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The Data Dependence for the Solutions of Darboux – Ionescu Problem Associated with a Hyperbolic Inclusion of Second Order

Georgeta TEODORU

Abstract. In this paper we consider the Darboux – Ionescu Problem for a second order hyperbolic inclusion with the modified argument of the form

$$\frac{\partial^2 z}{\partial x\,\partial y} \in F\left(x, y, z\left(g\left(x, y\right), h\left(x, y\right)\right)\right).$$

We prove three existence theorems and, as corollaries, we obtain the dependence data results for the solutions of the considered problem.

Key words and phrases: continuous multifunction, Lipschitzian multifunction, absolutely continuous function in Carathéodory's sense, Hausdorff – Pompeiu metric, hyperbolic inclusion.

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1 Introduction

On the 7^{th} of July 1927, D. V. Ionescu (1901 – 1984) brilliantly defended his Ph. D. Thesis in mathematics with the topic "Sur une classe d'équations fonctionnelles". In this dissertation he generalized the results of Darboux, Cauchy, Picard and Goursat for partial differential equations of hyperbolic type. The essential results of his thesis were published in Comptes Rendus de l'Académie des Sciences de Paris, T. 184 (1927), presented by E. Goursat and J. Hadamard [1].

In his Ph. D. thesis [2], [3], D. V. Ionescu studied for the first time in the mathematical literature, boundary value problems of Darboux, Cauchy, Picard and Goursat types for second order hyperbolic equations with modified argument.

More recently, a series of authors studied the same problems for second order hyperbolic equations with modified argument of various forms [4] – [23] and the Cauchy – Ionescu Problem for systems of hyperbolic type equations of first order with modified argument [21], [23], [24].

The Darboux – Ionescu Problem for third order hyperbolic equations with modified argument is studied in [21], [23], [25], [26]. The Darboux – Ionescu, Cauchy – Ionescu, Picard – Ionescu, Goursat – Ionescu problems for hyperbolic inclusions of second order with modified argument is studied in [27] – [30] and the Darboux – Ionescu problem for third order hyperbolic inclusions with modified argument is studied in [31].

The present paper is an extension of [32]. We consider the Darboux – Ionescu problem associated with hyperbolic inclusion of second order with modified argument

$$\frac{\partial^2 z}{\partial x\,\partial y} \in F\left(x,y,z\left(g(x,y)\,,h(x,y)\right)\right), \quad (x,y)\in D=[0,a]\times[0,b]\,,\ z\in\mathbb{R}^n, \tag{1.1}$$

$$\begin{cases} z\,(x,0)=\sigma\,(x)\,, & 0\le x\le a \\ z\,(0,y)=\tau\,(y)\,, & 0\le y\le b \end{cases}, \tag{1.2}$$

where $F: D\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a Lipschitzian multifunction with respect to z, $F\,(x,y,z)$ is a nonconvex set, $g\in C\,(D;[0,a])$, $h\in C\,(D;[0,b])$, $\sigma\in AC\,([0,a]\,;\mathbb{R}^n)$, $\tau\in AC\,([0,b]\,;\mathbb{R}^n)$, $\sigma\,(0)=\tau\,(0)$.

Under suitable assumptions we prove three existence theorems. The results are obtained by the successive approximations method, using two selection theorems [33], [34]. This method was applied by A. F. Filippov [35], Henry Hermes [33] and C. J. Himmelberg and F. S. Van Vleck [34] for the inclusion $\dot{x}\in R\,(t,x)$. Our results are similar to those reported in [32] – [35]. As corollaries we obtain the data dependence results for the solutions of the considered problem.

2 Preliminaries

The definitions and Theorems in this section are taken from [32] – [46].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi: X\to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x\in X$, a subset $\Phi\,(x)$ of Y is associated by the multifunction Φ. The set $\bigcup\limits_{x\in X}\Phi\,(x)$ is the *range* of Φ.

Definition 2.2. Let us consider $\Phi: X\to 2^Y$.

a) If $A\subset X$, the *image* of A by Φ is $\Phi\,(A)=\bigcup\limits_{x\in A}\Phi\,(x)$.

b) If $B\subset Y$, the *counterimage* of B by Φ is

$$\Phi^-\,(B)=\{x\in X|\Phi\,(x)\cap B\ne\varnothing\}$$

c) The graph of Φ, denoted graph Φ, is the set

$$\operatorname{graph}\Phi=\{(x,y)\in X\times Y|y\in\Phi\,(x)\}$$

Definition 2.3. A single-valued function $\varphi: X\to Y$ is said to be *selection* of $\Phi: X\to 2^Y$ if $\varphi\,(x)\in\Phi\,(x)$ for all $x\in X$.

Definition 2.4. Let X and Y be two topological spaces. The multifunction $\Phi: X\to 2^Y$ is *upper-semicontinuous* if, for any closed subset $B\subseteq Y$, the set $\Phi^-\,(B)$ is closed in X.

Definition 2.5. If X and Y are two topological spaces, the multifunction $\Phi: X\to 2^Y$ is *lower-semicontinuous* if, for every open subset $\Omega\subseteq Y$, the set $\Phi^-\,(\Omega)$ is open in X.

Definition 2.6. The multifunction $\Phi : X \to 2^Y$ is *continuous* if it is upper-semicontinuous and lower-semicontinuous.

Definition 2.7. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space the multifunction $\Phi : X \to 2^Y$ is *measurable* (*weakly measurable*) if

$$\Phi^{-}(B) \in \mathcal{F}$$

for every closed (open) subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^{-}(B)$ is measurable.

