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On the General Quadratic Functional Equation [1]

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Abstract. In 1940 and in 1968 S.M. Ulam proposed the **general problem**: "*When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?*". In 1941 D.H. Hyers solved this stability problem for linear mappings. In 1951 D.G. Bourgin was the second author to treat the same problem for additive mappings. According to P.M. Gruber (1978) this kind of stability problems are of particular interest in *probability theory* and in the case of *functional equations* of different types. In 1981 F. Skof was the first author solving the Ulam problem for quadratic mappings. In 1982-2002 we solved the above Ulam problem for linear and non linear mappings and established analogous stability problems even on restricted domains. Besides, we applied some of our recent results to the asymptotic behavior of functional equations of different types. In this paper we establish the stability of the Ulam problem for the general quadratic functional equation.

Keywords:Ulam problem, stability, general quadratic mapping.

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1 Introduction

In 1940 and in 1968 S. M. Ulam [27] proposed the general problem:

"*When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?*".

In 1941 D.H. Hyers [13] solved this stability problem for linear mappings. In 1951 D.G. Bourgin [3] was the second author to treat the same problem for additive mappings. According to P. M. Gruber [12] (1978) this kind of stability problems are of particular interest in *probability theory* and in the case of *functional equations* of different types. In 1978 Th.M. Rassias [22] employed Hyers' ideas to new additive mappings. In 1981

[1] The Editorial Board has to apologize for several errors that occured in the printed version of this paper, published in LIBERTAS MATHEMATICA, tomus XXIII (2003), 165-174. We are including its corrected version here.

and 1983 F. Skof ([23-24]) was the first author solving the Ulam problem for quadratic mappings. In 1982-2002 we ([16-21]) solved the above Ulam problem for linear and non linear mappings and established analogous stability problems even on restricted domains. Besides, we applied some of our recent results to the asymptotic behavior of functional equations of different types. In 1999 P. Gavruta [11] answered a question of ours [16] concerning the stability of Cauchy equation. In 1996 and 1998 we ([19-20]) solved the Ulam stability problem for quadratic mappings $Q : X \to Y$ satisfying the functional equation

$$Q(a_1x_1 + a_2x_2) + Q(a_2x_1 - a_1x_2) = (a_1^2 + a_2^2)[Q(x_1) + Q(x_2)].$$

for every $x_1, x_2 \in X$ and fixed reals $a_1, a_2 \neq 0$, where X and Y are real linear spaces.

In this paper we solve the Ulam stability problem for quadratic mappings $Q : X \to Y$ satisfying the more general functional equation

$$Q\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1\le i<j\le p} Q(a_j x_i - a_i x_j) = m \sum_{i=1}^{p} Q(x_i) \qquad (*)$$

for every $x_i \in X$ $(i = 1, 2, \dots, p)$, and fixed reals $a_i \neq 0$ $(i = 1, 2, \dots, p)$, where p is arbitrary but fixed and equals to 2 , 3 , 4 , . . . , such that

$$0 < m = \sum_{i=1}^{p} a_i^2.$$

If X and Y are normed linear spaces and Y complete, then we establish an approximation of approximately quadratic mappings $f : X \to Y$ by quadratic mappings $Q : X \to Y$, such that the corresponding approximately quadratic functional inequality

$$\left\| f\left(\sum_{i=1}^{p} a_i x_i\right) + \sum_{1\le i<j\le p} f(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(x_i)\right]\right\| \le c \prod_{i=1}^{p} \|x_i\|^{r_i} \qquad (**)$$

holds with constants $c \geq 0$ (independent of $x_i \in X : i = 1, 2, \dots, p$), and any fixed reals $a_i, r_i \neq 0$ $(i = 1, 2, \dots, p)$. Denote

$$I_1 = \{(r, m) \in R^2 : r < 2,\ m > 1 \text{ or } r > 2,\ 0 < m < 1\},$$

$$I_2 = \{(r, m) \in R^2 : r < 2,\ 0 < m < 1 \text{ or } r > 2,\ m > 1\},$$

$$I_3 = \{(r, m) \in R^2 : r < 2,\ m = 1 = pb^2,\ a_i = b = p^{-1/2} : i = 1, 2, \dots, p\},$$

where $r = \sum_{i=1}^{p} r_i \neq 0$, where p is arbitrary but fixed and equals to 2, 3, 4,... . Note that $m^{r-2} < 1$ if $(r, m) \in I_1$, $m^{2-r} < 1$ if $(r, m) \in I_2$, and $p^{r-2} < 1$ if $(r, m = 1) \in I_3$. Also denote $\gamma = \prod_{i=1}^{p} |a_i|^{r_i} > 0$. Besides denote

$$f_n(x) = \begin{cases} m^{-2n} f(m^n x), & \text{if } (r, m) \in I_1 \\ m^{2n} f(m^{-n} x), & \text{if } (r, m) \in I_2 \\ p^{-n} f(p^{n/2} x), & \text{if } (r, m = 1) \in I_3, \end{cases}$$

for all $x \in X$ and $n \in N : 2, 3, 4 \dots$.

