

Some Sequences of Type ∞^0 , All with Limit 1

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Abstract. In this Note we will prove the Propositions 1-4. All sequences which appear in the conclusions of these Propositions are of type ∞^0 and with limit 1. Several examples are given.

Everywhere in this Note we suppose that $\lambda_n \geq 0$ for all $n \in \{1, 2, \dots\}$ and that the sequence $\lambda_{n+1} - \lambda_n$ is bounded. (For example, $\lambda_n = \sqrt{n}$, ${}^{2k+1}\sqrt{n}$ with $k \in \{1, 2, \dots\}$, $\ln n$, and any convergent sequence of nonnegative numbers.)

Proposition 1. *Suppose that $f : (0, 1] \rightarrow (0, 1]$ is increasing, that exists $a \in (0, 1]$ such that $f(x) \leq ax$ for all $x \in (0, 1]$, and that exists $b > 1$ such that $\exists \lim_{x \rightarrow 0} \frac{ax - f(x)}{x^b} \in [0, \infty)$. Then the sequence*

$$x_n = \left(f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \dots + f\left(\frac{1}{(2n)^p}\right) + \lambda_n \right)^{1/n}$$

has the limit 1 for every $p \in \{2, 3, \dots\}$.

Proof. First step. We show that the sequence

$$\alpha_n = f(1) + f\left(\frac{1}{2}\right) + \dots + f\left(\frac{1}{n}\right) - a \ln n$$

is convergent. To this end we consider the sequences $\beta_n = a \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \right)$ which is known to be convergent to $a\gamma$ where γ is Euler's constant. We have

$$\beta_n - \alpha_n = (a1 - f(1)) + \left(a\frac{1}{2} - f\left(\frac{1}{2}\right) \right) + \dots + \left(a\frac{1}{n} - f\left(\frac{1}{n}\right) \right) = s_n$$

the sequence of the partial sums of the series $\sum_{n=1}^{\infty} \left(a\frac{1}{n} - f\left(\frac{1}{n}\right) \right)$ with $a\frac{1}{n} - f\left(\frac{1}{n}\right) \geq 0$ $\forall n \in \{1, 2, \dots\}$. By the hypothesis, $\exists b > 1$ such that $\exists \lim_{n \rightarrow \infty} \frac{a\frac{1}{n} - f\left(\frac{1}{n}\right)}{\frac{1}{n^b}} \in [0, \infty)$. By

the comparison test, the series $\sum_{n=1}^{\infty} \left(a\frac{1}{n} - f\left(\frac{1}{n}\right) \right)$ is convergent, hence $s_n = \beta_n - \alpha_n$ is convergent. So α_n is convergent.

Second step. We show that $\lim_{x \rightarrow 0} \frac{f(x)}{x} = a$. We have

$$\frac{f(x)}{x} = \frac{f(x) - ax + ax}{x} = -\frac{ax - f(x)}{x} + a = -\frac{ax - f(x)}{x^b} x^{b-1} + a.$$

So $\exists \lim_{x \rightarrow 0} \frac{f(x)}{x} = a$.

Third step.

$$\begin{aligned} a_n &= f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{(2n)^p}\right) + \lambda_n = \\ &= f(1) + f\left(\frac{1}{2}\right) + \cdots + f\left(\frac{1}{n}\right) + f\left(\frac{1}{n+1}\right) + \cdots + f\left(\frac{1}{(2n)^p}\right) - \\ &\quad - a \ln(2n)^p + a \ln(2n)^p - \left(f(1) + f\left(\frac{1}{2}\right) + \cdots + f\left(\frac{1}{n}\right)\right) + \\ &\quad + a \ln n - a \ln n + \lambda_n \rightarrow \infty \end{aligned}$$

since $a \ln(2n)^p - a \ln n = a \ln 2^p + a(p-1) \ln n \rightarrow \infty$. $x_n = e^{\ln x_n} = e^{\frac{\ln a_n}{n}}$. We investigate

