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The Essential Uniqueness of the Connes-Morley Theorem

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Abstract. We use work by A. Connes to explore possible generalizations of Morley's trisector theorem to triangles in arbitrary valued fields. We find that Morley's theorem is essentially an unique phenomenon. However, tri-sectioning procedures different from Morley's do exist for generic sets of triangles.

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1 Introduction

Around 1899 F. Morley proved a remarkable theorem on the elementary geometry of Euclidean triangles: In any triangle, the intersections of adjacent trisectors form the vertices of an equilateral triangle. For a detailed account of the fascinating history of this problem the reader is directed to A. Bogomolny's Web page [B].

During a recent debate one member of the deliberating panel mentioned this result, and (rightly) attributed it to A. Connes too. Connes has made fundamental contributions to mathematics and besides doing monumental work in Noncommutative Geometry has given a beautiful proof to Morley's theorem by considering the group of affine transformations of the line over an arbitrary field [C].

It was the first time we heard about Morley's result and when we went home, following past advice of H. Moscovici, we began to think about it by studying Connes' work first. Our only motivation besides curiosity was the obvious challenge: This is one of the rare achievements of Connes we should be able to compete with. After a few unsuccessful attempts and much hard work we realized that Connes' work can be generalized in two directions: first, by doubling the number of affine transformations, the concept of trisector can be replaced by arbitrary angle tri-sectioning, and second, the classical Morley set-up, which corresponds to the Euclidean plane (identified with the field of complex numbers) can be extended to arbitrary valued fields.

Our main finding (see Theorems 3.2 and 3.5 below) is that Morley's theorem is an unique phenomenon, in the following sense:

a) A valued field for which Morley's problem has a solution must necessarily be (isomorphic to) a subfield of the field of complex numbers, with absolute value equivalent to the standard one.

b) For a subfield of the field of complex numbers no other (continuously varying with the angle) procedure of internally tri-sectioning the angles of a triangle, except Morley's, yields equilateral triangles. Therefore, the subfield must contain the cube roots of all its elements of absolute value one.

We also find (Theorem 4.1), somewhat surprisingly, that for a generic set of complex triangles there are solutions to Morley's problem totally different from Morley's.

The main results of this paper have been announced in [A].

Acknowledgement. We would like to thank A. Connes for borrowing freely from his work [C].

2 A General Result

Let K be a *valued field*, that is a field of arbitrary characteristic equipped with an absolute value function $|\cdot|: K \to \mathbf{R}_+$, satisfying the following properties, for all $a, b \in K$,

$$\begin{gathered} |ab| = |a||b| \\ |a+b| \leq |a| + |b| \\ |a| = 0 \ \text{ if and only if } \ a = 0. \end{gathered} \tag{2.1}$$

Typical examples of valued fields will be the familiar fields $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, equipped with the standard absolute value $|\cdot|_0$, or $\mathbf{Q}$, equipped with the p-adic absolute value $|\cdot|_p$, p a prime number, defined by

$$\left|\frac{m}{n}\right|_p = p^{-\alpha}, \quad \text{if} \quad \frac{m}{n} = p^{\alpha}\frac{m'}{n'},$$

where $\frac{m}{n} \in \mathbf{Q}$ is irreducible and m', n' are relatively prime to p. All fields admit at least one absolute value, the trivial one, which assigns the value 1 to each non-zero element. Finite fields admit only the trivial absolute value.

An ordered triple (v_1, v_2, v_3) of points in K is said to form a *triangle* if the three possible triangle inequalities hold, e.g., $|v_1 - v_2| < |v_1 - v_3| + |v_3 - v_2|$, etc. v_1, v_2, v_3 must then be mutually distinct. A triangle will be *equilateral* if $|v_1 - v_2| = |v_2 - v_3| = |v_3 - v_1|$. Equilateral triangles admit a simple characterization. To this end denote by Δ the set of *standard* equilateral triangles, that is

$$\Delta = \{j \in K \mid |j| = |j+1| = 1\}.$$

Clearly, $j \in \Delta$ if and only if $(0, 1, -j)$ is an equilateral triangle. Then the following proposition is obvious.