Let (X, d) be a metric space and $\mathcal{P}(X)$ the set of subsets of X. For $A, B \subset X$, we denote

$$d(x, A) = \inf_{y \in A} d(x, y), \quad d(x, \varnothing) = \infty, \quad d^{*}(A, B) = \sup_{y \in A} d(y, b).$$

Definition 2.8. The function $d_H : \mathcal{P}(X) \to [0, +\infty]$,

$$d_H(A, B) = \max\{d^{*}(A, B), d^{*}(B, A)\} = \max\left\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\right\}$$

is the *Hausdorff – Pompeiu pseudometric.*

The function d_H defines a metric on the space $\mathcal{F}(X)$ of non-empty and closed subsets of X, called the *Hausdorff – Pompeiu metric.*

Definition 2.9. Let be (X, d) and (Y, ρ) two metric spaces. A multifunction $\Phi : X \to 2^Y$ with non-empty values is said to be *Lipschitzian* (*L-Lipschitz*) if there exists a real number $L > 0$ (Lipschitz constant) such that

$$\rho_H(\Phi(x), \Phi(y)) \leq L \cdot d(x, y)$$

for all $x, y \in X$. If $L < 1$, we say that Φ is a *multi-valued contraction.*

Notice that any Lipschitzian multifunction is lower-semicontinuous.

Definition 2.10. Let (X, d) be a metric space and $\mathcal{P}(X)$ the set of subsets of X.

$$N_{\varepsilon}(C) = \{x \in X \mid d(x, c) < \varepsilon \text{ for any } c \in C\}, \ \varepsilon > 0, \ C \in \mathcal{P}(X).$$

For $A, B \in \mathcal{P}(X)$,

$$h_d(A, B) = \begin{cases} \inf\{\varepsilon > 0 \mid A \subseteq N_{\varepsilon}(B) \text{ and } B \subseteq N_{\varepsilon}(A)\} & \text{if the infimum exists,} \\ \infty & \text{otherwise} \end{cases}$$

is the *Hausdorff - Pompeiu generalized pseudometric.*

Definition 2.11. [36] – [39] The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory's sense* [36; §565-570] if $u(x, y)$ is continuous on D, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x, y)$ is (posibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue integrable on D.

Theorem 2.1. [36], [37], [46] *The function $u : D \to \mathbb{R}^n$ is absolutely continuous on D if and only if there exist $f \in L^1(D; \mathbb{R}^n)$, $g \in L^1([0, a]; \mathbb{R}^n)$, $h \in L^1([0, b]; \mathbb{R}^n)$ such that*

$$u(x, y) = \int_0^x \int_0^y f(s, t)\, ds\, dt + \int_0^x g(s)\, ds + \int_0^y h(t)\, dt + u(0, 0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^{*}(D;\mathbb{R}^n)$, [38], [39]. In [37], this space is denoted by $AC(D;\mathbb{R}^n)$.

Theorem 2.2. [37] *The space* $C^{*}(D,\mathbb{R}^n)$ *endowed with the norm*

$$\|u(\cdot,\cdot)\| = \int_0^a \int_0^b \|u_{xy}(s,t)\|\, ds\, dt + \int_0^a \|u_x(s,0)\|\, ds + \int_0^b \|u_y(0,t)\|\, dt + \|u(0,0)\|,$$

where $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

We denote by $AC([t_1,t_2];\mathbb{R}^n)$ [37] the Banach space of absolutely continuous functions $f:[t_1,t_2]\to\mathbb{R}^n$ endowed with the norm

$$\|f\| = \sup_{t\in[t_1,t_2]} \|f(t)\| + \int_{t_1}^{t_2} \|f'(t)\|\, dt.$$

Definition 2.12. [40] The sequence of multifunctions $\{F_i\}_{i\in\mathbb{N}}$, $F_i : D \to 2^{\mathbb{R}^n}$, with values in open set $\Omega\subset\mathbb{R}^n$, converge to $F : D\to 2^{\mathbb{R}^n}$ if for every $\varepsilon>0$ and every $S\in \operatorname{comp}(\Omega)$ there exists a number N such that for every $n>N$,

$$d_H\left(\tilde{F}_n,\tilde{F}\right)<\varepsilon,$$

where d is a metric on $\mathbb{R}^n$, d_H is the Hausdorff – Pompeiu metric on $\mathcal{F}(\mathbb{R}^n)$, $\tilde{F}_n$ and $\tilde{F}$ denote the graphs of restrictions of multifunctions F_n and F on S.

$$\tilde{F}_n = \operatorname{graph} F_n\mid_S,\quad \tilde{F} = \operatorname{graph} F\mid_S .$$

Given a sequence F, F_1, F_2, $\ldots$ of measurable multifunctions, with complete values, from a measurable space $(X,\mathcal{F})$ into a separable metric space (Y,d), the following two theorems caracterize the possibility of pointwise approximating every measurable selection f of F by means of a sequence of functions $\{f_n\}$, each f_n being a measurable selection of F_n.

Theorem 2.3. [44] *The following assertions are equivalent:*

(*a*) *For every measurable selection* f *of* F *there exists a sequence of functions* $\{f_n\}$ *such that*

(*i*) $\lim\limits_{n\to\infty} f_n(x) = f(x)$ *for each* $x\in X$;

(*ii*) f_n *is a measurable selection of* F_n *for any* $n\in\mathbb{N}$.

(*b*) *For any* $x\in X$, $y\in F(x)$, $\lim\limits_{n\to\infty} d(y,F_n(x))=0$.

Theorem 2.4. [44] *Suppose that* μ *is a* σ*-finite, non-negative measure on* $\mathcal{F}$ *and the values of* F *are compacts. Then the following assertions are equivalent:*

a) *There exists a sequence* $\{X_k\}$ *in* $\mathcal{F}$, *with* $\mu\left(X-\bigcup\limits_{k=1}^{\infty}X_k\right)=0$, *such that for each measurable selection* f *of* F *there exists a sequence of functions* $\{f_n\}$ *such that:*

(*i*) $\lim\limits_{n\to\infty} f_n(x)=f(x)$ *uniformly in* X_k *for every* $k\in\mathbb{N}$;

(*ii*) f_n *is a measurable selection of* F_n *for every* $n\in\mathbb{N}$.

b) *There exists* $X^{*}\in\mathcal{F}$, *with* $\mu(X^{*})=0$, *such that for each* $x\in X\setminus X^{*}$, $y\in F(x)$, $\lim\limits_{n\to\infty} d(y,F_n(x))=0$.

