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Close to Convex Functions Associated with Some Hyperbola

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Abstract. In this paper we define a subclass of close to convex functions associated with some hyperbola and we obtain some properties regarding this class.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : \ f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : \ f \text{ is univalent in } U\}$.

We recall here the definitions of the well - known classes of starlike functions and close to convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\},$$

$$CC = \left\{ f \in A : \exists g \in S^*, \ \operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U \right\}.$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C}, \quad \operatorname{Re}\ a \geq 0. \tag{1.1}$$

For $a = 1$ we obtain the Libera integral operator, for $a = 0$ we obtain the Alexander integral operator and in the case $a = 1, 2, 3, \ldots$ we obtain the Bernardi integral operator.

The purpose of this note is to define a subclass of close to convex functions associated with some hyperbola and to obtain some estimations for the coefficients of the series expansion and some other properties regarding this class.

2 Preliminary results

Definition 2.1. [5] A function $f \in S$ is said to be in the class $SH(\alpha)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < \operatorname{Re}\left\{\sqrt{2}\,\frac{zf'(z)}{f(z)}\right\} + 2\alpha\left(\sqrt{2}-1\right),$$

for some α ($\alpha > 0$) and for all $z \in U$.

Remark 2.1. Geometric interpretation: Let $\Omega(\alpha) = \left\{\frac{zf'(z)}{f(z)} : z \in U,\ f \in SH(\alpha)\right\}$. Then $\Omega(\alpha) = \left\{w = u + i \cdot v : v^2 < 4\alpha u + u^2, u > 0\right\}$. Note that $\Omega(\alpha)$ is the interior of a hyperbola in the right half-plane which is symmetric with respect to the real axis and has vertex at the origin. If we denote with p_α the analytic and univalent function with the properties $p_\alpha(0) = 1$, ${p'}_\alpha(0) > 0$ and $p_\alpha(U) = \Omega(\alpha)$, then $f \in SH(\alpha)$ if and only if $\frac{zf'(z)}{f(z)} \prec p_\alpha$, where the symbol $\prec$ denotes the subordination in U.

Remark 2.2. We have $p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-z}} - 2\alpha,\ b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ and the branch of the square root $\sqrt{w}$ is chosen so that $\operatorname{Im}\sqrt{w} \geq 0$. If we consider $p_\alpha(z) = 1 + C_1 z + \ldots$, we have $C_1 = \frac{1+4\alpha}{1+2\alpha}$.

Theorem 2.1. [5] *Let $f \in SH(\alpha)$ and $f(z) = z + b_2 z^2 + b_3 z^3 + \ldots$. Then*

$$|b_2| \leq \frac{1+4\alpha}{1+2\alpha}, \quad |b_3| \leq \frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{4(1+2\alpha)^3}.$$

The next theorems are results of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [1], [2], [3]).

Theorem 2.2. *Let h convex in U and $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } p(z) \prec h(z).$$

Theorem 2.3. *Let q be convex in U and $j : U \to \mathbb{C}$ with $\operatorname{Re}[j(z)] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ and satisfies $p(z) + j(z) \cdot zp'(z) \prec q(z)$, then $p(z) \prec q(z)$.*

3 Main results

Definition 3.1. Let $f \in A$ and $\alpha > 0$. We say that the function f is in the class $CCH(\alpha)$ with respect to the function $g \in SH(\alpha)$ if

$$\left|\frac{zf'(z)}{g(z)} - 2\alpha\left(\sqrt{2}-1\right)\right| < \operatorname{Re}\left\{\sqrt{2}\,\frac{zf'(z)}{g(z)}\right\} + 2\alpha\left(\sqrt{2}-1\right),\ z \in U.$$

Remark 3.1. Geometric interpretation: $f \in CCH(\alpha)$ with respect to the function $g \in SH(\alpha)$ if and only if $\dfrac{zf'(z)}{g(z)}$ take all values in the convex domain $\Omega(\alpha)$ contained in the right half-plane. Using the function p_α (see Remark 2.2) we have: $f \in CCH(\alpha)$ with respect to the function $g \in SH(\alpha)$ if and only if $\dfrac{zf'(z)}{g(z)} \prec p_\alpha$.

