle a_1, a_2, \ldots, a_n + a_{n+1} \rangle = A.$$

Proof. Without loss of generality, assume $a_1 \in J(A)$. Hence if $\langle a_1 + a_{n+1}, a_2, \ldots, a_n \rangle \neq A$, then by Result 1.1 there exists a maximal submodule M of A such that $\langle a_1 + a_{n+1}, a_2, \ldots, a_n \rangle \subseteq M$ which implies $A \subseteq M$ and this contradicts the maximality of M. □

Theorem 1.2. *For any fixed integer $n \geq 1$, A is strongly n*-stable *if and only if any sequence of size $n+1$ is stable.*

Proof. A proof by induction is given for the sufficient part. Assume, $a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}$ is a sequence in A. Thus, $a_{n+2} \in \langle a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2} \rangle$ implies $a_{n+2} = \sum_{i=1}^{n+2} a_i x_i = \sum_{i=1}^{n} a_i x_i + l$ for some $x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2} \in R$ and $l = a_{n+1}x_{n+1} + a_{n+2}x_{n+2}$. Consequently, $a_{n+2} \in \langle a_1, a_2, \ldots, a_n, l \rangle$ and for appropriate $r_1, r_2, \ldots, r_n \in R$,

$$\begin{aligned} a_{n+2} &\in \{a_1 + r_1 l, a_2 + r_2 l, \ldots, a_n + r_n l\} \subseteq \\ &\subseteq \langle a_1 + r_1 x_{n+2} a_{n+2}, a_2 + r_2 x_{n+2} a_{n+2}, \ldots, a_n + r_n x_{n+2} a_{n+2}, a_{n+1} + 0 a_{n+2} \rangle . \end{aligned}$$

Theorem 1.3. *For a fixed integer $n \geq 1$, a cyclic R-module is n-stable if and only if any unimodular sequence of size $n+1$ is stable.*

Proof. Proof by induction on the number of the generators of A. Let $A = \langle a \rangle$ be a cyclic R-module and $a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}$ a unimodular sequence of A. Thus, by assuming $a \in \langle a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2} \rangle$, the pattern of the proof is parallel to the argument in the above theorem. □

Example. For any integer $n \geq 1$, it is clear that if $\{a_1, a_2, \ldots, a_n, a_{n+1}\}$ is a stable minimal generating set of a nontrivial R-module A, then for appropriate $r_1, r_2, \ldots, r_n \in R$, $\{a_1 + r_1 a_{n+1}, a_2 + r_2 a_{n+1}, \ldots, a_n + r_n a_{n+1}\}$ is also a minimal generating set of A. Consequently, a finitely generated R-module of rank $m \geq 2$ cannot be $(m-1)$-stable. Also, by virtue of Theorem 1.1, none of the elements of a minimal generating set of size m of an R-module A of rank m belongs to $J(A)$. Actually, Corollary 2.5 of [3] states that every element of any minimal generating set of a finitely generated R-module A lies outside $J(A)$.

Remark. In the above example, it is clear that for at least one $1 \leq i \leq n$, $r_i \ni \operatorname{ann}(A)$.

Theorem 1.4. *For any integer $m \geq 2$ if A is a finitely generated $(m-1)$-stable R-module, then $1 \leq \operatorname{rank}(A) \leq m-1$.*

Proof. The proof is an immediate consequence of the above example and the fact that any n-stable R-module is also m-stable for any integer $m \geq n$. □

Remark. From the above result, it is clear that a finitely generated stable module must be cyclic. Also, in the next section, we will show that the converse of this statement is not valid in general.

