

# Univalent Functions in Simply Connected Domains

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**Abstract.** In this paper we define the notions of starlikeness, spirallikeness and convexity for analytic functions in the simply connected domain and in the other hand we obtain sufficient conditions for an analytic function to be a convex, starlike and spirallike function in the simply connected domains. For applications we deduce conditions in the domains bordered by the Lamé curves:  $\left(\frac{x}{a}\right)^{2p} + \left(\frac{y}{b}\right)^{2p} = 1$ , where  $p \in \mathbb{Q}^* \setminus P$ , with  $P = \{p = \frac{2l+1}{2m} \in \mathbb{Q}; l \in \mathbb{Z}, m \in \mathbb{Z}^*\}$  and in an the domain which are the area  $\pi$ . We generalize the result of N. N. Pascu and coauthors [3].

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## 1 Starlike, spirallike and convex functions in the unit disk

Let  $g$  be a complex function,  $g : U \rightarrow \mathbb{C}$  defined in the unit disc  $U = \{z \in \mathbb{C}, |z| < 1\}$ , with  $g(z) = u(x, y) + iv(x, y)$ . By definition [1], [2] we say that the function  $g$  belongs to the class  $C^1(U)$ , respectively  $C^2(U)$  if the functions  $u(x, y) = \operatorname{Re} g(z)$  and  $v(x, y) = \operatorname{Im} g(z)$  of the real variables  $x$  and  $y$  have continuous first order, respectively second order partial derivatives in  $U$ . In this paper we use the following operators:  $Dg = z \frac{\partial g}{\partial z} - \bar{z} \frac{\partial g}{\partial \bar{z}}$  and  $Jg = \left| \frac{\partial g}{\partial z} \right|^2 - \left| \frac{\partial g}{\partial \bar{z}} \right|^2$ , where  $\frac{\partial g}{\partial z} = \frac{1}{2} \left( \frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y} \right)$  and  $\frac{\partial g}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} \right)$ .

The professor P. T. Mocanu [1], [2] obtained sufficient conditions for a non-analytic function in the  $U$ , to be univalent, starlike, spirallike,  $\alpha$ -convex and convex. We know the following definitions and theorems:

**Definition 1.1.** A function  $g \in C^1(U)$ , with  $g(0) = 0$  is a starlike, respectively spirallike  $\gamma$ -type function ( $|\gamma| < \frac{\pi}{2}$ ) if  $g$  is injective in  $U$  and  $f(U)$  is a starlike, respectively spirallike  $\gamma$ -type domain with respect to the origin.

**Definition 1.2.** A function  $g$  of the class  $C^1(U)$  is a convex function in  $U$  if it is univalent and  $g(U)$  is a convex domain.

**Theorem 1.1.** Let  $|\gamma| < \frac{\pi}{2}$ . If the function  $g \in C^1(U)$  satisfies the conditions:

- a)  $g(0) = 0$ ,  $g(z) \neq 0$ , for all  $z \in U \setminus \{0\}$ ;  
 b)  $Jg(z) > 0$ , for all  $z \in U$ ;  
 c)  $\operatorname{Re} \left[ \frac{Dg(z)}{g(z)} \right] > 0$ , respectively  $\operatorname{Re} \left[ e^{i\gamma} \frac{Dg(z)}{g(z)} \right] > 0$ , for all  $z \in U \setminus \{0\}$

then  $g$  is a starlike, respectively spirallike function in  $U$ .

**Theorem 1.2.** If the function  $g \in C^1(U)$  satisfies the conditions:

- a)  $g(0) = 0$ ,  $Dg \in C^1(U)$  and  $g(z)Dg(z) \neq 0$ , for all  $z \in U \setminus \{0\}$ ;  
 b)  $Jg(z) > 0$ , for all  $z \in U$ ;  
 c)  $\operatorname{Re} \left[ \frac{D^2g(z)}{Dg(z)} \right] > 0$ , for all  $z \in U \setminus \{0\}$

then  $g$  is a convex function in  $U$ .

