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On a Particular Second-Order Nonlinear Differential Subordination II

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Abstract. We find conditions on the complex-valued functions A, B, C, D in the unit disc U such that the differential inequality

$$|A(z)z^2p''(z)+B(z)zp'(z)+C(z)p^2(z)+D(z)|<M$$

implies $|p(z)|<N$, where p is analytic in U, with $p(0)=0$.

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1 Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U=\{z\in\mathbb{C}:\ |z|<1\}.$$

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$ we let

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}[U],\ f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots,\ z\in U\}$$

and

$$\mathcal{A}_n=\{f\in\mathcal{H}[U],\ f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\dots,\ z\in U\}$$

with $\mathcal{A}_1=\mathcal{A}$.

In [1] chapter IV, the authors have analyzed a second-order linear differential subordination

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z), \tag{1}$$

where A,B,C,D and h are complex-valued functions in the unit disc, where $p\in\mathcal{H}[0,n]$. A more general version of (1) is given by:

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\in\Omega,$$

where $\Omega\subset\mathbb{C}$.

In [2] we found conditions on the complex-valued functions A,B,C,D in the unit disc U and the positive numbers M and N such that

$$|A(z)zp'(z)+B(z)p^2(z)+C(z)p(z)+D(z)|<M$$

implies $|p(z)|<N$, where $p\in\mathcal{H}[0,n]$.

In this paper we shall consider the following particular second-order nonlinear differential subordination given by the inequality

$$|A(z)z^2p''(z)+B(z)zp'(z)+C(z)p^2(z)+D(z)|<M, \tag{2}$$

where $p\in\mathcal{H}[0,n]$.

We find conditions on complex-valued functions A,B,C,D and the positive numbers M and N such that (2) implies $|p(z)|<N$, where $p\in\mathcal{H}[0,n]$.

In order to prove the new results we shall use the following lemma:

Lemma A. [1, p. 34] *Let* $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ *and* $M>0$, $N>0$, n *positive integer, satisfy*

$$|\psi(Ne^{i\theta},Ke^{i\theta},L;z)|\geq M \tag{3}$$

whenever

$$\operatorname{Re}[Le^{-i\theta}]\geq(n-1)K,\quad K\geq nN,$$

$z\in U$ *and* $\theta\in\mathbb{R}$.

If $p\in\mathcal{H}[0,n]$ *and* $|\psi(p(z),zp'(z),z^2p''(z);z)|<M$ *then* $|p(z)|<N$.

2 Main results

Theorem. *Let* $M>0$, $N>0$, *and let* n *be a positive integer. Suppose that the functions* $A,B,C,D:U\to\mathbb{C}$ *satisfy* $A(z)\neq0$,

$$\operatorname{Re}\frac{B(z)}{A(z)}\geq\frac{M+N^2|C(z)|+|D(z)|}{N|A(z)|} \tag{4}$$

If $p\in H[0,n]$ *and*

$$|A(z)z^2p''(z)+B(z)zp'(z)+C(z)p^2(z)+D(z)|<M$$

then $|p(z)|<N$.

Proof. Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ be defined by

$$\psi(p(z),zp'(z),z^2p''(z);z)=A(z)z^2p''(z)+B(z)zp'(z)+C(z)p^2(z)+D(z). \tag{5}$$

From (2) we have

$$|\psi(p(z),zp'(z),z^2p''(z);z)|<M,\ \text{for } z\in U. \tag{6}$$

Using (4) in (5) we have

$$\begin{aligned}
|\psi(Ne^{i\theta},Ke^{i\theta},L;z)|&=|A(z)L+B(z)Ke^{i\theta}+C(z)N^2e^{2i\theta}+D(z)|=\\
&=|A(z)Le^{-i\theta}+B(z)K+C(z)N^2e^{i\theta}+D(z)e^{-i\theta}|=\\
&=|A(z)|\left|Le^{-i\theta}+K\frac{B(z)}{A(z)}+N^2\frac{C(z)}{A(z)}e^{i\theta}+\frac{D(z)}{A(z)}e^{-i\theta}\right|\geq\\
&\geq|A(z)|\left[\left|Le^{-i\theta}+K\frac{B(z)}{A(z)}+N^2\frac{C(z)}{A(z)}e^{i\theta}\right|-\left|\frac{D(z)}{A(z)}\right|\right]\geq\\
&\geq|A(z)|\left[\left|Le^{-i\theta}+K\frac{B(z)}{A(z)}\right|-N^2\left|\frac{C(z)}{A(z)}\right|-\left|\frac{D(z)}{A(z)}\right|\right]\geq
\end{aligned}$$