$$\begin{aligned} &\frac{\ln a_{n+1} - \ln a_n}{(n+1) - n} = \ln \frac{a_{n+1}}{a_n} = \\ &= \ln \frac{f\left(\frac{1}{n+2}\right) + f\left(\frac{1}{n+3}\right) + \cdots + f\left(\frac{1}{(2n+2)^p}\right) + \lambda_{n+1} + f\left(\frac{1}{n+1}\right) - f\left(\frac{1}{n+1}\right) + \lambda_n - \lambda_n}{a_n} = \\ &= \ln \left(1 + \frac{f\left(\frac{1}{(2n)^p+1}\right) + f\left(\frac{1}{(2n)^p+2}\right) + \cdots + f\left(\frac{1}{(2n+2)^p}\right) - \frac{f\left(\frac{1}{n+1}\right)}{a_n} + \frac{\lambda_{n+1} - \lambda_n}{a_n}}{a_n} \right). \\ &0 \leq \frac{f\left(\frac{1}{(2n)^p+1}\right) + f\left(\frac{1}{(2n)^p+2}\right) + \cdots + f\left(\frac{1}{(2n+2)^p}\right)}{a_n} \leq \\ &\leq \frac{1}{a_n} \frac{(C_p^1 2^p n^{p-1} + C_p^2 2^p n^{p-2} + \cdots + 2^p) f\left(\frac{1}{(2n)^p+1}\right)}{((2n)^p + 1) \frac{1}{(2n)^p+1}} \rightarrow 0. \\ &\frac{f\left(\frac{1}{n+1}\right)}{a_n} = \frac{f\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}} \frac{1}{(n+1)a_n} \rightarrow 0. \end{aligned}$$

So $\lim_{n \rightarrow \infty} \frac{\ln a_n}{n} = 0$ and thus $\lim x_n = 1$. \square

Remarks. 1. If $c \in (0, 1]$ then the functions $f(x) = cx, \sin cx, \operatorname{argsh} cx, \arctan cx, \tanh cx, \ln(1+cx)$ satisfy the hypotheses on f , with $a = c$ and $b = 2$.

2. If f and g satisfy the hypotheses of Proposition 1, with $a = 1$, then $g \circ f$ also satisfies them. Indeed, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{f(x)}{x} x\right) = 0$ and

$$\frac{x - g(f(x))}{x^b} = \frac{x - g(f(x)) + f(x) - f(x)}{x^b} = \frac{x - f(x)}{x^b} + \frac{f(x) - g(f(x))}{(f(x))^b} \left(\frac{f(x)}{x}\right)^b$$

and so $\lim_{x \rightarrow 0} \frac{x - g(f(x))}{x^b} \in [0, \infty)$.

Proposition 2. *Suppose that the function $f : (0, 1] \rightarrow (0, 1]$ has the following properties: f is increasing, exists $a \in (0, 1]$ such that $f(x) \leq ax$ for all $x \in (0, 1]$, and exists $b > 2$ such that $\exists \lim_{x \rightarrow 0} \frac{ax - f(x)}{x^b} \in [0, \infty)$. Then the sequence*

$$x_n = \left(f\left(\frac{1}{\sqrt{n+1}}\right) + f\left(\frac{1}{\sqrt{n+2}}\right) + \cdots + f\left(\frac{1}{\sqrt{(2n)^p}}\right) + \lambda_n \right)^{1/n}$$

has the limit 1 for every $p \in \{1, 2, \dots\}$.

Proof. First step. We show that the sequence

$$\alpha_n = f(1) + f\left(\frac{1}{\sqrt{2}}\right) + \cdots + f\left(\frac{1}{\sqrt{n}}\right) - a2\sqrt{n}$$

is convergent. To this end we consider the sequence

$$\beta_n = a \left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n} \right)$$

which is convergent to aI where I is Ioachimescu's constant, ([1]).

We have

$$\beta_n - \alpha_n = (a1 - f(1)) + \left(\frac{a}{\sqrt{2}} - f\left(\frac{1}{\sqrt{2}}\right) \right) + \cdots + \left(\frac{a}{\sqrt{n}} - f\left(\frac{1}{\sqrt{n}}\right) \right) = s_n$$

the sequence of the partial sums of the series $\sum_{n=1}^{\infty} \left(\frac{a}{\sqrt{n}} - f\left(\frac{1}{\sqrt{n}}\right) \right)$ with $\frac{a}{\sqrt{n}} - f\left(\frac{1}{\sqrt{n}}\right) \geq$

$0, \forall n \in \{1, 2, \dots\}$. By the hypothesis, $\exists b > 2$ such that $\exists \lim_{n \rightarrow \infty} \frac{\frac{a}{\sqrt{n}} - f\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{n^{b/2}}} \in [0, \infty)$.