Remark 3.2. From the geometric interpretation of the above definition we have $CCH(\alpha) \subset CC$.

Theorem 3.1. *If* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *belongs to the class* $CCH(\alpha)$, $\alpha > 0$, *with respect to the function* $g(z) \in SH(\alpha)$, $\alpha > 0$, $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, *then*

$$|a_2| \le \frac{1+4\alpha}{1+2\alpha},\ |a_3| \le \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{12(1+2\alpha)^3}.$$

Proof. We have $f(z) \in CCH(\alpha)$ with respect to the function $g(z) \in SH(\alpha)$ if and only if $h(z) = \dfrac{zf'(z)}{g(z)} \prec p_\alpha(z)$, where $p_\alpha(U) = \Omega(\alpha)$ (see Remark 3.1). Let $h(z) = 1 + c_1 z + c_2 z^2 + \dots$, $z \in U$ and $p_\alpha(z) = 1 + C_1 z + \dots$, $z \in U$, where $C_1 = \dfrac{1+4\alpha}{1+2\alpha}$. Taking account the Rogosinski subordination theorem (see [4]), we have $|c_j| \le \dfrac{1+4\alpha}{1+2\alpha}$, $j \ge 1$.

Using the hypothesis we have

$$\frac{z + \sum_{j=2}^{\infty} j a_j z^j}{z + \sum_{j=2}^{\infty} b_j z^j} = 1 + c_1 z + c_2 z^2 + \dots .$$

From the equality of the powers coefficients we obtain

$$2a_2 = b_2 + c_1 \text{ and } 3a_3 = b_3 + c_1 b_2 + c_2.$$

Using $|c_j| \le \dfrac{1+4\alpha}{1+2\alpha}$, $j \ge 1$ and the estimations from Theorem 2.1, by simple calculations, we obtain

$$|a_2| \le \frac{1+4\alpha}{1+2\alpha} \text{ and } |a_3| \le \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{12(1+2\alpha)^3}.$$

Theorem 3.2. *If* $F(z) \in SH(\alpha)$, *with* $\alpha > 0$, *then* $f(z) = L_a F(z) \in SH(\alpha)$, $\alpha > 0$, *where* L_a *is the integral operator defined by (1.1).*

Proof. By differentiating (1.1) we obtain

$$(1+a)F(z) = af(z) + zf'(z) \text{ and } (1+a)zF'(z) = (1+a)zf'(z) + z^2 f''(z).$$

From the above we have

$$\frac{zF'(z)}{F(z)} = \frac{(1+a)\dfrac{zf'(z)}{f(z)} + \dfrac{z^2 f''(z)}{f(z)}}{a + \dfrac{zf'(z)}{f(z)}}. \tag{3.1}$$

With notation $\dfrac{zf'(z)}{f(z)} = p(z)$, where $p(z) = 1 + p_1 z + \dots$, we have

$$zp'(z) = p(z)\left(1 - p(z)\right) + \frac{z^2 f''(z)}{f(z)}$$

and thus

$$\frac{z^2 f''(z)}{f(z)} = zp'(z) + p(z)\left(p(z) - 1\right).$$

Using (3.1) we have $\dfrac{zF'(z)}{F(z)} = p(z) + \dfrac{1}{p(z)+a} zp'(z)$. From Remark 2.1 we have $\dfrac{zF'(z)}{F(z)} \prec p_\alpha(z)$ and thus

$$p(z) + \frac{1}{p(z)+a} zp'(z) \prec p_\alpha(z).$$

We have from Remark 2.1 and from the hypothesis $\operatorname{Re}\left(p_\alpha(z) + a\right) > 0$, $z \in U$. In this conditions from Theorem 2.2 we obtain $p(z) \prec p_\alpha(z)$ or $\dfrac{zf'(z)}{f(z)} \prec p_\alpha(z)$. This means that $f(z) = L_a F(z) \in SH(\alpha)$.