$$\geq |A(z)|\left[\operatorname{Re} Le^{-i\theta} + K\operatorname{Re}\frac{B(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right| - \left|\frac{D(z)}{A(z)}\right|\right] \geq$$
$$\geq |A(z)|\left[(n-1)K + K\operatorname{Re}\frac{B(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right| - \left|\frac{D(z)}{A(z)}\right|\right] \geq$$
$$\geq |A(z)|\left[n(n-1)N + nN\operatorname{Re}\frac{B(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right| - \left|\frac{D(z)}{A(z)}\right|\right] \geq$$
$$\geq |A(z)|\left[nN\operatorname{Re}\frac{B(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right| - \left|\frac{D(z)}{A(z)}\right|\right] \geq$$
$$\geq |A(z)|\left[N\operatorname{Re}\frac{B(z)}{A(z)} - N^2\left|\frac{C(z)}{A(z)}\right| - \left|\frac{D(z)}{A(z)}\right|\right] \geq M.$$

Hence condition (3) holds and by Lemma A we deduce that (6) implies $|p(z)| < N$. □

Instead of prescribing the constant N in Theorem, in some cases we can use (4) to determine an appropriate $N = N(M, c, A, B, C, D)$ so that (2) implies $|p(z)| < N$. This can be accomplished by solving (4) for N by taking the supremum of the resulting function over U.

Condition (4) is equivalent to:

$$N^2|C(z)| - N|A(z)|\operatorname{Re}\frac{B(z)}{A(z)} + M + |D(z)| \leq 0. \tag{7}$$

If we suppose $C(z) \neq 0$, then the inequality (7) holds if

$$|A(z)|\operatorname{Re}\frac{B(z)}{A(z)} \geq 2\sqrt{|C(z)|(M + |D(z)|)} \tag{8}$$

If (8) holds, the roots of the trinomial in (7) are

$$N_{1,2} = \frac{|A(z)|\operatorname{Re}\frac{B(z)}{A(z)} \pm \sqrt{\left[|A(z)|\operatorname{Re}\frac{B(z)}{A(z)}\right]^2 - 4|C(z)|(M + |D(z)|)}}{2|C(z)|}.$$

We let

$$N = \frac{2(M + |D(z)|)}{|A(z)|\operatorname{Re}\frac{B(z)}{A(z)} + \sqrt{\left[|A(z)|\operatorname{Re}\frac{B(z)}{A(z)}\right]^2 - 4|C(z)|(M + |D(z)|)}}.$$

If this supremum is finite, the Theorem can be rewritten as follows:

Corollary 1. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$ and that the functions $A, B, C, D : U \to \mathbb{C}$, with $A(z) \neq 0$. If*

$$N = \sup_{|z|<1} \frac{2(M + |D(z)|)}{|A(z)|\operatorname{Re}\frac{B(z)}{A(z)} + \sqrt{\left[|A(z)|\operatorname{Re}\frac{B(z)}{A(z)}\right]^2 - 4|C(z)|(M + |D(z)|)}} < \infty$$

then

$$|A(z)z^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)| < M$$

implies $|p(z)| < N$.

Let $n = 1$, $A(z) = 4 + 3i$, $B(z) = 20 + 15i$, $C(z) = 12 - 9i$, $D(z) = \sqrt{3} - i$, $M = 8$, we find $N = \dfrac{2}{3}$.

In this case from Corollary 1 we deduce

Example 1. If $p \in \mathcal{H}[0, 1]$, then

$$|(4 + 3i)z^2p''(z) + (20 + 15i)zp'(z) + (12 - 9i)p^2(z) + \sqrt{3} - i| < 8$$

implies $|p(z)| < \dfrac{2}{3}$.

If $n = 2$, $A(z) = 6$, $B(z) = 24 - 24\sqrt{3}i$, $C(z) = 2\sqrt{3} - 2\sqrt{6}i$, $D(z) = 2\sqrt{2} + i$, $M = 15$, $N = \dfrac{3}{2}$.

In this case from Corollary 1 we deduce

Example 2. If $p \in \mathcal{H}[0, 2]$ then

$$|6z^2p''(z) + (24 - 24\sqrt{3}i)zp'(z) + (2\sqrt{3} - 2\sqrt{6}i)p^2(z) + 2\sqrt{2} + i| < 15$$

implies $|p(z)| < 1$.

If $A(z) = A > 0$ then the Theorem can be rewritten as follows:

Corollary 2. *Let* $M > 0$, $N > 0$ *and let* n *be a positive integer. Suppose that the functions* $B, C, D : U \to \mathbb{C}$ *satisfy*

$$\operatorname{Re} B(z) \geq \frac{M + N^2|C(z)| + |D(z)|}{N}.$$

If $p \in \mathcal{H}[0, n]$ *and*

$$|Az^2p''(z) + B(z)zp'(z) + C(z)p^2(z) + D(z)| < M$$

then $|p(z)| < N$. □

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Gh. Oros and G. I. Oros, *On a first-order nonlinear differential subordination I*, Analele Univ. Oradea Fasc. Matematica, Tome IX, p. 65-70.