Theorem 2.5. [45] *Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

(i) *the multifunction Φ is upper-semicontinuous;*

(ii) *the* graph Φ *is a closed subset of $X \times Y$;*

(iii) *any would be the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$, $y_n \to y$, it follows that $y \in \Phi(x)$.*

The set of all non-empty, compact subset of $\mathbb{R}^n$, with the topology induced by the Hausforff – Pompeiu metric, is a complete metric space denoted by Ω^n. A function R, defined on a real interval I with values in Ω^n is *measurable* (in the sense of Lebesgue) if, for every closed subset $D \subset \mathbb{R}^n$, $\{t \in T \mid R(t) \cap D \neq \emptyset\}$ is measurable [33].

Lemma 1.2. [33] *Let $R : I \to \Omega^n$ be a measurable multifunction, with values in a ball of radius ρ centered at origin, and let $w : I \to \mathbb{R}^n$ be a measurable point-valued function. Then there exists a measurable function r with values $r(t) \in R(t)$ for almost all t and such that $\|w(t) - r(t)\| = \rho(w(t), R(t))$.*

The folowing Proposition is a slight extension of the lemmas used by Filippov [35] and Hermes [33].

Proposition 1. [34] *Let $(T, \mathcal{A})$, $T = [t_0, t_1]$ be a measurable space, let $F : T \to 2^{\mathbb{R}^n}$ be a measurable function with closed values and let $w : T \to \mathbb{R}^n$ be a measurable function. Then there exists a measurable function $\nu : T \to \mathbb{R}^n$ such that $\nu(t) \in F(t)$ and $\|\nu(t) - w(t)\| = d(w(t), F(t))$ for $t \in T$.*

Let now T be a compact Hausdorff space with the positive Radon measure μ, let X be a Polish space (i.e. a metrizable separable space with complete metric), $F : T \times X \to 2^{\mathbb{R}^n}$ (X – a metric space with metric ρ) a multifunction such that $F(t, x)$ is measurable in t for any x and continuous (with respect to the pseudometric Haussdorf on $S(\mathbb{R}^n)$ – the set of non-empty subset of $\mathbb{R}^n$) at x for any t (conditions of Carathéodory type) (Theorem 1 [34]).

Corollary 1. [34] *Let F, T, X be as in Theorem 1 [34]. Let $x : T \to X$ be a measurable function and let us define the multifunction $G : T \to 2^{\mathbb{R}^n}$ by $G(t) = F(t, x(t))$. Then F and G are weakly measurable. If F has closed values then F and G are measurable.*

3 Results

Let $F : D \times 2^{\mathbb{R}^n}$ be a multifunction satisfying the hypotheses:

(H_1) The values of F are contained in the ball of radius r centered at the origin of $\mathbb{R}^n$;

(H_2) $F(x, y, z)$ is a compact set for every $(x, y, z) \in D \times \mathbb{R}^n$;

(H_3) F is a continuous multifunction;

(H_4) F is Lipschitz with respect to z, there exists a function $k : D \to \mathbb{R}_+$, $k \in L^1(D)$ such that

$$d_H(F(x, y, z), F(x, y, z')) \le k(x, y)\|z - z'\|, \quad (x, y) \in D,\ z, z' \in \mathbb{R}^n, \tag{3.1}$$

where $d(z,z')=\|z-z'\|$, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ and d_H is the Hausdorff – Pompeiu metric;

(H_5) The function $\sigma\in AC([0,a],\mathbb{R}^n)$, $\tau\in AC([0,b],\mathbb{R}^n)$ satisfy the condition $\sigma(0)=\tau(0)$;

(H_6) There exists an absolutely continuous in Carathéodory's sense function [36, §565-570], $\zeta\in D\to\mathbb{R}^n$, $\zeta\in C^*(D,\mathbb{R}^n)$, such that

$$\sup_{(x,y)\in D} d\left(\frac{\partial^2\zeta(x,y)}{\partial x\,\partial y},F(x,y,\zeta(x,y))\right)\leq M<+\infty,\ \text{for some } M>0; \tag{3.2}$$

(H_7) $g\in C(D;[0,a])$, $h\in C(D;[0,b])$, $0\leq g(x,y)\leq x\leq a$, $0\leq h(x,y)\leq y\leq b$;

(H_8) The functions $\alpha:D\to\mathbb{R}^n$, $\alpha_0:D\to\mathbb{R}^n$ defined by

$$\alpha(x,y)=\sigma(x)+\tau(y)-\sigma(0),\quad (x,y)\in D, \tag{3.3}$$

$$\alpha_0(x,y)=\zeta(x,0)+\zeta(0,y)-\zeta(0,0),\quad (x,y)\in D, \tag{3.4}$$

satisfy the condition

$$\|\alpha(x,y)-\alpha_0(x,y)\|\leq M_1, \tag{3.5}$$

$M_1>0$ is a constant.

Remark 3.1. Such function ζ in (H_6) exist; for example, if ζ is constant, M may be taken $=r$.

Remark 3.2. The function α and α_0 are absolutely continuous in Carathéodory's sense on D [36, §565-570], $\alpha,\alpha_0\in C^*(D,\mathbb{R}^n)$.