Therefore the series is convergent, $s_n = \beta_n - \alpha_n$ is convergent, and thus α_n is convergent.

Second step. We show that $\lim_{x \rightarrow 0} \frac{f(x)}{x} = a$.

$$\frac{f(x)}{x} = \frac{f(x) - ax + ax}{x} = -\frac{ax - f(x)}{x} + a = -\frac{ax - f(x)}{x^b} x^{b-1} + a.$$

From here we see that $\lim_{x \rightarrow 0} \frac{f(x)}{x} = a$.

Third step.

$$\begin{aligned} a_n &= f\left(\frac{1}{\sqrt{n+1}}\right) + f\left(\frac{1}{\sqrt{n+2}}\right) + \cdots + f\left(\frac{1}{\sqrt{(2n)^p}}\right) + \lambda_n = \\ &= f(1) + f\left(\frac{1}{\sqrt{2}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + \cdots + f\left(\frac{1}{\sqrt{n}}\right) + f\left(\frac{1}{\sqrt{n+1}}\right) + \cdots + f\left(\frac{1}{\sqrt{(2n)^p}}\right) - \\ &\quad - a2\sqrt{(2n)^p} + a2\sqrt{(2n)^p} - \left(f(1) + f\left(\frac{1}{\sqrt{2}}\right) + \cdots + f\left(\frac{1}{\sqrt{n}}\right) \right) + 2a\sqrt{n} - 2a\sqrt{n} + \lambda_n \rightarrow \infty \end{aligned}$$

since

$$\begin{aligned} 2a\sqrt{(2n)^p} - 2a\sqrt{n} &= 2a \left(\sqrt{(2n)^p} - \sqrt{n} \right) = 2a \frac{(2n)^p - n}{\sqrt{(2n)^p} + \sqrt{n}} = \\ &= 2a \frac{n^p}{n^{p/2}} \frac{2^p - \frac{1}{n^{p-1}}}{2^{p/2} + \frac{1}{n^{(p-1)/2}}} \rightarrow \infty. \end{aligned}$$

$x_n = e^{\ln x_n} = e^{\frac{\ln a_n}{n}}$. We have

$$\begin{aligned} \frac{\ln a_{n+1} - \ln a_n}{(n+1) - n} &= \ln \frac{a_{n+1}}{a_n} = \\ &= \ln \frac{f\left(\frac{1}{\sqrt{n+2}}\right) + f\left(\frac{1}{\sqrt{n+3}}\right) + \cdots + f\left(\frac{1}{\sqrt{(2n+2)^p}}\right) + \lambda_{n+1} + f\left(\frac{1}{\sqrt{n+1}}\right) - f\left(\frac{1}{\sqrt{n+1}}\right) + \lambda_n - \lambda_n}{a_n} = \\ &= \ln \left(1 + \frac{f\left(\frac{1}{\sqrt{(2n)^p+1}}\right) + f\left(\frac{1}{\sqrt{(2n)^p+2}}\right) + \cdots + f\left(\frac{1}{\sqrt{(2n+2)^p}}\right)}{a_n} + \frac{\lambda_{n+1} - \lambda_n}{a_n} - \frac{f\left(\frac{1}{\sqrt{n+1}}\right)}{a_n} \right). \\ &0 \leq \frac{f\left(\frac{1}{\sqrt{(2n)^p+1}}\right) + f\left(\frac{1}{\sqrt{(2n)^p+2}}\right) + \cdots + f\left(\frac{1}{\sqrt{(2n+2)^p}}\right)}{a_n} \leq \\ &\leq \frac{(C_p^1 2^p n^{p-1} + C_p^2 2^p n^{p-2} + \cdots + 2^p) f\left(\frac{1}{\sqrt{(2n)^p+1}}\right)}{((2n)^p - n) f\left(\frac{1}{\sqrt{(2n)^p}}\right)} = \\ &= \frac{C_p^1 2^p n^{p-1} + C_p^2 2^p n^{p-2} + \cdots + 2^p}{(2n)^p - n} \frac{f\left(\frac{1}{\sqrt{(2n)^p+1}}\right)}{\frac{1}{\sqrt{(2n)^p+1}}} \frac{\frac{1}{\sqrt{(2n)^p}}}{f\left(\frac{1}{\sqrt{(2n)^p}}\right)} \frac{\sqrt{(2n)^p}}{\sqrt{(2n)^p+1}} \rightarrow 0. \\ &\frac{f\left(\frac{1}{\sqrt{n+1}}\right)}{a_n} = \frac{f\left(\frac{1}{\sqrt{n+1}}\right)}{\frac{1}{\sqrt{n+1}}} \frac{1}{\sqrt{n+1}a_n} \rightarrow 0. \end{aligned}$$

