LIBERTAS MATHEMATICA vol XXVI (2006)

The Royden Boundary and the Dirichlet Problem in a Standard H-cone

Liliana POPA

Abstract. In the case of a self adjoint harmonic space F.Y.Maeda has constructed a Royden boundary and has solved the associated Dirichlet problem ([5]). The harmonic space is taken in the sense of Constantinescu - Cornea ([4]). An analogous situation in the global axiomatic theory of potential is the case of an autodual H-cone which satisfies the axiom D ([1]). So, in [6] we proved that in a standard autodual H-cone which satisfies the axiom D, a Royden boundary can be constructed and relative to it the Dirichlet problem can be solved. We drop the hypothesis of autoduality and the axiom D and we extend the previous results to this new situation. We prove that the solution of the Dirichlet problem relative to the Royden boundary minimizes the norm given by Dirichlet integral, in a suitable class of bounded fonctions with finite Dirichlet integrals.

We shall use the notations and definition of [1]. Let be $S = S^{**}$ *a standard H-cone of functions on a set X.* S_0 is the set of all universally continuous elements of S. X is endowed with the *natural topology* which is the coarsest topology for which the universally elements are continuous. X is a measurable space relative to the σ-algebra of the borelian sets.

We recall the definition of the *specific order*.

$$s, t \in S, \quad s \prec t \text{ if } \exists u \in S, \text{ such that } s + u = t.$$

In [2] were introduced and studied the subtractible elements and the pure potentials. An element $h \in S$ is called *subtractible* if for any $s \in S$ for which $h \leq s$, it results $h \prec s$. The set of all subtractible elements is denoted by S_H. An element $p \in S$ is called *pure potential* if for any subtractible element h, we have $s \curlywedge h = 0$. S_P is naturally solid in S and any $s \in S$ has a unique decomposition of the form

$$s = h + p, \quad h \in S_H, \quad p \in S_P.$$

We recall from [1] that for any pair of weak units u and u^*, there exists a *Green set* E, such that $X \setminus E$ is semi-polar and g is the associated *Green function.* The set E is a natural measurable set.

Let be $\mathcal{P}(E)$ the set of all *Green potentials*, that means the set of all elements $p \in S$

for which there exists a positive Borel measure μ on E, such that

$$p(x) = \int_E g(x,y)d\mu(y).$$

We shall suppose the following hypothesis:

1. *E is semi-saturated*
2. *S satisfies the axiom of polarity* on E
3. *axiom A holds*
4. *the natural sheaf property holds on E*
5. $1 \in S_H$.

The first hypothesis is equivalent with each of the following statements. $\mathcal{P}(E)$ is natural solid in S; or any pure potential of S belongs to $\mathcal{P}(E)$. The second implies that any balayage is representable. In this case it results that $S_0 \subset \mathcal{P}(E)$ and that the nearly continuous elements of S are in $\mathcal{P}(E)$. The third is equivalent with the inclusion $S_0 \subset S_P$. The hypothesis 4 implies the axiom D_0.

We consider a correspondence, denoted σ, which associates to each element of the cone $s \in S$, a measure denoted $\sigma(s) \in \mathcal{M}(E)$ (the set of all σ-finite measures on E). σ is additive, positively homogenous and have the property

$$s \geq 0 \Leftrightarrow \sigma(s) \geq 0.$$

Such a correspondence is called *measure of representation.*

The map $\sigma : S \to \mathcal{M}(E)$ defined for any $s = h + p$, with $h \in S_H, p \in S_P$ by

$$\sigma(s) := \mu \tag{1}$$

where μ is the measure from the definition of the Green potentials, is called *canonical measure of representation* and it was studied in [7]. In the presence of the previous hypothesis, we proved that this measure of representation has the property

$$\sigma(s)|_U = \sigma_U(s|_U)$$

where for any natural open set $U \subset E$, $\mathcal{S}(U)$ is the *localization* of the H-cone S and σ_U is the canonical measure of representation which corresponds to $\mathcal{S}(U)$. Let

$$S_b = \{s \in S \mid \exists M > 0,\ s \leq M\}$$

We consider the set denoted by $\mathcal{R}$ of all functions $f : X \to \mathbb{R}$ such that there exists a covering of natural open sets $U_i, i \in I$, such that $f|_{U_i} = s - t, s, t \in \mathcal{S}_b(U_i)$. We have proved in [6] that the set $\mathcal{R}$ is an algebra relative to the usual product of functions.