Proposition 2.1. *Let v_1, v_2, v_3, be points in K. Then (v_1, v_2, v_3) is an equilateral triangle if and only if $v_1 \neq v_2$ and*

$$\frac{v_3 - v_1}{v_1 - v_2} \in \Delta.$$

Δ is empty if $K = (\mathbf{Q}, |\cdot|_0)$, or $K = (\mathbf{R}, |\cdot|_0)$ (there are no triangles in these cases), and if $K = (\mathbf{C}, |\cdot|_0)$ it equals the set of nontrivial cube roots of unity, i.e., $\Delta = \{j \in \mathbf{C} \mid j^2 + j + 1 = 0\}$. Also, when $K = (\mathbf{Q}, |\cdot|_p)$ then

$$\Delta = \left\{\frac{m}{n} \in \mathbf{Q} \mid (m,p) = 1, (n,p) = 1, (m+n,p) = 1\right\},$$

so Δ is quite large in this case. Finally, if $K = \mathbf{F}_q$, the field with q elements, then $\Delta = \mathbf{F}_q \setminus \{0, -1\}$.

Let $\mathcal{G}$ denote the group of *affine transformations* of K, that is

$$\mathcal{G} = \{g : K \to K \mid g(x) = ax + b, a \in K \setminus \{0\}, b \in K\}$$

and let $\mathcal{T} = \{g \in \mathcal{G} \mid a = 1\}$ be its subgroup of *translations*. Clearly, $\mathcal{G}$ acts on the set of all triangles, leaving invariant the equilateral ones. Any element in $g \in \mathcal{G} \setminus \mathcal{T}$ admits one and only one fixed point, namely

$$\operatorname{fix}(g) = \frac{b}{1-a}.$$

Let also $\mathcal{R} = \{g \in \mathcal{G} \mid |a| = 1, a \neq 1\}$ be the set of *proper rotations* of K (about their fixed points).

Consider now a triangle (v_1, v_2, v_3) in K and six elements, g_i, g_i', $i = 1, 2, 3$, all in $\mathcal{R}$, such that

$$\operatorname{fix}(g_i) = \operatorname{fix}(g_i') = v_i, \quad i = 1, 2, 3.$$

g_i and g_i' will be interpreted as arbitrary "*trisectors*" of the "*angle at v_i*" of the triangle (v_1, v_2, v_3). Assume that the *adjacent* trisectors g_i' and g_{i+1}, $i = 1, 2, 3$, *intersect*, in the sense that $g_i' \circ g_{i+1} \notin \mathcal{T}$. (Obviously, here we permute the index i circularly, so $i + 1 = 1$ when $i = 3$). Specifically, we have, for $i = 1, 2, 3$,

$$g_i(x) = a_i x + b_i, \quad g_i'(x) = a_i' x + b_i', \quad a_i, a_i', b_i, b_i' \in K, \quad |a_i| = |a_i'| = 1, \quad a_i, a_i' \neq 1.$$

Then the restrictions imposed above amount to

$$\frac{b_i}{1-a_i} = \frac{b_i'}{1-a_i'} = v_i, \quad a_i' a_{i+1} \neq 1, \quad i = 1, 2, 3.$$

Let us denote by w_i the fixed point of $g_i' \circ g_{i+1}$, i.e., the intersection point of the adjacent trisectors g_i' and g_{i+1}. (See Fig. 1). Then

$$w_i = \frac{(1-a_i')v_i + a_i'(1-a_{i+1})v_{i+1}}{1-a_i' a_{i+1}}, \quad i = 1, 2, 3. \tag{2.2}$$

The goal is to find suitable conditions on a_i, a_i', under which (w_1, w_2, w_3) becomes an equilateral triangle. These conditions should depend only on the "angles" of the triangle (v_1, v_2, v_3) and not the triangle itself. It is therefore necessary to define a suitable notion of "angles of a triangle" in a valued field.