It is useful for the following, to observe that, from $(*)$ with $x_i = 0$ $(i = 1, 2, \dots, p)$, and $0 < m \neq 1$ we get

$$Q(0) = 0. \tag{1}$$

Note that for $m = 1$ we assume in addition that (1) holds.

Definition 1.1. *Let X and Y be real linear spaces. Let $a = (a_1, a_2, \dots, a_p) \neq (0, 0, \dots, 0)$ with $a_i \in R$ $(i = 1, 2, \dots, p)$, where R :=set of reals. Then a mapping $Q : X \to Y$ is called* quadratic *with respect to $a : |a| = \left(\sum_{i=1}^{p} a_i^2\right)^{1/2}$, if the quadratic functional equation $(*)$ holds for every $x_i \in X$ $(i = 1, 2, \dots, p)$.*

Denote

$$\overline{Q}(x) = \begin{cases} \dfrac{\sum_{i=1}^{p} Q(a_i x)}{\sum_{i=1}^{p} a_i^2}, & \text{if } \left(r, m = \sum_{i=1}^{p} a_i^2 = |a|^2\right) \in I_1 \\ \left(\sum_{i=1}^{p} a_i^2\right) \left[\sum_{i=1}^{p} Q\left(\dfrac{a_i x}{\sum_{i=1}^{p} a_i^2}\right)\right], & \text{if } \left(r, m = \sum_{i=1}^{p} a_i^2 = |a|^2\right) \in I_2 \end{cases} \tag{2}$$

holds for all $x \in X$.

2 Quadratic functional stability

Theorem 2.1. *Let X and Y be normed linear spaces. Assume that Y is complete. Assume in addition that mapping $f : X \to Y$ satisfies the approximately quadratic functional inequality $(**)$. Also assume $f(0) = 0$ in the case $m = 1$. Then the limit*

$$Q(x) = \lim_{n\to\infty} f_n(x) \tag{3}$$

exists for all $x \in X$ and $Q : X \to Y$ is the unique quadratic mapping, such that

$$\|f(x) - Q(x)\| \leq \|x\|^r \begin{cases} \gamma c/(m^2 - m^r), & \text{if } (r, m) \in I_1 \\ \gamma c/(m^r - m^2), & \text{if } (r, m) \in I_2 \\ c/(p - p^{r/2}), & \text{if } (r, m = 1) \in I_3, \end{cases} \tag{4}$$

holds for all $x \in X$ and $c \geq 0$ (real constant independent of $x \in X$).

Existence

Proof. It is useful for the following, to observe that, from $(**)$ with $x_i = 0 (i = 1, 2, .., p)$ and $0 < m \neq 1$, we get

$$|2 + p(p - 1) - 2mp| \, \|f(0)\| \leq 0,$$

or

$$f(0) = 0. \tag{5}$$

Note that for $m = 1$ we assume in addition that (5) holds. Now claim that for $n \in N$

$$\|f(x)-f_n(x)\| \le \|x\|^r \begin{cases} \dfrac{\gamma c}{m^2-m^r}\left(1-m^{n(r-2)}\right), & \text{if } (r,m)\in I_1 : m^{r-2}<1 \\ \dfrac{\gamma c}{m^r-m^2}\left(1-m^{n(2-r)}\right), & \text{if } (r,m)\in I_2 : m^{2-r}<1 \\ \dfrac{c}{p-p^{r/2}}\left(1-p^{n(r-2)/2}\right), & \text{if } (r,m=1)\in I_3 : p^{r-2}<1. \end{cases} \tag{6}$$

For $n=0$, it is trivial.