If $f, g \in \mathcal{R}$ we call *mutual Dirichlet Integral*

$$\delta_{[f,g]}(1) := \frac{1}{2}\left(\int_E f\, d\sigma(g) + \int_E g\, d\sigma(f) - \int_E d\sigma(fg)\right)$$

In [6] were studied the properties of Dirichlet integral. We recall the positivity, that means

$$\delta_f(1) := \delta_{[f,f]} \geq 0 \tag{2}$$

for any $f \in \mathcal{R}$. If $p \in S_P$ and $h \in S_H$ are bounded, natural continuous and $\sigma(p)(1) < \infty$, then

$$\delta_{[p,h]}(1) = 0. \tag{3}$$

The previous results were proved in a particular case of an autodual H-cone of functions on X, for which the axiom D holds. In the presence of the hypothesis 1-5 the statements are still true, as we have proved in [9]. So, we can also extend the results from [8] concerning the Royden boundary and the solution of Dirichlet problem in this more general framework.

We shall denote by

$$P_D =: \{p \in S_P \cap S_b | \int_E \int_E g(x,y) d\sigma(p)(x) d\sigma(p)(y) < \infty\}$$

$$\mathcal{H}_D := \{h \in S_H \cap S_b \mid \delta_h(1) < \infty\}$$

$$\mathcal{Q}_D := \{f \mid f = h_1 - h_2 + p_1 - p_2, h_i \in \mathcal{H}_D, p_i \in P_D, i = 1, 2 \text{ natural continuous}\}.$$

Because the natural topology on E is complectly regular, one can apply Constantinescu-Cornea's compactification [3] relative to the set $\mathcal{Q}_D$. So there exists a compact space E^* which contains E and satisfies the conditions

1. the restriction to E of the topology on E^* coincides with the natural topology;
2. $\overline{E} = E^*$;
3. any function $f \in \mathcal{Q}_D$ can be uniquely extended by continuity to E^*. We shall denote this extension also by f.

Definition. The set

$$\Gamma_h := \{\xi \in E^* \mid p(\xi) = 0, \forall p \in P_D, \text{natural continuous}\}.$$

is called *Royden harmonic boundary.*

We denote by $\mathcal{D}(\Gamma_h)$ the set of all functions $\varphi : \Gamma_h \to [0, \infty)$ for which there exists $f \in \mathcal{Q}_D$, such that

$$\lim_{x \to \xi, x \in E} f(y) = \varphi(\xi), \quad \forall \xi \in \Gamma_h.$$

That is, the functions from $\mathcal{D}(\Gamma_h)$ are the traces on the Royden boundary of the functions from $\mathcal{Q}_D$. Since the compact is considered with respect to $\mathcal{Q}_D$, it follows that any functions from $\mathcal{D}(\Gamma_h)$ is continuous on Γ_h.

Remarks. The set Γ_h has the following properties:

1. $\Gamma_h \cap E = \emptyset$. Indeed for any $x \in E$ there exists $s \in S_0$ such that $s(x) > 0$; it follows that $x \notin \Gamma_h$.
2. $\mathcal{H}_D$ separates the points of Γ_h.
3. The set Γ_h is a naturally compact set, because the elements of P_D are natural continuous.

Theorem 1. *The set* Γ_h *is nonempty.*

Proof. Let us suppose that $\Gamma_h = \emptyset$; then there exists $p \in P_D$, naturally continuous, such that $p > 0$ on E^*. Let $\alpha > 0$ be such that $p \geq \alpha$ on E^*; because p is a potential and $\alpha \in S_H$ it results that $\alpha = 0$, which is a contradiction. ■

The following lemma is analogous to a similar result from [5] and [6].

Lemma. *Let be a compact set $K \subset E^*$ such that $K \cap \Gamma_h = \emptyset$. There exists $p \in P_D$, naturally continuous such that $p = 1$ on K and $0 \leq p \leq 1$ on E^*.*

We can prove a principle of minimum relative to the Royden boundary.