Definition 2.1. Given a triangle (v_1, v_2, v_3) in a valued field $(K, |\cdot|)$, an ordered triple $(\theta_1, \theta_2, \theta_3)$ of elements of K, $|\theta_i| = 1$, $\theta_i \neq 1$, $i = 1, 2, 3$, is called a choice of angles for that triangle if the proper rotations $h_i(x) = \theta_i x + v_i(1 - \theta_i)$, $i = 1, 2, 3$, satisfy

$$h_1 \circ h_2 \circ h_3 = 1_{\mathcal{G}} \tag{2.3}$$

For the choice of angles $(\theta_1, \theta_2, \theta_3)$, θ_i will then be called the angle of the triangle at the vertex v_i.

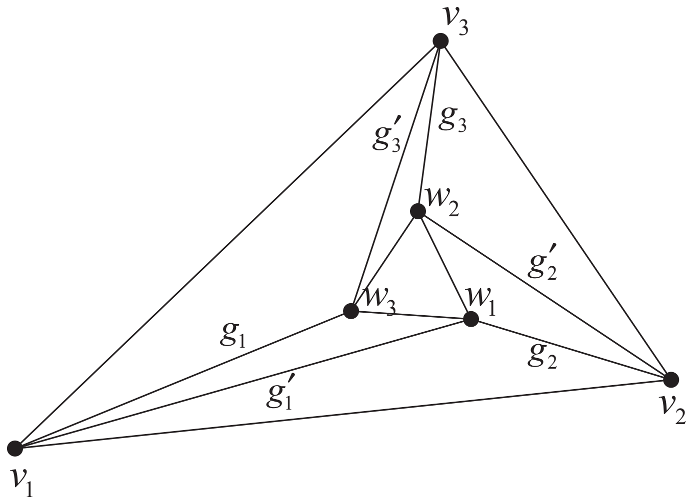

Fig. 1

This apparently strange definition is dictated by what happens in $(\mathbf{C}, |\cdot|_0)$, as it will be seen shortly.

A routine calculation shows that Equation (2.3) is equivalent to

$$\theta_1\theta_2\theta_3 = 1 \quad \text{and} \quad \frac{\theta_1(1-\theta_2)}{1-\theta_1\theta_2} = \frac{v_3 - v_1}{v_2 - v_1}. \tag{2.4}$$

It is clear from (2.4) that two triangles whose vertices correspond under an affine transformation have equal angles. A triangle may not have angles at all or have several choices of angles. For instance, the triangle $(0, 1, 2)$ in $\mathbf{Z}_3$ does not have any angles, while in $(\mathbf{Q}, |\cdot|_p)$, $\left(\theta_1, \frac{2-\theta_1}{\theta_1}, \frac{1}{2-\theta_1}\right)$ is always a choice of angles for the triangle $(0, 1, 2)$, if $|\theta_1|_p = |2 - \theta_1|_p = 1$ and $\theta_1 \neq 1$.

Proposition 2.2. *In $(\mathbf{C}, |\cdot|_0)$ each triangle (v_1, v_2, v_3) has an unique choice of angles $(\theta_1, \theta_2, \theta_3)$, given by*

$$\theta_i = \frac{v_{i+2} - v_i}{\overline{v_{i+2}} - \overline{v_i}} \frac{\overline{v_{i+1}} - \overline{v_i}}{v_{i+1} - v_i}, \quad i = 1, 2, 3, \tag{2.5}$$

where the bar *operation denotes complex conjugation. θ_i is related to the measure α_i, $-\pi < \alpha_i < \pi$, of the oriented Euclidean angle made by the Euclidean rays $\overrightarrow{v_iv_{i+1}}$ and $\overrightarrow{v_iv_{i+2}}$ by*

$$\theta_i = \exp\left(2\sqrt{-1}\alpha_i\right).$$

Proof. Assume that $(\theta_1, \theta_2, \theta_3)$ is a choice of angles for the triangle (v_1, v_2, v_3). It will also be a choice of angles for the triangle $(0, 1, v)$, where $v = \frac{v_3 - v_1}{v_2 - v_1}$, since the two triangles correspond under the affine transformation $g(x) = \frac{x - v_1}{v_2 - v_1}$. Notice that $v \notin \mathbf{R}$, or else $(0, 1, v)$ would not be a triangle. The second part of (2.4) is equivalent to