By replacing Q, $\overline{Q}$ of (2), with f, $\bar{f}$, respectively, one denotes:

$$\bar{f}(x) = \begin{cases} \dfrac{\sum\limits_{i=1}^{p} f(a_i x)}{\sum\limits_{i=1}^{p} a_i^2}, & \text{if } \left(r, m=\sum\limits_{i=1}^{p} a_i^2 = |a|^2\right)\in I_1 \\ \left(\sum\limits_{i=1}^{p} a_i^2\right)\left[\sum\limits_{i=1}^{p} f\left(\dfrac{a_i x}{\sum_{i=1}^{p} a_i^2}\right)\right], & \text{if } \left(r, m=\sum\limits_{i=1}^{p} a_i^2 = |a|^2\right)\in I_2, \end{cases} \tag{7}$$

holds for all $x \in X$.

From (5), (7) and (**), with $x_i = a_i x$ $(i=1,2,\ldots,p)$, we obtain

$$\left\| f(mx) + \binom{p}{2} f(0) - m\sum_{i=1}^{p} f(a_i x)\right\| \le \gamma c\|x\|^r.$$

or

$$\left\| f(mx) - m\left[\sum_{i=1}^{p} f(a_i x)\right]\right\| \le \gamma c\|x\|^r.$$

or

$$\|m^{-2} f(mx) - \bar{f}(x)\| \le \frac{\gamma c}{m^2}\|x\|^r. \tag{8}$$

if I_1 holds. Besides from (5), (7) and $(**)$, with $x_1 = x,\ x_j = 0$ $(j=2,3,\ldots,p)$, we get

$$\left\| f(a_1 x) + \sum_{j=2}^{p} f(a_j x) - m[f(x) + (p-1)f(0)]\right\| \le 0,$$

or

$$\left\| \sum_{i=1}^{p} f(a_i x) - m f(x)\right\| \le 0,$$

or

$$\bar{f}(x) = f(x), \tag{9}$$

if I_1 holds. Therefore from (8) and (9) we have

$$\|f(x) - m^{-2} f(mx)\| \le \frac{\gamma c}{m^2}\|x\|^r = \frac{\gamma c}{m^2-m^r}\left(1-m^{r-2}\right)\|x\|^r, \tag{10}$$

which is (6) for $n=1$, if I_1 holds.

Similarly, from (5), (7) and $(**)$, with $x_i = \dfrac{a_i}{m}x$ $(i=1,2,\ldots,p)$ we obtain

$$\left\| f(x) + \binom{p}{2} f(0) - m\sum_{i=1}^{p} f\left(\frac{a_i}{m}x\right)\right\| \le \frac{\gamma c}{m^2}\|x\|^r.$$

or

$$\|f(x) - \bar{f}(x)\| \leq \frac{\gamma c}{m^r}\|x\|^r, \tag{11}$$

if I_2 holds. Besides from (5), (7) and $(**)$, with $x_1 = \frac{x}{m}$, $x_j = 0$ $(j = 2, 3, \ldots, p)$ we get

$$\left\| f\left(\frac{a_1}{m}x\right) + \sum_{j=2}^{p} f\left(\frac{a_j}{m}x\right) - m[f(m^{-1}x) + (p-1)f(0)] \right\| \leq 0,$$

or

$$\left\| \sum_{i=1}^{p} f\left(\frac{a_i}{m}x\right) - mf(m^{-1}x) \right\| \leq 0,$$

or

$$\bar{f}(x) = m^2 f(m^{-1}x), \tag{12}$$

if I_2 holds. Therefore from (11) and (12) we have

$$\|f(x) - m^2 f(m^{-1}x)\| \leq \frac{\gamma c}{m^r}\|x\|^r = \frac{\gamma c}{m^r - m^2}(1 - m^{2-r})\|x\|^r, \tag{13}$$

which is (6) for $n = 1$, if I_2 holds.

Also, with $x_i = x$ $(i = 1, 2, \ldots, p)$ in $(**)$ and $a_i = b = p^{-1/2}$ $(i = 1, 2, \ldots, p)$, we obtain

$$\|f(pbx) - pf(x)\| \leq c\|x\|^r,$$

or

$$\|f(x) - p^{-1}f(p^{1/2}x)\| = \|f(x) - p^{-1}f((pb)^1 x)\| \leq \frac{c}{p}\|x\|^r = \frac{c}{p - p^{r/2}}[1 - p^{(r-2)/2}]\|x\|^r, \tag{14}$$

which is (6) for $n = 1$, if I_3 holds.