Definition 3.1. [27] The *Darboux-Ionescu Problem* for second order hyperbolic inclusion with modified argument

$$\frac{\partial^2 z}{\partial x\,\partial y}\in F(x,y,z(g(x,y),h(x,y))),\quad (x,y,z)\in D\times\mathbb{R}^n, \tag{3.6}$$

consists in the determination of *solution* of (3.6) which satisfies the conditions

$$\begin{cases} z(x,0)=\sigma(x), & 0\leq x\leq a\\ z(0,y)=\tau(y), & 0\leq y\leq b\end{cases}. \tag{3.7}$$

Definition 3.2. [27] A *solution of Darboux – Ionescu Problem* (3.6) + (3.7) is a function $z:D\to\mathbb{R}^n$, absolutely continuous in Carathéodory's sense [36, §565-570], $z\in C^*(D;\mathbb{R}^n)$, which satisfies a.e. for $(x,y)\in D$ the inclusion (3.6) and the conditions (3.7).

Theorem 3.1. *If the hypotheses* $(H_1)-(H_8)$ *are satisfied, the Darboux – Ionescu Problem* (3.6) + (3.7) *has a solution in* D.

Proof. We define the sequence of succesive approximations $\{z_i\}$, $i\in\mathbb{N}$;

$$z_0(x,y)=z_0(g(x,y),h(x,y))=\zeta(x,y),\quad (x,y)\in D. \tag{3.8$_0$}$$

According to Lemma 1.2 [33] there exists a measurable function $v_0 : D \to \mathbb{R}^n$ such that

$$v_0(x,y) = v_0(g(x,y), h(x,y)) \in F(x,y,z_0(g(x,y), h(x,y))), \text{ a.e. for } (x,y) \in D, \tag{3.9$_0$}$$

and

$$d\left(\frac{\partial^2 z_0(x,y)}{\partial x\,\partial y}, F(x,y,z_0(g(x,y),h(x,y)))\right) = \left\| v_0(x,y) - \frac{\partial^2 z_0(x,y)}{\partial x\,\partial y} \right\|, \quad (x,y) \in D. \tag{3.10$_0$}$$

We define the second approximaxion by

$$\begin{aligned} z_1(x,y) &= z_1(g(x,y),h(x,y)) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y v_0(\xi,\eta)\,d\xi\,d\eta = \\ &= \alpha(x,y) + \int_0^x \int_0^y v_0(\xi,\eta)\,d\xi\,d\eta, \quad (x,y) \in D \end{aligned} \tag{3.11$_1$}$$

from which it follows

$$\frac{\partial^2 z_1(x,y)}{\partial x\,\partial y} = v_0(x,y) \in F(x,y,z_0(g(x,y),h(x,y))), \text{ a.e. for } (x,y) \in D. \tag{3.12$_1$}$$

Again applying the cited Lemma, it follows the existence of a measurable function $v_1 : D \to \mathbb{R}^n$ having the properties

$$v_1(x,y) = v_1(g(x,y), h(x,y)) \in F(x,y,z_1(g(x,y), h(x,y))), \text{ a.e. for } (x,y) \in D, \tag{3.9$_1$}$$

and

$$d\left(\frac{\partial^2 z_1(x,y)}{\partial x\,\partial y}, F(x,y,z_1(g(x,y),h(x,y)))\right) = \left\| v_1(x,y) - \frac{\partial^2 z_1(x,y)}{\partial x\,\partial y} \right\|, \quad (x,y) \in D. \tag{3.10$_1$}$$

The third approximation is given by

$$z_2(x,y) = z_2(g(x,y),h(x,y)) = \alpha(x,y) + \int_0^x \int_0^y v_1(\xi,\eta)\,d\xi\,d\eta, \quad (x,y) \in D, \tag{3.11$_2$}$$

which implies

$$\frac{\partial^2 z_2(x,y)}{\partial x\,\partial y} = v_1(x,y) \in F(x,y,z_1(g(x,y),h(x,y))), \text{ a.e. for } (x,y) \in D. \tag{3.12$_2$}$$

In this way we obtain the function sequences $\{z_i\}_{i\in\mathbb{N}}$, $\{v_i\}_{i\in\mathbb{N}}$, $z_i, v_i : D \to \mathbb{R}^n$, which satisfy

$$v_i(x,y) \in F(x,y,z_i(g(x,y),h(x,y))), \text{ a.e. for } (x,y) \in D,\ i = 0,1,2,\dots \tag{3.9$_i$}$$

and

$$d\left(\frac{\partial^2 z_i(x,y)}{\partial x\,\partial y}, F(x,y,z_i(g(x,y),h(x,y)))\right) = \left\| v_i(x,y) - \frac{\partial^2 z_i(x,y)}{\partial x\,\partial y} \right\|, \tag{3.10$_i$}$$

$$(x,y) \in D,\ i = 0,1,2,\dots$$

where

$$z_i\left(x,y\right)=\alpha\left(x,y\right)+\int_0^x\int_0^y v_{i-1}\left(\xi,\eta\right)d\xi\,d\eta,\quad\left(x,y\right)\in D,\ i=1,2,\ldots\qquad(3.11_i)$$

From the preceding relation it follows

$$\frac{\partial^2 z_i\left(x,y\right)}{\partial x\,\partial y}=v_{i-1}\left(x,y\right)\in F\left(x,y,z_{i-1}\left(g\left(x,y\right),h\left(x,y\right)\right)\right),\ \text{a.e. for }\left(x,y\right)\in D,\ i=1,2,\ldots\qquad(3.12_i)$$