$$\theta_1\theta_2(v - 1) = v - \theta_1. \tag{2.6}$$

(2.6) gives then $|v-1|_0 = |v-\theta_1|_0$, or, $(v-1)(\overline{v}-1) = (v-\theta_1)(\overline{v}-\overline{\theta_1})$. Since in $(\mathbf{C}, |\cdot|_0)$, $\overline{\theta_1} = \frac{1}{\theta_1}$, this last equation becomes

$$(\theta_1 - 1)(\overline{v}\theta_1 - v) = 0.$$

However, $\theta_1 \neq 1$, so we must have $\theta_1 = \frac{v}{\bar{v}}$, which is exactly (2.5) for $i = 1$. The expressions for θ_2 and θ_3 can then be obtained from (2.4), or by permuting circularly the vertices of the triangle. It is easy to see that by obeying Equations (2.4) the assignment (2.5) does constitute a choice of angles for (v_1, v_2, v_3). Finally, the connection between our angles and the Euclidean angles is a simple consequence of the fact that if in polar coordinates $z = r\exp\left(\sqrt{-1}\alpha\right)$, then $\frac{z}{\bar{z}} = \exp\left(2\sqrt{-1}\alpha\right)$. □

We come now to the main result of this section.

Theorem 2.3. *Let (v_1, v_2, v_3) be a triangle in a valued field $(K, |\cdot|)$ and assume that $(\theta_1, \theta_2, \theta_3)$ is a choice of angles for it. Let $a_i, a_i' \in K$ be such that $a_i, a_i' \neq 1$, $a_i' a_{i+1} \neq 1$, $|a_i| = |a_i'| = 1$, $i = 1, 2, 3$. Let*

$$w_i = \frac{(1 - a_i')v_i + a_i'(1 - a_{i+1})v_{i+1}}{1 - a_i' a_{i+1}}, \quad i = 1, 2, 3.$$

be the trisector intersection points given by Equation (2.2). Then (w_1, w_2, w_3) is an equilateral triangle if and only if the following two conditions hold, for some element $j \in \Delta$:

$$\begin{aligned}
&(1 - a_3'a_1)(-1 + a_1' + a_2' - a_1'a_2'a_2 - a_1'a_2'a_3 + a_1'a_2'a_2a_3 - a_2'\theta_1 + a_2'a_3\theta_1 + \\
&a_1'a_2'a_2\theta_1 - a_1'a_2'a_2a_3\theta_1 + \theta_1\theta_2 - a_1'\theta_1\theta_2 - a_2'a_3\theta_1\theta_2 + a_1'a_2'a_3\theta_1\theta_2)j + \\
&(1 - a_2'a_3)(a_1' - a_1'a_2 - a_1'a_3'a_1 + a_1'a_3'a_1a_2 - \theta_1 + a_3'\theta_1 + a_1'a_2\theta_1 - a_1'a_3'a_2\theta_1 + \\
&\theta_1\theta_2 - a_1'\theta_1\theta_2 - a_3'\theta_1\theta_2 + a_1'a_3'a_1\theta_1\theta_2 + a_1'a_3'a_2\theta_1\theta_2 - a_1'a_3'a_1a_2\theta_1\theta_2) = 0.
\end{aligned} \tag{2.7}$$

$$a_1'(1 - a_3'a_1)(1 - a_2)(1 - \theta_1\theta_2) - (1 - a_1'a_2)(1 - a_3')\theta_1(1 - \theta_2) \neq 0. \tag{2.8}$$