Theorem 3.3. *If $F(z) \in CCH(\alpha)$, $\alpha > 0$, with respect to the function $G(z) \in SH(\alpha)$, $\alpha > 0$, and $f(z) = L_a F(z)$, $g(z) = L_a G(z)$, where L_a is the integral operator defined by (1.1), then $f(z) \in CCH(\alpha)$, $\alpha > 0$, with respect to the function $g(z) \in SH(\alpha)$ with $\alpha > 0$ (see the above theorem).*

Proof. By differentiating (1.1) we obtain

$$(1+a)G(z) = ag(z) + zg'(z) \text{ and } (1+a)zF'(z) = (1+a)zf'(z) + z^2 f''(z).$$

From the above we have

$$\frac{zF'(z)}{G(z)} = \frac{(1+a)\dfrac{zf'(z)}{g(z)} + \dfrac{z^2 f''(z)}{g(z)}}{a + \dfrac{zg'(z)}{g(z)}}. \tag{3.2}$$

With the notations $\dfrac{zf'(z)}{g(z)} = p(z)$ and $\dfrac{zg'(z)}{g(z)} = h(z)$, by a similarly calculus as the above theorem, it follows that

$$\frac{z^2 f''(z)}{g(z)} = p(z)\left(h(z) - 1\right) + zp'(z).$$

Using (3.2) we have $\frac{zF'(z)}{G(z)} = p(z) + \frac{1}{h(z)+a} zp'(z)$. From Remark 3.1 we have $\frac{zF'(z)}{G(z)} \prec p_\alpha(z)$ and thus

$$p(z) + \frac{1}{h(z)+a} zp'(z) \prec p_\alpha(z).$$

We have from Remark 2.1 and the above theorem that $\operatorname{Re} h(z) > 0$, $z \in U$. Using the hypothesis we have $\operatorname{Re} \frac{1}{h(z)+a} > 0$, $z \in U$. In this conditions from Theorem 2.3 we obtain $p(z) \prec p_\alpha(z)$ or $\frac{zf'(z)}{g(z)} \prec p_\alpha(z)$. This means that $f(z) \in CCH(\alpha)$, with respect to the function $g(z) \in SH(\alpha)$.

Theorem 3.4. *Let $a \in \mathbb{C}$, $\operatorname{Re} a \geq 0$, and $\alpha > 0$. If $F(z) \in CCH(\alpha)$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $f(z) = L_a F(z)$, $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L_a is the integral operator defined by (1.1), then*

$$|b_2| \leq \left| \frac{a+1}{a+2} \right| \cdot \frac{1+4\alpha}{1+2\alpha}, \quad |b_3| \leq \left| \frac{a+1}{a+3} \right| \cdot \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{12(1+2\alpha)^3}.$$

Proof. From $f(z) = L_a F(z)$ we have $(1+a)F(z) = af(z) + zf'(z)$. Using the above series expansions we obtain

$$(1+a)z + \sum_{j=2}^{\infty} (1+a) a_j z^j = az + \sum_{j=2}^{\infty} a b_j z^j + z + \sum_{j=2}^{\infty} j b_j z^j$$

and thus $b_j(a+j) = (1+a)a_j$, $j \geq 2$. From the above we have $|b_j| \leq \left| \frac{a+1}{a+j} \right| \cdot |a_j|$, $j \geq 2$. Using the estimations from Theorem 3.1 we obtain the needed results.

For $a = 1$, when the integral operator L_a become the Libera integral operator, we obtain from the above theorem:

Corollary 3.1. *Let $\alpha > 0$. If $F(z) \in CCH(\alpha)$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $f(z) = L(F(z))$, $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where L is Libera integral operator defined by $L(F(z)) = \frac{2}{z} \int_0^z F(t)dt$, then*

$$|b_2| \leq \frac{2+8\alpha}{3+6\alpha}, \quad |b_3| \leq \frac{(1+4\alpha)(11+56\alpha+72\alpha^2)}{24(1+2\alpha)^3}.$$

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