The functions $\left\{v_i\right\}_{i\in\mathbb{N}}$ are integrable in view of (3.12_i), (H_4) and (H_6). Moreover, from (3.1) we have

$$\begin{aligned}d\left(\frac{\partial^2 z_i\left(x,y\right)}{\partial x\,\partial y},F\left(x,y,z_i\left(g\left(x,y\right),h\left(x,y\right)\right)\right)\right)=&\\
=d\left(v_{i-1}\left(x,y\right),F\left(x,y,z_i\left(g\left(x,y\right),h\left(x,y\right)\right)\right)\right)\leq&\\
\leq d\left(v_{i-1}\left(x,y\right),F\left(x,y,z_{i-1}\left(g\left(x,y\right),h\left(x,y\right)\right)\right)\right)+&\\
+d_H\left(F\left(x,y,z_{i-1}\left(g\left(x,y\right),h\left(x,y\right)\right)\right),F\left(x,y,z_i\left(g\left(x,y\right),h\left(x,y\right)\right)\right)\right)\leq&\\
\leq k\left(x,y\right)\left\|z_{i-1}\left(g\left(x,y\right),h\left(x,y\right)\right)-z_i\left(g\left(x,y\right),h\left(x,y\right)\right)\right\|=&\\
=k\left(x,y\right)\left\|z_{i-1}\left(x,y\right)-z_i\left(x,y\right)\right\|,\quad\left(x,y\right)\in D,\ i=2,3,\ldots&\qquad(3.13_i)\end{aligned}$$

and for $i=1$ holds (3.2) and (3.8_0).

After some standard calculation, using the inequality

$$\begin{aligned}\int_0^x\int_0^y k\left(s,t\right)\int_0^s\int_0^t k\left(s_1,t_1\right)\int_0^{s_1}\int_0^{t_1}k\left(s_2,t_2\right)\cdots\int_0^{s_{n-1}}\int_0^{t_{n-1}}k\left(s_n,t_n\right)\cdot&\\
\cdot\,ds_n\,dt_n\cdots ds_1\,dt_1\,ds\,dt\leq&\\
\leq\frac{1}{\left(n+1\right)!}\left[\int_0^x\int_0^y k\left(u,v\right)du\,dv\right]^{n+1},\quad\left(x,y\right)\in D,&\qquad(3.14)\end{aligned}$$

we obtain the following basic estimations:

$$\begin{aligned}\left\|\frac{\partial^2 z_{i+1}\left(x,y\right)}{\partial x\,\partial y}-\frac{\partial^2 z_i\left(x,y\right)}{\partial x\,\partial y}\right\|=\left\|v_i\left(x,y\right)-v_{i-1}\left(x,y\right)\right\|\leq&\\
\leq k\left(x,y\right)\frac{M_1+Mab}{\left(i-1\right)!}\left[\int_0^x\int_0^y k\left(\xi,\eta\right)d\xi\,d\eta\right]^{i-1},\quad i=1,2,\ldots&\qquad(3.15_{i+1})\end{aligned}$$

and

$$\left\|z_{i+1}\left(x,y\right)-z_i\left(x,y\right)\right\|\leq\frac{M_1+Mab}{i!}\left[\int_0^x\int_0^y k\left(\xi,\eta\right)d\xi\,d\eta\right]^i,\quad i=0,1,2,\ldots\qquad(3.16_{i+1})$$

From (3.15_{i+1}) we conclude that the sequence $\left\{v_i\left(x,y\right)\right\}_{i\in\mathbb{N}}$ converges to $v:D\to\mathbb{R}^n$ in $L_\infty^n\left(D\right)$ and the sequence $\left\{z_i\left(x,y\right)\right\}_{i\in\mathbb{N}}$ is uniformly convergent to $z:D\to\mathbb{R}^n$.

Letting $i \to \infty$ in (3.10_i), (3.11_i), (3.12_i) and using the hypotheses (H_2), (H_3) it follows that the limit z is an absolutely continuous function in Carathéodory's sense, $z \in C^*(D;\mathbb{R}^n)$ and satisfies the Darboux – Ionescu Problem (3.6) + (3.7). We obtain

$$d\left(\frac{\partial^2 z(x,y)}{\partial x\,\partial y}, F(x,y,z(g(x,y),h(x,y)))\right) = \left\|v(x,y) - \frac{\partial^2 z(x,y)}{\partial x\,\partial y}\right\|, \quad (x,y) \in D, \tag{3.10}$$

$$z(x,y) = \alpha(x,y) + \int_0^x \int_0^y v(\xi,\eta)\,d\xi\,d\eta, \quad (x,y) \in D, \tag{3.11}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = v(x,y) \in F(x,y,z(g(x,y),h(x,y))), \text{ a.e. for } (x,y) \in D. \tag{3.12}$$

Taking into account (3.11) and (3.12), the function $z(x,y)$ given by (3.11) satisfies the Darboux – Ionescu Problem (3.6) + (3.7).

Theorem 3.2. *We suppose that F satisfies the hypotheses $(H_1)-(H_5)$, (H_7), (H_8) and*

(H_6') *There exists an absolutely continuous function in Carathéodory's sense* [36, §565-570], $\zeta : D \to \mathbb{R}^n$, $\zeta \in C^*(D;\mathbb{R}^n)$ *such that*

$$d\left(\frac{\partial^2 \zeta(x,y)}{\partial x\,\partial y}, F(x,y,\zeta(x,y))\right) < \varepsilon, \quad (x,y) \in D, \text{ for some } \varepsilon > 0. \tag{3.2'}$$

Then there exists a solution $z \in C^*(D;\mathbb{R}^n)$ of the Darboux – Ionescu Problem (3.6) + (3.7) *satisfying*

$$\|z(x,y) - \zeta(x,y)\| \le (M_1 + \varepsilon ab)\exp\left[\int_0^x \int_0^y k(\xi,\eta)\,d\xi\,d\eta\right], \quad (x,y) \in D. \tag{3.17}$$

The proof is similar to that one of Theorem 3.1; we obtain

$$\|z_{i+1}(x,y) - z_i(x,y)\| \le \frac{M_1 + \varepsilon ab}{i!}\left[\int_0^x \int_0^y k(\xi,\eta)\,d\xi\,d\eta\right]^i, \quad i = 0,1,2,\dots \tag{$3.16_i'$}$$

and using the elementary inequality

$$\sum_{n=0}^{j} \frac{u^n}{n!} \le e^u, \quad u \ge 0,$$

it follows for every $j = 0,1,2,\dots$

$$\|z_j(x,y) - \zeta(x,y)\| \le (M_1 + \varepsilon ab)\exp\left[\int_0^x \int_0^y k(\xi,\eta)\,d\xi\,d\eta\right], \quad (x,y) \in D. \tag{$3.16_j''$}$$

The conclusion follows letting $j \to \infty$ in the preceding relation, $z_j(x,y) \to z(x,y)$, $z : D \to \mathbb{R}^n$, uniformly for $j \to \infty$.