Proof. In matrix form the points w_1, w_2, w_3 are related to the triangle (v_1, v_2, v_3) by $\mathbf{w} = A\mathbf{v}$, where

$$A = \begin{bmatrix} \frac{1-a_1'}{1-a_1'a_2} & \frac{a_1'(1-a_2)}{1-a_1'a_2} & 0 \\ 0 & \frac{1-a_2'}{1-a_2'a_3} & \frac{a_2'(1-a_3)}{1-a_2'a_3} \\ \frac{a_3'(1-a_1)}{1-a_3'a_1} & 0 & \frac{1-a_3'}{1-a_3'a_1} \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

For $g \in \mathcal{G}$ define $\Gamma : K^3 \to K^3$ by $\Gamma(\mathbf{x}) = \mathbf{y}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ g(x_3) \end{bmatrix}.$$

Notice that A commutes with Γ since the entries on each row of A add up to 1. Thus, $\Gamma(\mathbf{w}) = \Gamma(A\mathbf{v}) = A\Gamma(\mathbf{v})$.

(w_1, w_2, w_3) is an equilateral triangle if and only if $(g(w_1), g(w_2), g(w_3))$ is an equilateral triangle. We want to show that for a suitable g, $(g(w_1), g(w_2), g(w_3)) = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$, where

$$\begin{aligned}
\widetilde{w}_1 &= \frac{a_1'(1 - a_2)}{1 - a_1'a_2} \\
\widetilde{w}_2 &= \frac{(1 - a_2')(1 - \theta_1\theta_2) + a_2'(1 - a_3)\theta_1(1 - \theta_2)}{(1 - a_2'a_3)(1 - \theta_1\theta_2)} \\
\widetilde{w}_3 &= \frac{(1 - a_3')\theta_1(1 - \theta_2)}{(1 - a_3'a_1)(1 - \theta_1\theta_2)}.
\end{aligned} \tag{2.9}$$

Let $g(x) = \frac{x-v_1}{v_2-v_1}$. The above observation applied to this particular g gives, via (2.4),

$$\begin{bmatrix} g(w_1) \\ g(w_2) \\ g(w_3) \end{bmatrix} = A\Gamma(\mathbf{v}) = A \begin{bmatrix} 0 \\ 1 \\ \frac{v_3-v_1}{v_2-v_1} \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \\ \frac{\theta_1(1-\theta_2)}{1-\theta_1\theta_2} \end{bmatrix} = \begin{bmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \widetilde{w}_3 \end{bmatrix}.$$

Notice now that Equations (2.7) and (2.8) are a convenient implementation of Proposition 2.1 for the triangle $(\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$. Indeed, (2.7) is equivalent to $(\widetilde{w}_1-\widetilde{w}_2)j+\widetilde{w}_1-\widetilde{w}_3 = 0$, while (2.8) simply states that $\widetilde{w}_1 - \widetilde{w}_3 \neq 0$, which is equivalent to $\widetilde{w}_1 \neq \widetilde{w}_2$, in the presence of (2.7). □

Corollary 2.4. *Assume now that the valued field in Theorem 2.3 is a subfield of* $(\mathbf{C}, |\cdot|_0)$ *containing the cube roots of unity. With the same notations and hypotheses as there, if* (w_1, w_2, w_3) *is an equilateral triangle then there exists* $j \in \mathbf{C}$, $j^2 + j + 1 = 0$, *such that*

$$\begin{aligned} (a_3 - 1)P(\theta_1, \theta_2, a_1, a_2, a_1', a_2', j) &= Q(\theta_1, \theta_2, a_1, a_2, a_1', a_2', j), \\ (a_3' - 1)P(\theta_2, \theta_1, a_2', a_1', a_2, a_1, j) &= Q(\theta_2, \theta_1, a_2', a_1', a_2, a_1, j), \end{aligned} \tag{2.10}$$

where P *and* Q *are polynomial expressions in* $\theta_1, \theta_2, a_1, a_2, a_1', a_2'$, j, *given by*

