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On Some Classes of Analytic Functions with Negative Coefficients

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Abstract. We obtain a characterization theorem, coefficients bounds, inclusion relations and integral properties concerning a new class of analytic functions with negative coefficients defined using a differential operator.

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1 Preliminaries

Let $\mathbb{N}^* = \{1, 2, ...\}$, let $\mathbb{N} = \mathbb{N}^* \cup \{0\}$ and let $\mathcal{N}_k$, $k \in \mathbb{N}^*$ be the class of functions of the form

$$f(z) = z - \sum_{j=k+1}^{\infty} a_j z^j\ ,\ a_j \geq 0,\ j \in \mathbb{N},\ \ j \geq k+1 \tag{1}$$

that are analytic in the unit complex disc $U = \{z; |z| < 1\}$.

Definition 1. [4] *We define the differential operator* $D^n : \mathcal{N}_k \to \mathcal{N}_k\ , n \in \mathbb{N}$, *by*
a) $D^0 f(z) = f(z)$;
b) $D^1 f(z) = Df(z) = zf'(z)$;
c) $D^n f(z) = D(D^{n-1} f(z)), \quad z \in U$.

Definition 2. *Let* $\alpha \in [0, 1)$, $\lambda \in [0, 1]$, *let* $B \in [-1, 0)$, $-1 \leq B < A \leq 1$ *and let* $n \in N$; *we define the class* $\mathcal{A}_{n,k,\lambda}(A, B, \alpha)$ *by*

$$\mathcal{A}_{n,k,\lambda}(A, B, \alpha) = \left\{ f \in N_k : \left| \frac{J_{n,\lambda}(z) - 1}{(A - B)(1 - \alpha) - B[J_{n,\lambda}(z) - 1]} \right| < 1, \quad z \in U,\ n \in \mathbb{N} \right\}, \tag{2}$$

where

$$J_{n,\lambda}(z) = \left[\lambda D^n f(z) + (1 - \lambda) D^{n+1} f(z)\right] / z. \tag{3}$$

Remark 1. The class $\mathcal{A}_{0,k,0}(1, -1, 0) = \mathcal{A}_{1,k,1}(1, -1, 0)$ is the well-known class of starlike functions with negative coefficients introduced and studied by H. Silverman [2] and [3].

Remark 2. If in (2) and (3) we replace the operator D^n by Ruscheweyh's operator $\mathcal{D}^n$, where

$$\mathcal{D}^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}, \quad n \in \mathbb{N}.$$

we obtain the definition of the class $T^{\lambda}_{n,k}(A,B,\alpha)$ defined and studied by J. Patel and P. Sahoo in [1].

2 Characterization theorem

Theorem 1. *Let* $\alpha \in [0,1)$, $\lambda \in [0,1]$, *let* $A \in (-1,1]$, $B \in [-1,0)$, *with* $-1 \leq B < A \leq 1$ *and let* $n \in \mathbb{N}$; *the function* $f \in N_k$, *is in* $\mathcal{A}_{n,k,\lambda}(A,B,\alpha)$ *if and only if*

$$\sum_{j=k+1}^{\infty} (1-B)j^n[j-\lambda(j-1)]a_j \leq (A-B)(1-\alpha). \tag{4}$$

The result is sharp and the extremal functions are

$$f_j(z) = z - \frac{(A-B)(1-\alpha)}{j^n[j-\lambda(j-1)]} z^j \ , \ j \in \{k+1, k+2, ...\}. \tag{5}$$

Proof. We suppose that (4) holds. Then we have

$$|J_{n,k}(z)-1| - |(A-B)(1-\alpha) - B[J_{n,k}(z)-1]| =$$

$$= \left| \sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1} \right| - \left| (A-B)(1-\alpha) + B \sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1} \right|$$

Let $|z| = 1$; then

$$|J_{n,k}(z)-1| - |(A-B)(1-\alpha) - B[J_{n,k}(z)-1]| \leq$$

$$\leq \sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j - (A-B)(1-\alpha) - B \sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j =$$

$$= \sum_{j=k+1}^{\infty} (1-B)j^n[\lambda(1-j)+j]a_j - (A-B)(1-\alpha) \leq 0,$$

where we used (4).

Hence, by maximum modulus theorem and by (2), $f \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$.

Conversely, we assume that $f \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$. Then

$$\left| \frac{J_{n,\lambda}(z)-1}{(A-B)(1-\alpha) - B[J_{n,\lambda}(z)-1]} \right| =$$

$$= \left| \frac{\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1}}{(A-B)(1-\alpha) + B\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1}} \right| < 1 \tag{6}$$

For $z \in [0,1)$ (z real number) the inequality (6) can be rewritten

$$\frac{\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1}}{(A-B)(1-\alpha) + B\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1}} < 1 \tag{7}$$

We note that

$$E(z) = (A-B)(1-\alpha) + B\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j z^{j-1} > 0,\ z \in [0,1)$$

because $E(z) \neq 0$ for $z \in [0,1)$ (by (6)) and $E(0) > 0$.

Upon clearing the denominator in (7) and by letting $z \to 1^-$ $(z \in [0,1))$ we deduce

$$\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j \leq (A-B)(1-\alpha) + B\sum_{j=k+1}^{\infty} j^n[\lambda(1-j)+j]a_j$$

and this gives (5).