Theorem 3.3. *Let $F : D \times R^n \to 2^{\mathbb{R}^n}$ be a multifunction satisfying the hypotheses:*

a) $F(x,y,z)$ is closed for every $(x,y,z) \in D \times \mathbb{R}^n$;

b) $F(\cdot,\cdot,z)$ is measurable for each $z \in R^n$, with respect to the Lebesgue measure on D;

c) $F(x,y,\cdot)$ *is Lipschitzian with respect to z; there exists a function* $k : D \to \mathbb{R}^n$, $k \in L^1(D)$, *such that*

$$h_d\left(F(x,y,z), F(x,y,z')\right) \le k(x,y)\left\|z - z'\right\|, \quad (x,y) \in D,\ z, z' \in \mathbb{R}^n, \tag{3.1$'$}$$

where h_d *is the Hausdorff – Pompeiu generalized pseudometric, and* $(H_5) - (H_8)$*.*

Then, the Darboux – Ionescu Problem (3.6) + (3.7) *has a solution in D.*

The proof is similar to that one of Theorem 3.1, and use the Proposition 1 [34] and the Corollary 1 [34] to ensure the existence of measurable functions $\nu_i : D \to \mathbb{R}^n$, $i = 0, 1, 2, \ldots$, with

$$\nu_i(x,y) \in F\left(x,y,z_i\left(g(x,y), h(x,y)\right)\right), \text{ a.e. for } (x,y) \in D, \tag{3.9$'_i$}$$

$$d\left(\frac{\partial^2 z_i(x,y)}{\partial x\, \partial y}, F\left(x,y,z_i\left(g(x,y), h(x,y)\right)\right)\right) = \left\|\nu_i(x,y) - \frac{\partial^2 z_i(x,y)}{\partial x\, \partial y}\right\|, \tag{3.10$'_i$}$$

$$(x,y) \in D, \quad i = 0, 1, 2, \ldots$$

$$z_i(x,y) = \alpha(x,y) + \int_0^x \int_0^y \nu_{i-1}(\xi,\eta)\, d\xi\, d\eta, \quad (x,y) \in D, \quad i = 0, 1, 2, \ldots \tag{3.11$'_i$}$$

$$\frac{\partial^2 z_{i+1}(x,y)}{\partial x\, \partial y} = \nu_i(x,y) \in F\left(x,y,z_i\left(g(x,y), h(x,y)\right)\right), \tag{3.12$'_i$}$$

$$\text{a.e. for } (x,y) \in D, \quad i = 0, 1, 2, \ldots\ .$$

The sequence $\{z_i(x,y)\}_{i \in \mathbb{N}}$ is uniformly convergent to $z : D \to \mathbb{R}^n$, and the sequence $\{\nu_i(x,y)\}_{i \in \mathbb{N}}$ converges in $L^n_\infty(D)$ to $\nu : D \to \mathbb{R}^n$.

Letting $i \to \infty$ in (3.11$'_i$) and (3.12$'_i$), we obtain

$$z(x,y) = \alpha(x,y) + \int_0^x \int_0^y \nu(\xi,\eta)\, d\xi\, d\eta, \quad (x,y) \in D, \tag{3.11$'$}$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \nu(x,y) \in F\left(x,y,z\left(g(x,y), h(x,y)\right)\right), \text{ a.e. for } (x,y) \in D. \tag{3.12$'$}$$

To obtain the relation (3.12$'$) we make use of the fact that the graph F is closed, according to the hypotheses *a*), *c*). The function z given by (3.11$'$) satisfies (3.7), is an absolutely continuous in Carathéodory's sense function [36, §565-570], $z \in C^*(D;\mathbb{R}^n)$ and from (3.12$'$), z is a solution of the Darboux – Ionescu Problem (3.6) + (3.7).

We now study the data dependence of solutions of the Darboux – Ionescu Problem (3.6) + (3.7).

Corollary 3.1.

(*i*) *In the Darboux – Ionescu Problem* (3.6) + (3.7) *the functions satisfy the same hypotheses as in Theorem 3.1 and let* $z^* \in C^*(D;\mathbb{R}^n)$ *be a solution of this problem;*

(*ii*) *Let* $F_n : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be a sequence of multifunctions convergent to F [40];*

(*iii*) *Let* $z_n^* \in C^*(D;\mathbb{R}^n)$ *a solution of the Darboux – Ionescu Problem*

$$\frac{\partial^2 z}{\partial x\, \partial y} \in F_n\left(x,y,z\left(g(x,y), h(x,y)\right)\right), \quad (x,y) \in D,\ z \in \mathbb{R}^n, \tag{3.17}$$

$$\begin{cases} z(x,0) = \sigma(x), & 0 \le x \le a \\ z(0,y) = \tau(y), & 0 \le y \le b \end{cases}, \tag{3.18}$$

where F_n *and all functions satisfy the same hypotheses as in Theorem 3.1.* *Then* $z_n^* \to z^*$ *in* $C^*(D;\mathbb{R}^n)$.