Assume (6) is true if $(r, m) \in I_1$. From (10), with $m^{-n}x$ on place of x, and the triangle inequality, we have

$$\begin{aligned} \|f(x) - f_{n+1}(x)\| &= \|f(x) - m^{-2(n+1)}f(m^{n+1}x)\| \leq \|f(x) - m^{-2n}f(m^n x)\| + \\ &\quad + \|m^{-2n}f(m^n x) - m^{-2(n+1)}f(m^{n+1}x)\| \leq \\ &\leq \frac{\gamma c}{m^2 - m^r}[(1 - m^{n(r-2)}) + m^{-2n}(1 - m^{r-2})m^{nr}]\|x\|^r = \\ &= \frac{\gamma c}{m^2 - m^r}(1 - m^{(n+1)(r-2)})\|x\|^r, \end{aligned} \tag{15}$$

if I_1 holds.

Similarly assume (6) is true if $(r, m) \in I_2$. From (13), with $m^{-n}x$ on place of x, and the triangle inequality, we have

$$\begin{aligned} \|f(x) - f_{n+1}(x)\| &= \|f(x) - m^{2(n+1)}f(m^{-(n+1)}x)\| \leq \|f(x) - m^{2n}f(m^{-n}x)\| + \\ &\quad + \|m^{2n}f(m^{-n}x) - m^{2(n+1)}f(m^{-(n+1)}x)\| \leq \\ &\leq \frac{\gamma c}{m^r - m^2}[(1 - m^{n(2-r)}) + m^{2n}(1 - m^{2-r})m^{-nr}]\|x\|^r = \\ &= \frac{\gamma c}{m^r - m^2}(1 - m^{(n+1)(2-r)})\|x\|^r, \end{aligned} \tag{16}$$

if I_2 holds.

Also, assume (6) is true if $(r, m = 1) \in I_3$. From (14), with $(pb)^n x (= p^{n/2}x)$ on place of x, and the triangle inequality, we have

$$\begin{aligned}\|f(x) - f_{n+1}(x)\| &= \left\|f(x) - p^{-(n+1)} f\left(p^{\frac{n+1}{2}} x\right)\right\| = \|f(x) - p^{-(n+1)} f((pb)^{n+1}x)\| \le \\ &\le \|f(x) - p^{-n} f((pb)^n x)\| + \|p^{-n} f((pb)^n x) - p^{-(n+1)} f((pb)^{n+1}x)\| \le \\ &\le \frac{c}{p - p^{r/2}} \left\{\left[1 - p^{n(r-2)/2}\right] + p^{-n}\left[1 - p^{(r-2)/2}\right](pb)^{nr}\right\} \|x\|^r = \\ &= \frac{c}{p - p^{r/2}} \left[1 - p^{(n+1)(r-2)/2}\right] \|x\|^r,\end{aligned} \tag{17}$$

if I_3 holds.

Therefore inequalities (15), (16) and (17) prove inequality (6) for any $n \in N$.

Claim now that the sequence $\{f_n(x)\}$ converges.

To do this it suffices to prove that it is a Cauchy sequence. Inequality (6) is involved if $(r, m) \in I_1$. In fact, if $i > j > 0$, and $h_1 = m^j x$, we have:

$$\begin{aligned}\|f_i(x) - f_j(x)\| &= \|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\| = m^{-2j} \|m^{-2(i-j)} f(m^{i-j} h_1) - f(h_1)\| \le \\ &\le m^{-2j} \frac{\gamma c}{m^2 - m^r} \left(1 - m^{(i-j)(r-2)}\right) \|x\|^r < \frac{\gamma c}{m^2 - m^r} m^{-2j} \|x\|^r \stackrel{j\to\infty}{\longrightarrow} 0,\end{aligned} \tag{18}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly, if $h_2 = m^{-j} x$ in I_2, we have:

$$\begin{aligned}\|f_i(x) - f_j(x)\| &= \|m^{2i} f(m^{-i} x) - m^{2j} f(m^{-j} x)\| = m^{2j} \|m^{2(i-j)} f(m^{-(i-j)} h_2) - f(h_2)\| \le \\ &\le m^{2j} \frac{\gamma c}{m^r - m^2} \left(1 - m^{(i-j)(2-r)}\right) \|x\|^r < \frac{\gamma c}{m^r - m^2} m^{2j} \|x\|^r \stackrel{j\to\infty}{\longrightarrow} 0,\end{aligned} \tag{19}$$

if I_2 holds: $m^{2-r} < 1$.