$$\begin{aligned} P(\theta_1, \theta_2, &a_1, a_2, a_1', a_2', j) \\ &= a_2'\theta_1 j(-1 + a_1 + a_2 - a_1'a_1a_2 - a_1\theta_2 + a_1'a_1a_2\theta_2 + \theta_1\theta_2 - a_2\theta_1\theta_2) \\ &+ a_1'a_2'a_1 - a_2'\theta_1 + \theta_1\theta_2 - \theta_1^2\theta_2 + a_2'\theta_1^2\theta_2 + a_1'a_2\theta_1^2\theta_2 - a_1'a_2'a_2\theta_1^2\theta_2 \\ &- a_1'a_2'a_1a_2 + a_1'a_2'a_2\theta_1 - a_1'a_2\theta_1\theta_2 - a_1'a_2'a_1\theta_1\theta_2 + a_1'a_2'a_1a_2\theta_1\theta_2 \end{aligned}$$

$$\begin{aligned} Q(\theta_1, &\theta_2, a_1, a_2, a_1', a_2', j) \\ &= (1 - a_2')(1 - \theta_1\theta_2)[(a_1 - \theta_1 + a_2\theta_1 - a_1'a_1a_2)j + a_1'a_1 - a_1'a_1a_2 - \theta_1 + a_1'a_2\theta_1] \end{aligned}$$

Four more equations similar to (2.10) *hold true. Formally, they can be obtained by permuting circularly the indices* 1, 2, *and* 3 *in* (2.10).

Proof. Recall that in $(\mathbf{C}, |\cdot|_0)$, $\Delta = \{j \in \mathbf{C} \mid j^2 + j + 1 = 0\}$. Since $j \in \Delta \iff j^2 \in \Delta$, Equation (2.7), with j replaced by j^2, can be rewritten as

$$a_3'E(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j) = F(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j),$$

where E and F are polynomial expressions in $\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j$. The above equation implies

$$\begin{aligned} &a_3'E(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j)\overline{a_3'E(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j)} \\ &= F(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j)\overline{F(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j)}, \end{aligned}$$

where *bar* indicates complex conjugation. In turn, this is equivalent to

$$\begin{aligned} &E(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j)E\left(\theta_1^{-1}, \theta_2^{-1}, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_1'^{-1}, a_2'^{-1}, j^{-1}\right) \\ &= F(\theta_1, \theta_2, a_1, a_2, a_3, a_1', a_2', j)F\left(\theta_1^{-1}, \theta_2^{-1}, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_1'^{-1}, a_2'^{-1}, j^{-1}\right). \end{aligned}$$

Keeping in mind that $j^2 = -j - 1$, this last equation simplifies to give

$$\Omega[(a_3 - 1)P(\theta_1, \theta_2, a_1, a_2, a_1', a_2', j) - Q(\theta_1, \theta_2, a_1, a_2, a_1', a_2', j)] = 0,$$

where Ω is the non-zero quantity

$$\Omega := \frac{(1 - \theta_2)(1 - a_1)(1 - a_1' a_2)(1 - a_2' a_3)}{a_1' a_2' a_1 a_2 a_3 \theta_1 \theta_2 (j + 1)}.$$

Thus, the first equation (2.10) follows. The second can be obtained in a similar manner, by switching the roles of a_3 and a_3'. Of course, by solving (2.7) for a_1, a_1', respectively a_2, a_2', one gets the other four equations mentioned in the statement of the corollary. □

Remark 2.1. The result of the above corollary is geometrically clear. Since four trisectors determine two vertices of an equilateral triangle, they also determine the third vertex, i.e., the remaining two trisectors, up to orientation, which is accounted for by the presence of a nontrivial cube root of unity j. It is important to stress that j is the same in all six equations of type (2.10).

3 Applications

In this section we will present applications to Theorem 2.3 and Corollary 2.4.