## 2 Starlike, spirallike and convex functions in simply connected domains

Let  $\Delta$  a simply connected domain,  $\Delta \neq \mathbb{C}$  and let the functions  $g : U \rightarrow \mathbb{C}$ ,  $h : U \rightarrow \Delta$ ,  $f : \Delta \rightarrow \mathbb{C}$  such that  $g = f \circ h$ . Clearly there exists a function  $h \in C^1(U)$  such that  $h(U) = \Delta$ . If we calculate the expressions in the above theorems for the function  $g$ , we have:

$$Jg(z) = \left| \frac{\partial g}{\partial z} \right|^2 - \left| \frac{\partial g}{\partial \bar{z}} \right|^2 = |f'(h(z))|^2 \cdot Jh(z), \quad Dg(z) = f'(h(z)) \cdot Dh(z),$$

$$\frac{Dg(z)}{g(z)} = Dh(z) \frac{f'(h(z))}{f(h(z))}, \quad D^2g(z) = f''(h(z)) [Dh(z)]^2 + f'(h(z)) D^2h(z),$$

$$\frac{D^2g(z)}{Dg(z)} = Dh(z) \cdot \frac{f''(h(z))}{f'(h(z))} + \frac{D^2h(z)}{Dh(z)}.$$

**Definition 2.1.** A function  $f$ , with  $f(0) = 0$  is a starlike, respectively spirallike  $\gamma$ -type function ( $|\gamma| < \frac{\pi}{2}$ ) if  $f$  is injective in  $\Delta$  and  $f(\Delta)$  is a starlike, respectively spirallike  $\gamma$ -type domain with respect to the origin.

**Definition 2.2.** A function  $f$  is a convex function in  $\Delta$  if it is univalent and  $f(\Delta)$  is a convex domain.

**Theorem 2.1.** If the functions  $h : U \rightarrow \Delta$ ,  $f : \Delta \rightarrow \mathbb{C}$  satisfy the conditions:

- a)  $(f \circ h)(0) = 0$ ,  $f(h(z)) \neq 0$  and  $f'(h(z)) \neq 0$  for all  $z \in U \setminus \{0\}$ ;  
 b)  $Jh(z) > 0$ , for all  $z \in U$ ;  
 c)  $\operatorname{Re} \left[ \frac{Dh(z)f'(h(z))}{f(h(z))} \right] > 0$ , respectively  $\operatorname{Re} \left[ e^{i\gamma} \frac{Dh(z)f'(h(z))}{f(h(z))} \right] > 0$ , for all  $z \in U \setminus \{0\}$   
 and  $|\gamma| < \frac{\pi}{2}$ ,

then  $f$  is a starlike, respectively spirallike function in  $\Delta$ .

**Theorem 2.2.** If the functions  $h : U \rightarrow \Delta$ ,  $f : \Delta \rightarrow \mathbb{C}$  satisfy the conditions:

- a)  $(f \circ h)(0) = 0$  and  $f'(h(z)) \neq 0$  for all  $z \in U \setminus \{0\}$ ;

b)  $Jh(z) > 0$ , for all  $z \in U$ ;

c)  $\operatorname{Re} \left[ \frac{Dh(z)f'(h(z))}{f'(h(z))} \right] + \operatorname{Re} \left[ \frac{D^2h(z)}{Dh(z)} \right] > 0$ , for all  $z \in U \setminus \{0\}$ ,

then  $f$  is a convex function in  $\Delta$ .

**Remark 2.1.** The proofs of above theorems are immediately because if we consider the function  $g = f \circ h$ , then it is enough to show that  $g$  satisfies the conditions of Theorem 1.1 and Theorem 1.2.