Also, if $h_3 = p^{j/2} x$ in I_3, we have:

$$\begin{aligned}\|f_i(x) - f_j(x)\| &= \|p^{-i} f(p^{i/2} x) - p^{-j} f(p^{j/2} x)\| = p^{-j} \|p^{-(i-j)} f(p^{(i-j)/2} h_3) - f(h_3)\| \le \\ &\le p^{-j} \frac{c}{p - p^{r/2}} \left(1 - p^{(i-j)(r-2)/2}\right) \|x\|^r < \frac{c}{p - p^{r-2}} p^{-j} \|x\|^r \stackrel{j\to\infty}{\longrightarrow} 0,\end{aligned} \tag{20}$$

if I_3 holds: $p^{r-2} < 1$.

Then inequalities (18), (19) and (20) define a mapping $Q : X \to Y$ in p-variables $x_i \in X$ $(i = 1, 2, \ldots, p)$, given by (3).

Claim that from $(**)$ and (3) we can get $(*)$, or equivalently that the afore-mentioned *well-defined mapping* $Q : X \to Y$ *is quadratic with respect to* a $(\neq 0)$.

In fact, it is clear from the functional inequality $(**)$ and the limit (3) for $(r, m) \in I_1$ that the following functional inequality

$$\begin{aligned}m^{-2n} \left\| f\left(\sum_{i=1}^{p} a_i m^n x_i\right) + \sum_{1\le i<j\le p} f\left(a_j m^n x_i - a_i m^n x_j\right) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} f(m^n x_i)\right]\right\| \le \\ \le m^{-2n} c \prod_{i=1}^{p} \|m^n x_i\|^{r_i},\end{aligned}$$

holds for all vectors $(x_1, x_2, .., x_p) \in X^p$, and all $n \in N$ with $p = 2, 3, 4, \dots$ and $f_n(x) = m^{-2n} f(m^n x) : I_1$ holds. Therefore

$$\left\| \lim_{n\to\infty} f_n \left(\sum_{i=1}^p a_i x_i\right) + \lim_{n\to\infty} \sum_{1\le i<j\le p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^p a_i^2\right) \left[\sum_{i=1}^p \lim_{n\to\infty} f_n(x_i)\right] \right\| \le$$

$$\le \left(\lim_{n\to\infty} m^{n(r-2)}\right) c \prod_{i=1}^p \|x_i\|^{r_i} = 0,$$

because $m^{r-2} < 1$ or

$$\left\| Q\left(\sum_{i=1}^p a_i x_i\right) + \sum_{1\le i<j\le p} Q(a_j x_i - a_i x_j) - \left(\sum_{i=1}^p a_i^2\right) \left[\sum_{i=1}^p Q(x_i)\right] \right\| = 0, \qquad (21)$$

or mapping Q satisfies the quadratic functional equation $(*)$.

Similarly, from $(**)$ and (3) for $(r, m) \in I_2$ we get that

$$m^{2n} \left\| f\left(\sum_{i=1}^p a_i m^{-n} x_i\right) + \sum_{1\le i<j\le p} f\big(a_j m^{-n} x_i - a_j m^{-n} x_j\big) - \left(\sum_{i=1}^p a_i^2\right) \left[\sum_{i=1}^p f(m^{-n} x_i)\right] \right\| \le$$

$$\le m^{2n} c \prod_{i=1}^p \|m^{-n} x_i\|^{r_i},$$

holds for all vectors $(x_1, x_2, \dots, x_p) \in X^p$, and all $n \in N$ with $f_n(x) = m^{2n} f(m^{-n} x) : I_2$ holds. Thus

$$\left\| \lim_{n\to\infty} f_n \left(\sum_{i=1}^p a_i x_i\right) + \lim_{n\to\infty} \sum_{1\le i<j\le p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^p a_i^2\right) \left[\sum_{i=1}^p \lim_{n\to\infty} f_n(x_i)\right] \right\| \le$$

$$\le \left(\lim_{n\to\infty} m^{n(2-r)}\right) c \prod_{i=1}^p \|x_i\|^{r_i} = 0,$$

because $m^{2-r} < 1$, or (21) holds or mapping Q satisfies $(*)$.

Also, from $(**)$ and (3) for $(r, m = 1) \in I_3$ we obtain that

$$p^{-n} \left\| f\left(\sum_{i=1}^p a_i p^{n/2} x_i\right) + \sum_{1\le i<j\le p} f\left(a_j p^{n/2} x_i - a_i p^{n/2} x_j\right) - \left(\sum_{i=1}^p a_i^2\right) \left[\sum_{i=1}^p f(p^{n/2} x_i)\right] \right\| \le$$

$$\le p^{-n} c \prod_{i=1}^p \|p^{n/2} x_i\|^{r_i},$$

holds for all vectors $(x_1, x_2, \dots, x_p) \in X^p$, and all $n \in N$ with $f_n(x) = p^{-n} f(p^{n/2} x) : I_3$ holds.