Morley's theorem. *Let $(K, |\cdot|)$ be a valued field such that every element of $\{a \in K \mid |a| = 1\}$ admits three distinct cube roots in K. Let (v_1, v_2, v_3) be a triangle in K with a choice of angles $(\theta_1, \theta_2, \theta_3)$. Let t_i be a cube root of θ_i, $i = 1, 2, 3$, with the property that $t_1 t_2 t_3 \neq 1$. Consider the trisectors $g_i = g_i'$, $i = 1, 2, 3$, given by $g_i(x) = g_i'(x) = t_i x + v_i(1 - t_i)$. Then the intersections w_i of the adjacent trisectors g_i' and g_{i+1}, $i = 1, 2, 3$, form the vertices of an equilateral triangle.*

Proof. By hypothesis, $a_i = a_i' = t_i$ and $\theta_i = t_i^3$, $i = 1, 2, 3$. It is obvious that $a_i, a_i' \neq 1$, $a_i' a_{i+1} \neq 1$, $|a_i| = |a_i'| = 1$, $i = 1, 2, 3$. The proof will consist in showing that this assignment of trisector angles satisfies Equations (2.7) and (2.8) of Theorem 2.3, if the element j is taken to be $j = \frac{1}{t_1 t_2 t_3}$. Such j belongs to Δ because

$$j^3 = \frac{1}{t_1^3 t_2^3 t_3^3} = \frac{1}{\theta_1 \theta_2 \theta_3} = 1,$$

and since $j \neq 1$, this is equivalent to $j^2 + j + 1 = 0$. However, $j^3 = 1$ and $j + 1 = -j^2$ imply $|j| = |j + 1| = 1$, i.e. $j \in \Delta$.

(2.8) becomes then

$$\frac{t_1}{t_3}(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_1 t_2)(1 - t_3 t_1) \neq 0,$$

which holds since t_i is a cube root of $\theta_i \neq 1$, $i = 1, 2, 3$. By setting now

$$a_i = a_i' = t_i, \quad \theta_i = t_i^3, \quad i = 1, 2, \quad a_3 = a_3' = \frac{1}{j t_1 t_2},$$

in Equation (2.7) this becomes, after a straightforward calculation,

$$\frac{(j^2+j+1)(1-t_1)(1-t_2)(1-t_1t_2)(1-t_1t_2^2-t_2j+t_1^2t_2^2j)}{t_2j^2}=0.$$

The proof of Morley's theorem is complete. □

The above theorem brings up a very natural problem.

Morley's problem. Given a valued field $(K, |\cdot|)$ find functions $a_i(T)$, $a_i'(T)$, $i=1,2,3$, defined on the set of all triangles $T=(v_1(T), v_2(T), v_3(T))$ of K, with values in $\{a \in K|\ |a|=1, a\neq 1\}$, which satisfy the following two conditions:

a) For each $i=1,2,3$, $a_i'(T)a_{i+1}(T)\neq 1$, and the adjacent trisector intersection points $w_1(T), w_2(T), w_3(T)$, given by Equation (2.2), form the vertices of an equilateral triangle.

b) For each $i=1,2,3$, $a_i(T), a_i'(T)$ depend only on the "i^{th} angle of the triangle T", in the sense that if T_1 and T_2 are two triangles in K and $\Theta_1=(\theta_1^1,\theta_2^1,\theta_3^1)$, respectively $\Theta_2=(\theta_1^2,\theta_2^2,\theta_3^2)$, are choices of angles for T_1, respectively T_2, such that $\theta_i^1=\theta_i^2$, then $a_i(T_1)=a_i(T_2)$ and $a_i'(T_1)=a_i'(T_2)$.

Notice that Morley's problem makes sense regardless whether triangles have several choices of angles or none. Notice also that the whole discussion is irrelevant if the absolute value $|\cdot|$ is trivial since all triangles are equilateral in this case.