### 3 An application for domains bordered by Lamé curves

In this section we consider the Lamé curve:  $\left(\frac{x}{a}\right)^{2p} + \left(\frac{y}{b}\right)^{2p} = 1$ , where  $p \in \mathbb{Q}^* \setminus P$ , with  $P = \{p = \frac{2l+1}{2m} \in \mathbb{Q}; l \in \mathbb{Z}, m \in \mathbb{Z}^*\}$ . We know that the above curve are closed and the domains bordered by this curve are simply connected domains. On the other hand the parametrization of this curve is the following:

$$\begin{cases} x = a(\cos t)^{\frac{1}{p}} \\ y = b(\sin t)^{\frac{1}{p}} \end{cases}.$$

So it is clearly that the function which transform the unit disk in

$$E_p = \left\{ z = x + iy \in \mathbb{C} : \left(\frac{x}{a}\right)^{2p} + \left(\frac{y}{b}\right)^{2p} - 1 < 0, p \in \mathbb{Q}^* \setminus P \right\}$$

is given by

$$h_p(z) = a\left(\frac{z + \bar{z}}{2}\right)^{\frac{1}{p}} + ib\left(\frac{z - \bar{z}}{2i}\right)^{\frac{1}{p}}.$$

**Theorem 3.1.** If the analytic function  $f : E_p \rightarrow \mathbb{C}$  satisfies the conditions

a)  $f(0) = 0$ ,  $f(z) \neq 0$  and  $f'(z) \neq 0$ ,  $\forall z \in E_p \setminus \{0\}$ ;

b) the inequality  $\operatorname{Re} \left[ \frac{Q_1(z)f'(z)}{f(z)} \right] - \operatorname{Im} \left[ \frac{Q_2(z)f'(z)}{f(z)} \right] > 0$ , respectively the inequality

$$\operatorname{Re} \left[ e^{i\gamma} \frac{Q_1(z)f'(z)}{f(z)} \right] - \operatorname{Im} \left[ e^{i\gamma} \frac{Q_2(z)f'(z)}{f(z)} \right] > 0 \text{ (where we denote by } Q_1(z) = \left(\frac{z - \bar{z}}{2i}\right) \cdot$$

$$\left(\frac{b}{a} \frac{z + \bar{z}}{z - \bar{z}} i\right)^p \text{ and } Q_2(z) = \left(\frac{z + \bar{z}}{2}\right) \cdot \left(\frac{a}{b} \frac{z - \bar{z}}{z + \bar{z}} \frac{1}{i}\right)^p, |\gamma| < \frac{\pi}{2}) \text{ holds for all } z \in E_p \setminus \{0\},$$

then  $f$  is starlike, respectively spirallike function in  $E_p$ .

**Proof.** Let the function  $h_p : U \rightarrow E_p$  be the function defined by

$$h_p(z) = a\left(\frac{z + \bar{z}}{2}\right)^{\frac{1}{p}} + ib\left(\frac{z - \bar{z}}{2i}\right)^{\frac{1}{p}}.$$

Then  $h_p \in C^1(U)$  and  $h$  is a univalent function in  $U$ ,  $h_p(U) = E_p$ . We consider the function  $g : U \rightarrow \mathbb{C}$ ,  $g = f \circ h_p$ . In order to prove that  $f$  is a starlike, respectively spirallike function in  $E_p$  it is sufficient to show that the function  $g$  satisfies the conditions from Theorem 1.1. We have

$$Dg(z) = \frac{1}{2p} f'(u) [(A + B)z - (A - B)\bar{z}],$$

where  $u = h_p(z) \in E_p$  and  $A = a\left(\frac{z+\bar{z}}{2}\right)^{\frac{1}{p}-1}$ ,  $B = b\left(\frac{z-\bar{z}}{2i}\right)^{\frac{1}{p}-1}$ . Since  $f(u) \neq 0$ ,  $\forall u \in E_p$ , then  $g(z) \neq 0$ ,  $\forall z \in U \setminus \{0\}$ . The Jacobian of  $g$  is

$$Jg(z) = \frac{1}{2p} |f'(u)|^2 [|A+B|^2 - |A-B|^2] > 0, \quad \forall z \in U.$$