Thus $\frac{\ln a_n}{n} \rightarrow 0$ and $x_n \rightarrow 1$. \square

Remarks. 1. If $c \in (0, 1]$ then the functions $f(x) = cx$, $\sin cx$, $\operatorname{argsh} cx$, $\arctan cx$, $\tanh cx$ satisfy the hypotheses on f , with $a = c$ and $b = 3$.

2. If f and g satisfy the hypotheses of Proposition 2, with $a = 1$, then $g \circ f$ also satisfies them (what can be checked as in Proposition 1).

Proposition 3. *Suppose that the function $f : (0, a) \rightarrow \mathbb{R}$ has the following properties: it is increasing, $f(x) \geq x$ for all $x \in (0, a)$, and $\exists \lim_{x \rightarrow 0} \frac{f(x)}{x} \in [0, \infty)$. Then the sequence*

$$x_n = \left(f\left(\frac{1}{(n+1)^\alpha}\right) + f\left(\frac{1}{(n+2)^\alpha}\right) + \cdots + f\left(\frac{1}{((2n)^p)^\alpha}\right) + \lambda_n \right)^{1/n}$$

has the limit 1 for every $p \in \{1, 2, \dots\}$ and any $\alpha \in (0, 1)$.

Proof.

$$\begin{aligned} a_n &= f\left(\frac{1}{(n+1)^\alpha}\right) + f\left(\frac{1}{(n+2)^\alpha}\right) + \cdots + f\left(\frac{1}{((2n)^p)^\alpha}\right) + \lambda_n \geq \\ &\geq \frac{1}{(n+1)^\alpha} + \frac{1}{(n+2)^\alpha} + \cdots + \frac{1}{(2n)^{p\alpha}} \geq \\ &\geq \frac{(2n)^p - n}{(2n)^{p\alpha}} = n^{p(1-\alpha)} \frac{2^p - \frac{1}{n^{p-1}}}{2^{p\alpha}} \rightarrow \infty. \end{aligned}$$

$x_n = e^{\ln x_n} = e^{\frac{\ln a_n}{n}}$. We have

$$\begin{aligned} &\frac{\ln a_{n+1} - \ln a_n}{(n+1) - n} = \ln \frac{a_{n+1}}{a_n} = \\ &= \ln \frac{f\left(\frac{1}{(n+2)^\alpha}\right) + f\left(\frac{1}{(n+3)^\alpha}\right) + \cdots + f\left(\frac{1}{((2n+2)^p)^\alpha}\right) + f\left(\frac{1}{(n+1)^\alpha}\right) + \lambda_n + \lambda_{n+1} - \lambda_n}{a_n} = \\ &= \ln \left(1 + \frac{f\left(\frac{1}{((2n)^p+1)^\alpha}\right) + f\left(\frac{1}{((2n)^p+2)^\alpha}\right) + \cdots + f\left(\frac{1}{((2n+2)^p)^\alpha}\right) - \frac{f\left(\frac{1}{(n+1)^\alpha}\right)}{a_n} + \frac{\lambda_{n+1} - \lambda_n}{a_n}}{a_n} \right). \\ &0 \leq \frac{f\left(\frac{1}{((2n)^p+1)^\alpha}\right) + f\left(\frac{1}{((2n)^p+2)^\alpha}\right) + \cdots + f\left(\frac{1}{((2n+2)^p)^\alpha}\right)}{a_n} \leq \\ &\leq \frac{(C_p^1 2^p n^{p-1} + C_p^2 2^p n^{p-2} + \cdots + 2^p) f\left(\frac{1}{((2n)^p+1)^\alpha}\right)}{((2n)^p - n) \frac{1}{((2n)^p)^\alpha}} \rightarrow 0. \\ &0 \leq \frac{f\left(\frac{1}{(n+1)^\alpha}\right)}{a_n} = \frac{f\left(\frac{1}{(n+1)^\alpha}\right)}{\frac{1}{(n+1)^\alpha}} \frac{1}{(n+1)^\alpha a_n} \rightarrow 0. \end{aligned}$$

So $\frac{\ln a_n}{n} \rightarrow 0$ and $x_n \rightarrow 1$. \square

Remarks. 1. The functions $f(x) = x$, $\sinh x$, $\arcsin x$, $\tan x$, $\operatorname{argth} x$, satisfy the hypotheses of Properties 3 on f .