Corollary 3.2.

(i) *In the Darboux – Ionescu Problem* (3.6) + (3.7) *the functions satisfy the same hypotheses as in Theorem 3.1 and let* $z^* \in C^*(D;\mathbb{R}^n)$ *be a solution of this problem;*

(ii) *Let* $g_n \in C([0,a];\mathbb{R}^n)$, $h_n \in C([0,b];\mathbb{R}^n)$ *be a functions,* $g_n \to g$, $h_n \to h$*;*

(iii) *Let* $z_n^* \in C^*(D;\mathbb{R}^n)$ *a solution of the Darboux – Ionescu Problem*

$$\frac{\partial^2 z}{\partial x\,\partial y} \in F\left(x,y,z\left(g_n(x,y),h_n(x,y)\right)\right), \quad (x,y)\in D,\ z\in\mathbb{R}^n, \tag{3.19}$$

$$\begin{cases} z(x,0)=\sigma(x), & 0\le x\le a \\ z(0,y)=\tau(y), & 0\le y\le b \end{cases}, \tag{3.20}$$

Then $z_n^* \to z^*$ *in* $C^*(D;\mathbb{R}^n)$.

Corollary 3.3.

(i) *In the Darboux – Ionescu Problem* (3.6) + (3.7) *the functions satisfy the same hypotheses as in Theorem 3.1 and let* $z^* \in C^*(D;\mathbb{R}^n)$ *be a solution of this problem;*

(ii) The functions $\sigma_n \in AC([0,a];\mathbb{R}^n)$, $\tau_n \in AC([0,b];\mathbb{R}^n)$ satisfy $\sigma_n \to \sigma$, $\tau_n \to \tau$ and $\sigma_n(0)=\tau_n(0)$;

(iii) *Let* $z_n^* \in C^*(D;\mathbb{R}^n)$ *a solution of the Darboux – Ionescu Problem*

$$\frac{\partial^2 z}{\partial x\,\partial y} \in F_n\left(x,y,z\left(g(x,y),h(x,y)\right)\right), \quad (x,y)\in D,\ z\in\mathbb{R}^n, \tag{3.21}$$

$$\begin{cases} z(x,0)=\sigma_n(x), & 0\le x\le a \\ z(0,y)=\tau_n(y), & 0\le y\le b \end{cases}, \tag{3.22}$$

Then $z_n^* \to z^*$ *in* $C^*(D;\mathbb{R}^n)$.

References

[1] G. Micula, *D. V. Ionescu – a Romanian forerunner of modern numerical analysis*, Libertas Mathematica, T. XXI (2001), 1-4.

[2] D. V. Ionescu, *Sur une classe d'équations fonctionelles*, Thèse, Paris (1927).

[3] D. V. Ionescu, *Sur une classe d'équations fonctionelles*, Ann. Fac. Sc. Toulouse, T. 19 (1927), 39-92.

[4] M. Kwapisz and J. Turo, *On the existence and uniqueness of solutions of the Darboux problem for partial differential - functional equations in a Banach space*, Annales Polonici Mathematici, T. 29, no.2 (1974), 89-118.

[5] M. C. Alicu, *The Darboux problem for partial differential functional equations of hyperbolic type*, Mathematica, T. 19(42), no.2 (1977), 117-122.

[6] R. P. Agarwal and E. Thandapani, *On the uniqueness of solutions of hyperbolic delay differential equations*, Proc. Matscience Conference on Math. Methods in Phys (Diff. Equations), T. 96B (1978), 22-30.

[7] R. P. Agarwal and E. Thandapani, *On the uniqueness of solutions of hyperbolic delay differential equations*, Math. Sem. Notes, T. 6 (1978), 531-536.

[8] R. P. Agarwal and E. Thandapani, *Existence and uniqueness of solutions of hyperbolic delay differential equations*, Math. Sem. Notes, T. 8 (1980), 531-541.

[9] I. A. Rus, *On a problem of D. V. Ionescu*, Itinerant Seminar on functional equations, approximation and convexity, Cluj-Napoca, 177-183, 17-19 May 1979.

[10] I. A. Rus, *On the problem of Darboux – Ionescu*, "Babes-Bolyai" University, Faculty of Mathematics, Research Seminars, Cluj-Napoca, Preprint nr. 1, 1-32, 1981.

[11] I. A. Rus, *On a problem of Darboux – Ionescu*, Studia Univ. Babes-Bolyai, Mathematica, T.26, f.2 (1981), 43-45.

[12] D. Rădulescu, *Sur un problème de D. V. Ionescu*, Studia Univ. Babes-Bolyai, Mathematica, T.26, f.2 (1981), 51-55.

[13] P. Pavel, *Sur une équation aux dérivées partielles*, "Babes-Bolyai" University, Faculty of Mathematics and Physics, Research Seminars, Seminar on Differential Equations, Preprint nr.8 (1988), 71-86.

[14] P. Pavel, *Sur un système d'équations aux dérivées partielles*, "Babes-Bolyai" University, Faculty of Mathematics and Physics, Research Seminars, Seminar on Differential Equations, Preprint nr.3 (1989), 121-138.

[15] V. Berinde, *On the problem of Darboux – Ionescu using a generalised Lipschitz condition*, "Babes-Bolyai" University, Faculty of Mathematics, Research Seminars, Seminar on Fixed Point Theory, Preprint nr.3 (1992), 19-28.

[16] M. Zima, *On an integral equation with deviated arguments*, Demonstratio Mathematica, T.28 (1995), 967-973.

[17] D. Bugajewski and M. Zima, *On the Darboux problem of neutral type*, Bull. Austral. Math. Soc., Vol. 54 (1996), 451-458.

[18] G. Dezsö, *On the Cauchy – Ionescu problem*, Seminarul "Theodor Angheluţă", Cluj-Napoca (1983), 87-92.