We also have

$$\frac{Dg(z)}{g(z)} = \frac{1}{2p} \frac{f'(u)}{f(u)} [(A+B)z - (A-B)\bar{z}].$$

But  $u = h_p(z) = a\left(\frac{z+\bar{z}}{2}\right)^{\frac{1}{p}} + ib\left(\frac{z-\bar{z}}{2i}\right)^{\frac{1}{p}}$  and  $\bar{u} = a\left(\frac{z+\bar{z}}{2}\right)^{\frac{1}{p}} - ib\left(\frac{z-\bar{z}}{2i}\right)^{\frac{1}{p}}$ , it follows that  $z = \left(\frac{u+\bar{u}}{2a}\right)^p + i\left(\frac{u-\bar{u}}{2bi}\right)^p$  and  $\bar{z} = \left(\frac{u+\bar{u}}{2a}\right)^p - i\left(\frac{u-\bar{u}}{2bi}\right)^p$ . A simple calculation yields that  $A = a\left(\frac{z+\bar{z}}{2}\right)^{\frac{1}{p}-1} = 2^{p-1}a^p(u+\bar{u})^{1-p}$  and  $B = b\left(\frac{z-\bar{z}}{2i}\right)^{\frac{1}{p}-1} = (2i)^{p-1}b^p(u-\bar{u})^{1-p}$ . So we have that the expression

$$(A+B)z - (A-B)\bar{z} = i^{p-1}(u-\bar{u})Q(u) + \frac{1}{i^{p-1}}(u+\bar{u})\frac{1}{Q(u)},$$

where  $Q(u) = \left(\frac{b}{a} \frac{u+\bar{u}}{u-\bar{u}}\right)^p$ . But out of this expression follows that

$$\frac{Dg(z)}{g(z)} = \frac{1}{p} \frac{f'(u)}{f(u)} [Q_1(u) + iQ_2(u)]$$

and by the assumption result that  $\operatorname{Re} \frac{Dg(z)}{g(z)} > 0$ , respectively  $\operatorname{Re} e^{i\gamma} \frac{Dg(z)}{g(z)} > 0$ . So  $g$  satisfies the conditions of Theorem 1.1.  $\square$

**Remark 3.1.** For  $p = 1$  we obtain the Lamé curve:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  from boundary. In this case we have  $A = a$ ,  $B = b$  and  $z = \frac{1}{2ab} [(a+b)u - (a-b)\bar{u}]$ . We obtain that

$$(A+B)z - (A-B)\bar{z} = (a+b)z - (a-b)\bar{z} = \frac{1}{ab} [(a^2 + b^2)u - (a^2 - b^2)\bar{u}],$$

it follows that

$$\frac{Dg(z)}{g(z)} = \frac{1}{2ab} \frac{f'(u)}{f(u)} [(a^2 + b^2)u - (a^2 - b^2)\bar{u}].$$

So if  $f$  with  $f(0) = 0$ ,  $f(u) \neq 0$  and  $f'(u) \neq 0$  for all  $u \in E_1 \setminus \{0\}$  satisfies

$$(a^2 + b^2) \operatorname{Re} \frac{uf'(u)}{f(u)} - (a^2 - b^2) \operatorname{Re} \frac{\bar{u}f'(u)}{f(u)} > 0, \quad \forall u \in E_1 = E,$$

respectively

$$(a^2 + b^2) \operatorname{Re} e^{i\gamma} \frac{uf'(u)}{f(u)} - (a^2 - b^2) \operatorname{Re} e^{i\gamma} \frac{\bar{u}f'(u)}{f(u)} > 0, \quad \forall u \in E_1 = E,$$

then it is starlike, respectively spirallike in the elliptical domain

$$E_1 = E = \left\{ z = x + iy \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0 \right\}.$$

For  $a = b$  ( $E_1 = E = U$ ) the condition from above are the same with the well-known conditions of starlikeness, respectively spirallikeness for analytic functions in the unit disk.