2. If $f : (0, a) \rightarrow (0, b)$ and $g : (0, b) \rightarrow \mathbb{R}$ have the properties in the hypotheses then $g \circ f : (0, a) \rightarrow \mathbb{R}$ has the same properties. Indeed,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{f(x)}{x} x \right) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{g(f(x))}{x} = \lim_{x \rightarrow 0} \frac{g(f(x))}{f(x)} \frac{f(x)}{x} \in [0, \infty).$$

Proposition 4. Suppose that $f : (0, a) \rightarrow \mathbb{R}$ satisfies the following properties: it is increasing, $f(x) \geq x$ for all $x \in (0, a)$, and $\lim_{x \rightarrow 0} \frac{f(x)}{x} \in [0, \infty)$. Then the sequence

$$x_n = \left(f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{(2n)^p}\right) + \lambda_n \right)^{1/n}$$

has the limit 1 for every $p \in \{2, 3, \dots\}$.

Proof.

$$\begin{aligned} a_n &= f\left(\frac{1}{n+1}\right) + f\left(\frac{1}{n+2}\right) + \cdots + f\left(\frac{1}{(2n)^p}\right) + \lambda_n \geq \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{(2n)^p} = \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{(2n)^p} - \ln(2n)^p + \ln 2^p + (p-1) \ln n + \\ &\quad + \ln n - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \rightarrow \infty. \end{aligned}$$

$x_n = e^{\ln x_n} = e^{\frac{\ln a_n}{n}}$. We consider

$$\begin{aligned} &\frac{\ln a_{n+1} - \ln a_n}{(n+1) - n} = \ln \frac{a_{n+1}}{a_n} = \\ &= \ln \frac{f\left(\frac{1}{n+2}\right) + f\left(\frac{1}{n+3}\right) + \cdots + f\left(\frac{1}{(2n+2)^p}\right) + \lambda_{n+1} + f\left(\frac{1}{n+1}\right) - f\left(\frac{1}{n+1}\right) + \lambda_n - \lambda_n}{a_n} = \\ &= \ln \left(1 + \frac{f\left(\frac{1}{(2n)^{p+1}}\right) + f\left(\frac{1}{(2n)^{p+2}}\right) + \cdots + f\left(\frac{1}{(2n+2)^p}\right)}{a_n} + \frac{\lambda_{n+1} - \lambda_n}{a_n} - \frac{f\left(\frac{1}{n+1}\right)}{a_n} \right). \\ &0 \leq \frac{f\left(\frac{1}{(2n)^{p+1}}\right) + f\left(\frac{1}{(2n)^{p+2}}\right) + \cdots + f\left(\frac{1}{(2n+2)^p}\right)}{a_n} \leq \\ &\leq \frac{(C_p^1 2^p n^{p-1} + C_p^2 2^p n^{p-2} + \cdots + 2^p) f\left(\frac{1}{(2n)^{p+1}}\right)}{((2n)^p - n) \frac{1}{(2n)^p}} \rightarrow 0. \\ &\frac{f\left(\frac{1}{n+1}\right)}{a_n} = \frac{f\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}} \frac{1}{(n+1) a_n} \rightarrow 0. \end{aligned}$$

So $\frac{\ln a_n}{n} \rightarrow 0$ and $x_n \rightarrow 1$. □

Remarks. 1. The functions $f(x) = x, \sinh x, \arcsin x, \tan x, \operatorname{argth} x$, satisfy the hypotheses of Proposition 4 on f .

2. If $f : (0, a) \rightarrow (0, b)$ and $g : (0, b) \rightarrow \mathbb{R}$ verify the hypotheses of Proposition 4 then $g \circ f : (0, a) \rightarrow \mathbb{R}$ also satisfies the hypotheses (what can be checked as in Proposition 3).

References

- [1] Gazeta Matematică, 2, 1895, București.