2 Some Basic Algebraic Properties

Theorem 2.1. *The homomorphic image of a strongly n-stable R-module is strongly n-stable.*

Proof. Let B be a submodule of a strongly n-stable R-module A and $a_1 + B, a_2 + B, \ldots, a_n + B, a_{n+1} + B$ a sequence in A/B. Thus,

$$a_{n+1} = \sum_{i=1}^{n+1} r_i a_i + b = \sum_{i=1}^{n} r_i a_i + r_{n+1} a_{n+1} + b = \sum_{i=1}^{n} r_i a_i + l$$

for some $r_1, r_2, \ldots, r_n, r_{n+1} \in R$ and $b \in B$ with $l = r_{n+1} a_{n+1} + b$. Consequently, for appropriate $s_1, s_2, \ldots, s_n \in R$, $a_{n+1} \in \langle a_1 + s_1 l, a_2 + s_2 l, \ldots, a_n + s_n l \rangle$ implies

$$a_{n+1} + B \in \langle a_1 + s_1 r_{n+1} a_{n+1} + B, a_2 + s_2 r_{n+1} a_{n+1} + B, \ldots, a_n + s_n r_{n+1} a_{n+1} + B \rangle .$$

□

Theorem 2.2. *Let $B \subseteq J(A)$ be a submodule of a finitely generated R-module A. Then A is n-stable if and only if A/B is n-stable.*

Proof. The proof can be followed directly from the definition and Result 1.3 above. □

Theorem 2.3. *The homomorphic image of an n-stable cyclic R-module is n-stable.*

Proof. Let B be a submodule of an n-stable cyclic R-module $A = \langle a \rangle$ and $a_1 + B$, $a_2 + B, \ldots, a_n + B, a_{n+1} + B$ a unimodular sequence in A/B. Hence, for appropriate

$r_1, r_2, \ldots, r_n, r_{n+1} \in R$, $a+B = \sum_{i=1}^{n+1} r_i a_i + B$ implies $a = \sum_{i=1}^{n} r_i a_i + r_{n+1}a_{n+1} + b$ for some $b \in B$. Now, the rest of the proof is similar to the argument in Theorem 2.1 above. □

Theorem 2.4. *A torsion-free cyclic R-module is strongly n-stable (resp., n-stable) if and only if R is a strongly n-stable (resp., an n-stable) ring.*

Proof. The proof can be followed directly from the definition. Here, we just make an argument for the necessary part of the strongly n-stable case. Let $A = \langle a \rangle$ be a torsion-free cyclic R-module and $(r_1, r_2, \ldots, r_n, r_{n+1})$ an ideal of R. Therefore, $r_{n+1} \in (r_1, r_2, \ldots, r_n, r_{n+1})$ implies $r_{n+1} = \sum_{i=1}^{n+1} r_i s_i$ for some $s_1, s_2, \ldots, s_n, s_{n+1} \in R$. Thus, $r_{n+1}a = \sum_{i=1}^{n+1} r_i s_i a = \sum_{i=1}^{n+1} s_i a_i$ where $a_i = r_i a$. Consequently, for appropriate $t_1, t_2, \ldots, t_n \in R$,

$$r_{n+1}a \in \langle a_1, a_2, \ldots, a_n, a_{n+1} \rangle = \langle a_1 + t_1 a_{n+1}, a_2 + t_2 a_{n+1}, \ldots, a_n + t_n a_{n+1} \rangle$$

which implies $r_{n+1} \in (r_1 + t_1 r_{n+1}, r_2 + t_2 r_{n+1}, \ldots, r_n + t_n r_{n+1})$. □

Example. In Theorem 3.4 of [2], it is shown that any n-dimensional commutative integral domain (resp.,ring) is $N = 1$- (resp., $n + 2$-) stable. Thus, by applying the above theorem, it is clear that any torsion-free cyclic R-module over an n-dimensional integral domain (resp., ring) R is $n + 1$- (resp., $n + 2$-) stable.

Example. In Theorem 4 of [6], it is shown that $R[X]$ the ring of polynomials is not stable for any commutative ring R. Hence, by virtue of Theorem 2.4, we can conclude that a torsion-free cyclic $R[X]$-module is never a stable module.

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