**Theorem 3.2.** *If the analytic function  $f : E_p \rightarrow \mathbb{C}$  satisfies the conditions*

a)  $f(0) = 0$  and  $f'(z) \neq 0, \forall z \in E_p$ ;

b) the inequality  $\operatorname{Re} \left[ \frac{S(z)f''(z)}{f'(z)} \right] + \operatorname{Re} \left[ \frac{2z+M(z)}{S(z)} \right] > 0$  holds for all  $z \in E_p$ ,

then  $f$  is a convex function in  $E_p$ . In the above inequality we denote by  $S(z)$ ,  $M(z)$  the followings

$$S(z) = i^{1-p} \left[ (z + \bar{z}) \frac{1}{Q(z)} + (-1)^{p-1} (z - \bar{z}) Q(z) \right],$$

$$M(z) = \frac{1-p}{p} (-1)^{1-p} \left[ (z + \bar{z}) \frac{1}{Q^2(z)} + (z - \bar{z}) Q^2(z) \right]$$

and  $Q(z) = \left( \frac{b}{a} \frac{z+\bar{z}}{z-\bar{z}} \right)^p$ .

**Proof.** We know that for the function  $g = f \circ h_p$  we have

$$Dg(z) = \frac{1}{2p} f'(u) [(A+B)z - (A-B)\bar{z}],$$

where  $u = h_p(z) \in E_p$ . It follows that

$$D^2g(z) = z \frac{\partial Dg(z)}{\partial z} - \bar{z} \frac{\partial Dg(z)}{\partial \bar{z}} =$$

$$= \frac{1}{2p} f''(u) [(A+B)z - (A-B)\bar{z}]^2 + \frac{1}{2p} f'(u) \{ [(A+B)z - (A-B)\bar{z}] + M(z) \},$$

where

$$M(z) = \left( \frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} \right) z^2 - \left( \frac{\partial A}{\partial z} + \frac{\partial A}{\partial \bar{z}} - \frac{\partial B}{\partial z} + \frac{\partial B}{\partial \bar{z}} \right) z\bar{z} + \left( \frac{\partial A}{\partial \bar{z}} - \frac{\partial B}{\partial \bar{z}} \right) \bar{z}^2.$$

So we obtain that

$$\frac{D^2g(z)}{Dg(z)} = \frac{f''(u)}{f'(u)} [(A+B)z - (A-B)\bar{z}] + \frac{(A+B)z + (A-B)\bar{z} + M(z)}{(A+B)z - (A-B)\bar{z}}.$$

Since  $f'(u) \neq 0$ , for all  $u \in E_p$ , then  $g(z)Dg(z) \neq 0$ , for all  $z \in U \setminus \{0\}$ . We apply Theorem 1.2 and show that  $g$  satisfies the conditions of Theorem 1.2. But we know that  $Jg(z) > 0$  for all  $z \in U$  and  $z = \left( \frac{u+\bar{u}}{2a} \right)^p + i \left( \frac{u-\bar{u}}{2bi} \right)^p$ ,  $\bar{z} = \left( \frac{u+\bar{u}}{2a} \right)^p - i \left( \frac{u-\bar{u}}{2bi} \right)^p$ . On the other hand we have  $A = a \left( \frac{z+\bar{z}}{2} \right)^{\frac{1}{p}-1} = 2^{p-1} a^p (u+\bar{u})^{1-p}$  and  $B = b \left( \frac{z-\bar{z}}{2i} \right)^{\frac{1}{p}-1} = (2i)^{p-1} b^p (u-\bar{u})^{1-p}$ . A long calculation yields