Hence

$$\left\| \lim_{n\to\infty} f_n\left(\sum_{i=1}^{p} a_i x_i\right) + \lim_{n\to\infty} \sum_{1\le i<j\le p} f_n(a_j x_i - a_i x_j) - \left(\sum_{i=1}^{p} a_i^2\right)\left[\sum_{i=1}^{p} \lim_{n\to\infty} f_n(x_i)\right] \right\| \le$$

$$\le \left(\lim_{n\to\infty} p^{n(r-2)/2}\right) c \prod_{i=1}^{p} \|x_i\|^{r_i} = 0,$$

because $p^{r-2} < 1$, or (21) holds or mapping Q satisfies $(*)$.

Therefore (21) holds if I_j $(j = 1, 2, 3)$ hold or mapping Q satisfies the quadratic functional equation $(*)$, completing the proof that Q *is a quadratic mapping with respect to* a in X.

It is now clear from (6) with $n \to \infty$, as well as from the formula (3) that the functional inequality (4) holds in X. This completes *the existence proof* of the aforementioned Theorem 2.1.

Uniqueness

Let $Q' : X \to Y$ *be a quadratic mapping with respect to* a *satisfying* (4), *as well as* Q. *Then* $Q' = Q$.

Proof. Condition

$$Q(x) = \begin{cases} m^{-2n} Q(m^n x), & \text{if } (r, m) \in I_1 \\ m^{2n} Q(m^{-n} x), & \text{if } (r, m) \in I_2 \\ p^{-n} Q(p^{n/2} x), & \text{if } (r, m) \in I_3 \end{cases} \tag{22}$$

holds for all $x \in X$ and $n \in N$ where p is arbitrary but fixed and equals to $2, 3, 4, \ldots,$ as a consequence of (6) with $c = 0$. Remember Q' satisfies (22), as well, for $(r, m) \in I_1$, too. Then for every $x \in X$ and $n \in N$,

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|m^{-2n} Q(m^n x) - m^{-2n} Q'(m^n x)\| \le \\ &\le m^{-2n} \{\|Q(m^n x) - f(m^n x)\| + \|Q'(m^n x) - f(m^n x)\|\} \le \\ &\le m^{-2n} \frac{2\gamma c}{m^2 - m^r} \|m^n x\|^r = m^{n(r-2)} \frac{2\gamma c}{m^2 - m^r} \|x\|^r \to 0, \text{ as } n \to \infty, \end{aligned} \tag{23}$$

if I_1 holds: $m^{r-2} < 1$.

Similarly for $(r, m) \in I_2$, we establish

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|m^{2n} Q(m^{-n} x) - m^{2n} Q'(m^{-n} x)\| \le \\ &\le m^{2n} \{\|Q(m^{-n} x) - f(m^{-n} x)\| + \|Q'(m^{-n} x) - f(m^{-n} x)\|\} \le \\ &\le m^{2n} \frac{2\gamma c}{m^r - m^2} \|m^{-n} x\|^r = m^{n(2-r)} \frac{2\gamma c}{m^r - m^2} \|x\|^r \to 0, \text{ as } n \to \infty, \end{aligned} \tag{24}$$

if I_2 holds: $m^{2-r} < 1$.

Also for $(r, m = 1) \in I_3$, we get

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \|p^{-n} Q(p^{n/2} x) - p^{-n} Q'(p^{n/2} x)\| \le \\ &\le p^{-n} \left\{\|Q(p^{n/2} x) - f(p^{n/2} x)\| + \|Q'(p^{n/2} x) - f(p^{n/2} x)\|\right\} \le \\ &\le p^{-n} \frac{2c}{p - p^{r/2}} \|p^{n/2} x\|^r = p^{n(r-2)/2} \frac{2c}{p - p^{r/2}} \|x\|^r \to 0, \text{ as } n \to \infty, \end{aligned} \tag{25}$$

if I_3 holds: $p^{r-2} < 1$. Thus from (23), (24) and (25) we find $Q(x) = Q'(x)$ for all $x \in X$.

This completes the proof of the *uniqueness* and the *stability* of the quadratic functional equation $(*)$. ∎

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