[19] G. Dezsö, *On the Picard – Ionescu problem*, Research Seminars of the Faculty of Mathematics, "Babes-Bolyai" University, Cluj-Napoca, Preprint nr.3 (1984), 29-35.

[20] G. Dezsö, *On the Goursat – Ionescu problem*, Seminarul Naţional "Realizări şi perspective în domeniul traductoarelor pentru echipamente de măsură", Institutul Politehnic Cluj-Napoca, vol.II (1986), 151-158.

[21] G. Deszö, *Principii de punct fix şi aplicaţii în teoria ecuaţiilor hiperbolice cu argument modificat*, Teză de doctorat (2000), Cluj-Napoca, 1-199.

[22] G. Deszö, *On the Darboux – Ionescu problem in a Banach space*, International Conference on Nonlinear Operators, Differential Equations and Applications, September 12-15, 2001, Cluj-Napoca.

[23] G. Deszö, *Ecuaţii hiperbolice cu argument modificat. Tehnica punctului fix*, Presa Universitară Clujeană, Cluj-Napoca, 2003.

[24] G. Deszö, Problema Cauchy pentru un sistem de ecuaţii cu derivate parţiale de tip hiperbolic cu argument modificat, Al II-lea Seminar Naţional "Realizări şi perspective în domeniul traductoarelor", Institutul Politehnic Cluj-Napoca, (1988), 126-134.

[25] G. Deszö, *The Darboux – Ionescu problem for a third order system of hyperbolic equations*, Libertas Mathematica, T. 21 (2001), 27-34.

[26] G. Deszö, *The Darboux – Ionescu problem for a third order hyperbolic equation with modified argument*, Automation, Computers, Applied Mathematics (ACAM), Vol.11, no.2 (2002), 69-72.

[27] G. Teodoru, *The Darboux – Ionescu problem for hyperbolic inclusions with modified argument*, Acta Technica Napocensis, Section: Applied Mathematics and Mechanics, no. 44, vol. I (2001), 89-96.

[28] G. Teodoru, The Cauchy – Ionescu problem for hyperbolic inclusions with modified argument, Libertas Mathematica, T. 21 (2001), 5–14.

[29] G. Teodoru, *The Goursat – Ionescu problem for hyperbolic inclusions with modified argument*, Seminar on Fixed Point Theory, Cluj-Napoca, Proceedings of the International Conference on Nonlinear Operators, Differential Equations and Applications, September 2001, Cluj-Napoca, "Babeş-Bolyai", University of Cluj-Napoca, vol. III (2002), 381-388.

[30] G. Teodoru, *The Picard – Ionescu problem for hyperbolic inclusions with modified argument*, Ann. Polon. Math. T.83, f. 1 (2004), 21-33.

[31] G. Teodoru, *The Darboux – Ionescu problem for third order hyperbolic inclusions with modified argument*, Fixed Point Theory, An International Journal on Fixed Point Theory, Computation and Applications, House of the book of science, Vol. 5, no. 2 (2004), 379-391.

[32] G. Teodoru, *The Darboux problem asociated with a Lipschitzian hyperbolic multivalued equation*, Bul. Inst. Polit. Iaşi, T. XXXVI (XL), f. 1-4 (1990), Secţ. I Matematică, Mecanică teoretică, Fizică, 35-40.

[33] H. Hermes, *The Generalized Differential Equation* $\acute{x} \in R(t, x)$, Advanced in Mathematics, T. 4 (1970), 149-169.

[34] C. J. Himmelberg and F. S. Van Vleck, *Lipschitzian Generalized Differential Equa-*

tions, Rend. Sem. Mat. Univ. Padova, T. 48 (1973), 159-169.

[35] A. F. Filippov, *Classical Solutions of Differential Equations with Multivalued Right-Hand Side*, Vestnik Moskovskogo Universiteta, no. 3 (1967), 16-26, SIAM J. Control, Vol.5 (1967), 609-621.

[36] C. Carathéodory, *Vorlesungen über Reelle Funktionen*, Chelsea Publishing Company, New York, 1968, 3 Ed.

[37] A. Cernea, *Incluziuni diferenţiale hiperbolice şi control optimal*, Editura Academiei Române, Bucureşti, 2001.

[38] K. Deimling, *A Carathéodory theory for system of integral equations*, Ann. Mat. Pura Appl. T. (4) 86 (1970), 217-260.

[39] K. Deimling, Das Picard – Problem für $u_{xy} = f\,(x, y, u, u_x, u_y)$ unter Caratéodory Voraussetzungen, Math. Z., 114 (1970), 303-312.

[40] D. Tr. Docev, D. D. Bainov, *Existenţa soluţiilor unei clase de ecuaţii diferenţiale multivoce cu argument deviat*, Izvestia Akad. Nauk Kaz SSR, seria fizico-matematică (1976), no. 1, 34-37 (l. rusă).

[41] S. Marano, *Generalized Solutions of Partial Differential Inclusions Depending on a Parameter*, Rend. Accad. Naz. Sc. XL, Mem. Mat., 13 (1989), 281-295.

[42] S. Marano, *Classical Solutions of Partial Differential Inclusions in Banach Space*, Appl. Anal., no. 2 (1991), 127-143.

[43] S. Marano, *Controllability of Partial Differential Inclusions Depending on a Parameter and Distributed Parameter Control Process*, Le Matematiche, Vol. XLV, Fasc. II (1990), 283-300.

[44] B. Ricceri, *Sur l'approximation des sélections mesurables*, C. R. Acad. Sc. Paris, T. 295 (1982), Série I, 527-530.

[45] I. A. Rus, *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.

[46] W. Sosulski, *On neutral partial functional-differential inclusions of hyperbolic type*, Demonstratio Mathematica, Vol. XXIII, No. 4 (1990), 893-909.