$$(A+B)z + (A-B)\bar{z} = 2u,$$

$$(A+B)z - (A-B)\bar{z} = i^{1-p} \left[ (u+\bar{u}) \frac{1}{Q(u)} + (-1)^{p-1} (u-\bar{u}) Q(u) \right] = S(u),$$

$$\frac{\partial B}{\partial z} = -\frac{\partial B}{\partial \bar{z}} = b \left( \frac{1}{p} - 1 \right) \frac{1}{2i} \left( \frac{z-\bar{z}}{2i} \right)^{\frac{1}{p}-2} = \frac{1-p}{p} (2i)^{2p-2} b^{2p} (u-\bar{u})^{1-2p},$$

$$\frac{\partial A}{\partial z} = \frac{\partial A}{\partial \bar{z}} = a \left( \frac{1}{p} - 1 \right) \frac{1}{2} \left( \frac{z+\bar{z}}{2} \right)^{\frac{1}{p}-2} = \frac{1-p}{p} 2^{2p-2} a^{2p} (u+\bar{u})^{1-2p}.$$

It follows that

$$\begin{aligned} M(z) &= \frac{\partial A}{\partial z}(z - \bar{z})^2 + \frac{\partial B}{\partial z}(z + \bar{z})^2 = \\ &= \frac{1-p}{p}(-1)^{1-p} \left[ (u + \bar{u}) \frac{1}{Q^2(u)} + (z - \bar{u}) Q^2(u) \right] = M(u), \end{aligned}$$

where  $Q(u) = \left( \frac{b}{a} \frac{u+\bar{u}}{u-\bar{u}} \right)^p$ . Finally we obtain that  $\frac{D^2g(z)}{Dg(z)} = \frac{S(u)f''(u)}{f'(u)} + \frac{2u+M(u)}{S(u)}$  and by the condition  $\operatorname{Re} \frac{D^2g(z)}{Dg(z)} > 0$  which follows that the function  $g$  satisfies the conditions of Theorem 1.2.  $\square$

**Remark 3.2.** If we consider  $p = 1$ , we obtain the result of authors in [3]:  $M(u) = 0$ ,  $S(u) = \frac{1}{ab} [(a^2 + b^2)u - (a^2 - b^2)\bar{u}]$  and if the function  $f$  with  $f(0) = 0$  and  $f'(u) \neq 0$  for all  $u \in E_1 = E$  satisfies the following inequality

$$(a^2 + b^2) \operatorname{Re} \left[ \frac{uf''(u)}{f'(u)} + 1 \right] - (a^2 - b^2) \operatorname{Re} \left[ \frac{\bar{u}f''(u)}{f'(u)} + 1 \right] > 0, \quad \forall u \in E,$$

then it is convex in elliptical domain  $E$ . If  $a = b$  ( $E_1 = E = U$ ) the condition from above are the same with the well-know conditions for convexity for analytic functions in the unit disk.

## 4 Another application for domain with area $\pi$

Let the following geometric place problem: a point  $M$  of the circle  $C(O, 1)$  has the projections on the diameters orthogonals  $AA'$  and  $BB'$  in the points  $M_1$  and  $M_2$ , respectively on  $M_1M_2$  in  $M_0$ . In the segment  $M_2M$  let the point  $P$  such that  $MP = MM_0$  and let  $P$  in exterior or interior of the segment  $M_2M$  according to  $M$  is in semicircle  $ABA'$  or  $A'B'A$ . The place of  $P$  when  $M$  traverse the circle  $C$  is a quartic universal.

