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Applications of Sălăgean Differential Operator at the Class of Meromorphic Functions

Gh. OROS and Georgia Irina OROS

Abstract. By using the Sălăgean operator $D^n f(z)$, $z \in U$, we introduce a class of holomorphic functions denoted by $\Sigma_n(\alpha)$ and we obtain some inclusion relations.

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1 Introduction and Preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\dot{U} = U - \{0\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic function in U.

We let:

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$.

Let Σ denote the class of functions in U of the form

$$f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \dots, \quad z \in \dot{U}.$$

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We use the following subordination results.

Lemma A. (Hallenbeck and Ruscheweyh [2, p. 71]) *Let h be a convex function with $h(0) \equiv a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\mathrm{Re}\gamma \geq 0$*. If $p \in \mathcal{H}(U)$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z) \prec h(z)$$

where

$$g(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function g is convex and is the best dominant.

(The definition of best dominant is given in [2].)

Lemma B. (Miller and Mocanu [1]) *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z),$$

where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_n z^n + \dots$ is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Definition 1. [4] For $f \in A$ and $n \in \mathbb{N}^* \cup \{0\}$ the operator $D^n f$ is defined by

$$D^0 f(z) = f(z)$$

$$D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U.$$

Remark 1. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U$$

then

$$D^n f(z) = z + \sum_{j=2}^{\infty} J^n a_j z^j, \quad z \in U.$$

2 Main Results

Definition 2. If $0 \le \alpha < 1$, and $n \in \mathbb{N}$, let $\Sigma_n(\alpha)$ denote the class of function $f \in \Sigma$ which satisfy the inequality

$$\text{Re}\left[D^n\left(-\frac{z^2 f'(z)}{f(z)}\right)\right]' > \alpha, \quad z \in U.$$

Theorem 1. *If $0 \le \alpha < 1$ and $n \in \mathbb{N}$, then*

$$\Sigma_{(n+1)}(\alpha) \subset \Sigma_n(\delta),$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt.$$

Proof. Let $f \in \Sigma_{(n+1)}(\alpha)$. By using the properties of the operator $D^n f$ we have

$$D^{n+1}\left(-\frac{z^2 f'(z)}{f(z)}\right) = z\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]', \quad z \in U. \tag{1}$$

Differentiating (1), we obtain

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = \left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' + z\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]'', \quad z \in U. \tag{2}$$

If we let $p(z) = \left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]'$, then (2) becomes

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = p(z) + zp'(z), \quad z \in U. \tag{3}$$

Since $f \in \Sigma_{(n+1)}(\alpha)$, by using Definition 2 we have

$$\mathrm{Re}[p(z) + zp'(z)] > \alpha \tag{4}$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1+(2\alpha-1)z}{1+z} \equiv h(z), \quad z \in U.$$

By using Lemma A, we have

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

where

$$g(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t} dt = 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z}.$$

The function g is convex and is the best dominant. Since $p(z) \prec g(z)$, it results that

$$\mathrm{Re}\, p(z) > \mathrm{Re}\, g(1) = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2$$

from which we deduce that $\Sigma_{(n+1)}(\alpha) \subset \Sigma_n(\delta)$.

Theorem 2. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z).$$

If $f \in \Sigma$ and verifies the differential subordination

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec h(z), \quad z \in U \tag{5}$$

then

$$\left[D^{n}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. By using the properties of operator $D^n f$ we have

$$D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right) = z\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]', \quad z \in U.$$

By differentiating, we obtain

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = \left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' + z\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]''.$$

If we let

$$p(z) = \left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]', \quad z \in U$$

then we obtain

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = p(z) + zp'(z), \quad z \in U,$$

and (5) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z), \quad z \in U.$$

By using Lemma B, we have

$$p(z) \prec g(z),$$

i.e.

$$\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec g(z), \quad z \in U. \qquad \square$$

Theorem 3. *Let g be a convex function $g(0) = 1$, and*

$$h(z) = g(z) + zg'(z).$$

If $f \in \Sigma$ and verifies the differential subordination

$$\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec h(z), \quad z \in U, \tag{6}$$

then

$$\frac{D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)}{z} \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. We let

$$p(z) = \frac{D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)}{z}, \quad z \in U$$

and we obtain

$$D^n\left(\frac{-z^2 f'(z)}{f(z)}\right) = zp(z), \quad z \in U.$$

By differentiating, we obtain

$$\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = p(z) + zp'(z), \quad z \in U.$$

Then (6) becomes

$$p(z) + zp'(z) \prec h(z) \equiv g(z) + zg'(z), \quad z \in U.$$

By using Lemma B we have

$$p(z) \prec g(z),$$

i.e.

$$\frac{D^n\left(\dfrac{-z^2 f'(z)}{f(z)}\right)}{z} \prec g(z), \quad z \in U. \qquad \square$$

Theorem 4. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in \Sigma$ and verifies the differential subordination

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec h(z), \quad z \in U, \tag{7}$$

then

$$\left[D^{n}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec g(z), \quad z \in U$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [2, Corollary 2.6.g.2, p. 66] shows that the function g is convex.

From

$$D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right) = z\left[D^{n}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]', \quad z \in U,$$

we have

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = \left[D^{n}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' + z\left[D^{n}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]''.$$

If we let

$$p(z) = \left[D^{n}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]',$$

then we obtain

$$\left[D^{n+1}\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = p(z) + zp'(z), \quad z \in U$$

and (7) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

By using Lemma A we have

$$p(z) \prec g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U,$$

i.e.

$$\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec \frac{1}{z}\int_0^z h(t)dt, \quad z \in U. \qquad \square$$

Theorem 5. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in \Sigma$ and verifies the differential subordination

$$\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' \prec h(z), \quad z \in U \tag{8}$$

then

$$\frac{D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)}{z} \prec g(z), \quad z \in U,$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [2, Corollary 2.6.g.2, p. 66] shows that the function g is convex.

We let

$$p(z) = \frac{D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)}{z}, \quad z \in U,$$

and we obtain

$$D^n\left(\frac{-z^2 f'(z)}{f(z)}\right) = zp(z), \quad z \in U.$$

By differentiating, we obtain

$$\left[D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)\right]' = p(z) + zp'(z), \quad z \in U.$$

Then (8) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

By using Lemma A we have

$$p(z) \prec g(z), \quad z \in U,$$

i.e.

$$\frac{D^n\left(\frac{-z^2 f'(z)}{f(z)}\right)}{z} \prec g(z), \quad z \in U$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

The function g is convex and is best dominant.

Corollary 1. *If $f \in \Sigma_n(\alpha)$, then*

$$\operatorname{Re} \frac{D^n\left(\dfrac{-z^2 f'(z)}{f(z)}\right)}{z} > 2\alpha - 1 + 2(1-\alpha)\ln 2, \quad z \in U.$$

Proof. From Theorem 5 we deduce

$$\frac{D^n\left(\dfrac{-z^2 f'(z)}{f(z)}\right)}{z} \prec g(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U,$$

where

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}, \quad z \in U.$$

Hence

$$\operatorname{Re} \frac{D^n\left(\dfrac{-z^2 f'(z)}{f(z)}\right)}{z} > \operatorname{Re} g(1) = 2\alpha - 1 + 2(1-\alpha)\ln 2. \qquad \square$$

We remark that in the class of meromorphic functions similar results were obtained by M. Pap in [3].

References

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