Theorem 3. *If $u \in \mathcal{H}_D$ and $u \geq 0$ on Γ_h, then $u \geq 0$ on E.*
Proof. Let $u \in \mathcal{H}_D, u \geq 0$ on Γ_h and $\varepsilon > 0$. We consider the set

$$K_\varepsilon = \{x \in E^* \mid u(x) + \varepsilon \leq 0\}.$$

K_ε is a compact set in E^*; because we have supposed $u \geq 0$ on Γ_h it results that $K_\varepsilon \cap \Gamma_h = \emptyset$. From previous Lemma there exists $p \in P_M$ such that $p = 1$ on K_ε and $0 \leq p \leq 1$ on E^*. Let be now $M > 0$ such that $\mid u \mid \leq M$. Then we have $u = \varepsilon + Mp \geq 0$ on E^*. We get $u + \varepsilon \geq 0$ on E; since ε is arbitrary, we deduce that $u \geq 0$ on E. ■

Remark. It follows that if $|u| \leq M$ on Γ_h, then $|u| \leq M$ on E.

The next results shows that a certain form of the Dirichlet problem for the Royden boundary can be solved.

Theorem 4. *For each $\varphi \in \mathcal{D}(\Gamma_h)$ there exists $u_\varphi \in \mathcal{H}_D$ such that $\forall \xi \in \Gamma_h$, $\lim_{x \to \xi, x \in E} u_\varphi(x) = \varphi(\xi)$. Then if $\varphi \geq 0$, it results that $u_\varphi \geq 0$.*
Proof. Let $f \in \mathcal{Q}_D$ be such that $\lim_{x \in E, x \to \xi} f(x) = \varphi(\xi)$, $\forall \xi \in \Gamma_h$. From the relation $f = u + g$, $u \in [\mathcal{H}_D]$, $g \in [P_D]$ it results that $\lim_{x \in E, x \to \xi} u(x) = \varphi(\xi)$, $\forall \xi \in \Gamma_h$. The uniqueness and the others implications are valid from the previous theorem. ■

Remark. If $|\varphi| \leq M$, then $|u_\varphi| \leq M$.

Theorem 5. *The set $\mathcal{D}(\Gamma_h)$ is dense in $\mathcal{C}(\Gamma_h)$.*
Proof. We prove that $\mathcal{D}(\Gamma_h)$ is an algebra; then the assertion follows from the fact that it contains the constants and separates the points. It is sufficient to prove that if $f \in \mathcal{D}(\Gamma_h)$, then $f^2 \in \mathcal{D}(\Gamma_h)$. Let be $f = h + g$, with $h \in [\mathcal{H}_D]$, $g \in [P_D]$. Then $f|_{\Gamma_h} = h|_{\Gamma_h}$ and $f^2|_{\Gamma_h} = h^2|_{\Gamma_h}$. It results that $h^2|_{\Gamma_h} \in \mathcal{D}(\Gamma_h)$. ■

For any $x \in E$ the map $\varphi \to u_\varphi(x)$ is a bounded linear and positive functional on $\mathcal{D}(\Gamma_h)$, as it results from Theorem 3. Because $\mathcal{D}(\Gamma_h)$ is dense in $\mathcal{C}(\Gamma_h)$, this functional can be extended to a unique linear and positive functional on $\mathcal{C}(\Gamma_h)$, hence there exists a nonnegative measure μ_x on Γ_h, which will be called *harmonic measure*, such that for any $x \in E$ and any $\varphi \in \mathcal{D}(\Gamma_h)$

$$\int_{\Gamma_h} \varphi \, d\mu_x = u_\varphi(x).$$

Theorem 6. *For any $\varphi \in \mathcal{C}(\Gamma_h)$*

$$u_\varphi(x) = \int_{\Gamma_h} \varphi \, d\mu_x, \quad \forall x \in E \tag{4}$$

defines an element of $[S_H \cap S_b]$. The following assertion holds $\forall \xi \in \Gamma_h$:

$$\lim_{x \in E, x \to \xi} u_\varphi(x) = \varphi(\xi).$$

Proof. Because $\mathcal{D}(\Gamma_h)$ is dense in $\mathcal{C}(\Gamma_h)$ we can chose $\varphi_n \in \mathcal{D}(\Gamma_h)$ such that $\varphi_n \to \varphi$ uniformly on Γ_h. From the Theorem 4, it results that u_{φ_n} is uniformly convergent on E. But

$$u_\varphi(x) = \int_{\Gamma_h} \varphi \, d\mu_x = \lim_{n\to\infty} \int_{\Gamma_h} \varphi_n \, d\mu_x = \lim_{n\to\infty} u_{\varphi_n}(x)$$

for any $x \in E$, hence $u_\varphi \in [S_H \cap S_b]$.

Because u_{φ_n} is continuous on E^*, it results that $u_{\varphi_n}|_{\Gamma_h} = \varphi_n$ and $\varphi_n \to \varphi$ uniformly on Γ_h. We observe that u_φ, extended by φ to Γ_h, is continuous on $E \cup \Gamma_h$. Hence $u_\varphi(x) \to \varphi(\xi)$, if $x \to \xi$, $\xi \in \Gamma_h$ ■

Using the properties of the Dirichlet integral we can prove that the solution of the Dirichlet problem minimises the norm given by the Dirichlet integral.

Theorem 7. *For any $\varphi \in \mathcal{D}(\Gamma_h)$ the next assertion holds*

$$\delta_{u_\varphi}(1) = \min\{\delta_f(1) \mid \forall f \in \mathcal{Q}_D, f|_{\Gamma_h} = \varphi\}. \tag{5}$$

Proof. Let us observe that for any $\varphi \in \mathcal{D}(\Gamma_h)$, the function u_φ given by (4) defines an element of $[\mathcal{H}_D] \subset \mathcal{Q}_D$, hence the minimum is attained. Let $f = h_1 - h_2 + p_1 - p_2$ be naturally continuous and $\lim_{x\in E, x\to\xi} f(x) = \varphi(\xi),\ \xi \in \Gamma_h$, then

$$f|_{\Gamma_h} = \varphi = (h_1 - h_2)|_{\Gamma_h}.$$

Hence

$$f = u_\varphi + p_1 - p_2.$$

If we use (3), the value of the Dirichlet integral is

$$\delta_f(1) = \delta_{u_\varphi + p_1 - p_2}(1) = \delta_{u_\varphi}(1) + \delta_{p_1 - p_2}(1) \geq \delta_{u_\varphi}(1).$$

The last statement is valid because of the positivity of the Dirichlet integral (2). ■

References

[1] N.Boboc, Gh.Bucur, A.Cornea, *Order and Convexity in Potential Theory: H-Cones*, Lecture Notes 853, Springer Verlag Berlin, Heidelberg New York 1981.

[2] N. Boboc, Gh. Bucur, *Potentials and Pure Potentials in H-Cones*, Rev. Roum. Pures Appl. 27, 1982, nr 5, 529-549.

[3] C. Constantinescu, A. Cornea, *Ideale Rander Riemannscher Flachen* Springer Verlag Berlin, Heidelberg New York 1963.

[4] C. Constantinescu, A. Cornea, *Potential Theory on Harmonic Spaces*, Springer Verlag Berlin, Heidelberg New York 1972.

[5] F.Y.Maeda, *Dirichlet Integrals on Harmonic Spaces*, Lecture Notes 803. Berlin, New York, 1980.

[6] L.Popa, *Dirichlet Integrals on Standard H–Cones*, Bull. Math. Soc. Sci. Math. Roum. 37 (79) nr 2, 1987, 153-161.

[7] L.Popa, *Measures of representation on a Standard H-Cone*, Rev. Roum. Math. Pures Appl., XLI, 7-8, 1996.

[8] L.Popa, *Integrala Dirichlet pe H–conuri standard autoduale II*, Studii şi cercetări matematice 37, nr 6, 1985, 500-531.

[9] L.Popa, *Dirichlet Integral on Standard H-Cones*, Ed Demiurg, Iasi 2005, ISBN 973-7603-12-5.