3 Coefficients bounds

The next theorem follows from Theorem 1.

Theorem 2. *If* $f \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$, *then*

$$a_j \leq \frac{(A-B)(1-\alpha)}{(1-B)j^N[j-\lambda(j-1)]},\ \ j \in \{k+1, k+2, \cdots\}.$$

The estimates are sharp and the extremal functions are f_j *given by* (5).

4 Inclusion results

Theorem 3. *a*). *If* $0 \leq \lambda_1 < \lambda_2 \leq 1$, *then*

$$\mathcal{A}_{n,k,\lambda_1}(A,B,\alpha) \subset \mathcal{A}_{n,k,\lambda_2}(A,B,\alpha); \tag{8}$$

b). *If* $-1 \leq B_1 < B_2 < 0$, $-1 < A_1 < A_2 \leq 1$ *and* $B_2 < A_1$, *then*

$$\mathcal{A}_{n,k,\lambda}(A_1,B_2,\alpha) \subset \mathcal{A}_{n,k,\lambda_2}(A_2,B_2,\alpha) \subset \mathcal{A}_{n,k,\lambda_2}(A_2,B_1,\alpha). \tag{9}$$

Proof. *a*). For $0 \leq \lambda_1 < \lambda_2 \leq 1$, we have $j - \lambda_2(j-1) < j - \lambda_1(j-1)$, $j \geq k+1$ and by (4) we obtain that (8) holds.

b). If A_1, A_2, B_1, B_2 satisfy the hypotheses of the theorem, then

$$\frac{1-B_1}{A_2-B_1} \leq \frac{1-B_2}{A_2-B_2} \leq \frac{1-B_2}{A_1-B_2},$$

because $A_2 > A_1 > B_2 > B_1$ and, by (4), we have that (9) holds.

Theorem 4. *If* $n \in \mathbb{N}$ *and*

$$A^* = \frac{A-B}{k+1} + B, \tag{10}$$

then $\mathcal{A}_{n+1,k,\lambda}(A,B,\alpha) \subset \mathcal{A}_{n,k,\lambda}(A^*,B,\alpha)$ *and the result is sharp.*

Proof. Let f given by (1) be in the class $\mathcal{A}_{n+1,k,\lambda}(A,B,\alpha)$. Then by (4) we have

$$\sum_{j=k+1}^{\infty} (1-B)j^{n+1}[j-\lambda(j-1)]a_j \leq (A-B)(1-\alpha).$$

We find the largest A^* such that

$$\sum_{j=k+1}^{\infty}(1-B)j^n[j-\lambda(j-1)]a_j \leq (A^*-B)(1-\alpha)$$

holds. But this is implied by

$$\frac{j^n}{A^*-B} \leq \frac{j^{n+1}}{A-B}, \quad j \geq k+1,$$

or, equivalently,

$$A^* \geq \frac{A-B}{j}+B, \quad j \geq k+1.$$

The last inequality is satisfied for A^* given by (10). We note that $A < A^* \leq 1$ and $B < A^* < A$. The extremal function is f_{k+1} given by (5).

5 Integral properties

Let $I_c : \mathcal{N}_k \to \mathcal{N}_k$ be the integral operator defined by $g = I_c(f)$, where $c \in (-1, \infty)$, $f \in \mathcal{N}_k$ and

$$g(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\,. \tag{11}$$

We note that if $f \in N_k$ is a function of the form (1), then

$$g(z) = I_c(f)(z) = z - \sum_{j=k+1}^{\infty}\frac{c+1}{c+j}a_j z^j \tag{12}$$

and by using Theorem 1 we obtain that $f \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$ implies $g \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$. The following theorem improve this result.

Theorem 5. *If* $f \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$ *and* $g = I_c(f)$, *then* $g \in \mathcal{A}_{n,k,\lambda}(A^*,B,\alpha)$, *where*

$$A^* = \frac{(c+1)A+kB}{c+k+1}. \tag{13}$$

The result is sharp.

Proof. Let $f \in \mathcal{A}_{n,k,\lambda}(A,B,\alpha)$ be given by (1); then

$$F(z) = z - \sum_{j=k+1}^{\infty}\frac{c+1}{c+j}a_j z^j.$$

We find the smallest A^* such that

$$\sum_{j=k+1}^{\infty} j^n[j-\lambda(j-1)]a_j \leq (A^*-B)(1-\alpha)$$

But this inequality is implied by

$$\frac{c+1}{A^*-Bc+j} \leq \frac{1}{A-B}, \quad j \geq k+1,$$

because f satisfies (4). We obtain

$$A^* \geq \frac{(c+1)A+(j-1)B}{c+j}, \quad j \geq k+1$$

and this inequality is satisfied for A^* given by (13), because the function

$$h : [k+1, \infty) \to \mathbb{R}, \;\; h(z) = [(c+1)A + (x-1)B]/(c+x)$$

is a decreasing function of x.

The result is sharp for the extremal function $f = f_{k+1}$ given by (5).

We note that $A^* < A$ and, according to Theorem 3 *b*), we have $\mathcal{A}_{n,k,\lambda}(A^*, B, \alpha) \subset \mathcal{A}_{n,k,\lambda}(A, B, \alpha)$.

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