If we denote by  $\alpha = \widehat{AOM}$ ,  $A(1, 0)$  and  $B(0, 1)$ , then the curve has the parametrization

$$\begin{cases} x = (1 + \sin \alpha) \cos \alpha \\ y = \sin \alpha \end{cases}$$

or has the following implicity equality  $(y + 1)^2(y^2 - 1) + x^2 = 0$ . It is clear that this curve is symmetric with respect to  $BB'$  and the area of the domain bordered by this curve coincide the area of the unit disk.

$$\text{Area} = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin^3 t) dt = \pi \cdot 1^2 = \text{Area}_{\text{disk}}.$$

Like in the above section we consider the domain bordered by this curve

$$G = \{z = x + iy \in \mathbb{C} : (y + 1)^2(y^2 - 1) + x^2 < 0\}.$$

The function

$$h(z) = \frac{1}{4} [4z - i(z^2 - \bar{z}^2)]$$

transform the unit disk  $U$  in the domain  $G$ . If we calculate the expressions in the above

theorems, we have for the function  $g = f \circ h : U \rightarrow \mathbb{C}$  the followings

$$Dg(z) = \frac{1}{2} [2z - i(z^2 + \bar{z}^2)], \quad Jg(z) = |f'(u)|^2 \left( 1 + \frac{z - \bar{z}}{2i} \right),$$

$$\frac{Dg(z)}{g(z)} = \frac{1}{2} \frac{f'(u)}{f(u)} [2z - i(z^2 + \bar{z}^2)]$$

and

$$\frac{D^2g(z)}{Dg(z)} = \frac{1}{2} \left\{ \frac{f''(u)}{f'(u)} [2z - i(z^2 + \bar{z}^2)] + \frac{z - i(z^2 - \bar{z}^2)}{2z - i(z^2 + \bar{z}^2)} \right\},$$

where  $u = h(z)$ . But we get  $z = \frac{1}{2} \frac{(u-\bar{u})^2 + 4iu}{2i+u-\bar{u}}$  and  $\bar{z} = \frac{1}{2} \frac{4i\bar{u} - (u-\bar{u})^2}{2i+u-\bar{u}}$ . Using this expressions and the Theorems in the section Preliminaries we have the followings theorems for this domain.

**Theorem 4.1.** *If the analytic function  $f : G \rightarrow \mathbb{C}$  satisfies the conditions*

- a)  $f(0) = 0, f(z) \neq 0$  and  $f'(z) \neq 0, \forall z \in G \setminus \{0\}$ ;
- b) the inequality  $\operatorname{Re} \left[ \frac{T(z)f'(z)}{f(z)} \right] > 0$ , respectively the analogous inequality

$$\operatorname{Re} \left[ e^{i\gamma} \frac{T(z)f'(z)}{f(z)} \right] > 0 \text{ (where we denote by}$$

$$T(z) = \frac{1}{4} \frac{1}{(2i + z - \bar{z})^2} [-i(z - \bar{z})^4 + 6(z - \bar{z})^3 + 4i(5z^2 - z\bar{z} + 3\bar{z}^2) - 16z],$$

$$|\gamma| < \frac{\pi}{2}) \text{ holds for all } z \in G \setminus \{0\},$$

then  $f$  is starlike, respectively spirallike function in  $G$ .

**Theorem 4.2.** *If the analytic function  $f : G \rightarrow \mathbb{C}$  satisfies the conditions*

- a)  $f(0) = 0$  and  $f'(z) \neq 0, \forall z \in G$ ;
- b) the inequality  $\operatorname{Re} \left[ \frac{T(z)f''(z)}{f'(z)} \right] + \operatorname{Re} \left[ \frac{N(z)}{T(z)} \right] > 0$  holds for all  $z \in G$ ,

then  $f$  is a convex function in  $G$ . In the above inequality we denote by  $T(z), N(z)$  the followings

$$T(z) = \frac{1}{4} \frac{1}{(2i + z - \bar{z})^2} [-i(z - \bar{z})^4 + 6(z - \bar{z})^3 + 4i(5z^2 - 2z\bar{z} + 3\bar{z}^2) - 16z],$$

$$N(z) = \frac{1}{4} \frac{1}{2i + z - \bar{z}} (5z^2 - 2z\bar{z} - 3\bar{z}^2 + 4iz).$$

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