Theorem 3.1. *Morley's problem has no solution in a non-Archimedean field $(K, |\cdot|)$ with non-trivial absolute value.*

Proof. Recall that $(K, |\cdot|)$ is a non-Archimedean valued field if the triangle inequality in (2) is replaced by a stronger condition, the ultrametric inequality:

$$|a+b|\le \max\{|a|,|b|\}, \qquad a,b\in K. \tag{3.11}$$

It follows immediately from (3.11) that

$$|a+b|=|a|, \quad \text{if} \quad |b|<|a|. \tag{3.12}$$

In non-Archimedean valued fields every three distinct points form a triangle. We will show that if the absolute value is non-trivial then every triangle admits a choice of angles. By making use of (2.4) as in the proof of Proposition 2.2 it is easy to see that a triangle (v_1,v_2,v_3) admits a choice of angles if and only if there exists $\theta_1\in K$, $|\theta_1|=1$, $\theta_1\neq 1$ such that

$$|v-1|=|v-\theta_1|, \quad \text{where} \quad v=\frac{v_3-v_1}{v_2-v_1}. \tag{3.13}$$

θ_1 will then be the angle at v_1. Since the absolute value $|\cdot|$ is non-trivial there is an element $\epsilon\in K$ such that $0<|\epsilon|<1$. Then $|1+\epsilon|=1$, by (3.12), and so the set $\{a\in K|\ |a|=1, a\neq 1\}$ is non-empty.

If $|v|\neq 1$, then *any* $\theta_1\in\{a\in K|\ |a|=1, a\neq 1\}$ satisfies $|v-1|=|v-\theta_1|$, again by (3.12). If $|v|=1$ then $\theta_1=1+\epsilon^n$ will work, if n is large enough.

Assume now that Morley's problem has a solution $a_i(T), a_i'(T)$, $i=1,2,3$, T triangle in K. If $(\theta_1,\theta_2,\theta_3)$ is a choice of angles for T, then θ_1 is also the first angle of a suitable choice of angles for the particular triangle $(0,1,\epsilon)$, by the above discussion.

Thus, $a_1(T) = a_1((0,1,\epsilon))$, $a'_1(T) = a'_1((0,1,\epsilon))$, i.e., a_1 and a'_1 are constant functions. By permuting circularly vertices and angles of triangles this argument applies also to a_2, a'_2, a_3, a'_3. As a result, any solution of the Morley's problem must necessarily consist in constant functions $a_i(T) = a_i$, $a'_i(T) = a'_i$, $i = 1, 2, 3$.

Notice that

$$\frac{a'_1(1-a_2)}{1-a'_1a_2} = \frac{1-a'_3}{1-a'_3a_1}, \tag{3.14}$$

otherwise the elements $\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3$ given by Equations (2.9), associated to the triangle $\left(0, 1, \frac{a'_1(1-a_2)}{1-a'_1a_2} \frac{1-a'_3a_1}{1-a'_3}\right)$, cannot be the vertices of a triangle, because $\widetilde{w}_1 = \widetilde{w}_3$. Similarly,

$$\frac{a'_1(1-a_2)}{1-a'_1a_2} = \frac{1-a'_2}{1-a'_2a_3}, \tag{3.15}$$

or else a triangle in K would exist with $\widetilde{w}_1 = \widetilde{w}_2$.

Now (3.14) and (3.15) show that all the non-zero elements of the matrix A appearing in the proof of Theorem 2.3 must equal $\frac{1}{2}$, otherwise a triangle in K could be found for which $\widetilde{w}_2 = \widetilde{w}_3$. However, this is a contradiction since $(0, 1, \epsilon)$ generates a triangle $(\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)$ which obviously is not equilateral. The proof is complete. □

Theorem 3.2. *If Morley's problem has a solution in a valued field $(K, |\cdot|)$ then $(K, |\cdot|)$ must be isomorphic to a subfield of the field of complex numbers, equipped with an absolute value equivalent to the standard one.*

Proof. By Theorem 3.1, $(K, |\cdot|)$ must be an Archimedean field. By Ostrovski's theorem [R], an Archimedean field is isomorphic (as valued field) with a subfield of $(\mathbf{C}, |\cdot|_0^q)$, $0 < q \leq 1$. □

Proposition 3.3. *If K is a subfield of $\mathbf{C}$, then Morley's problem has a solution in $(K, |\cdot|_0^q)$, $0 < q \leq 1$, if and only if it has a solution in $(K, |\cdot|_0)$.*

Proof. It is easy to see that if $v \in K$ then $(0, 1, v)$ is a triangle in $(K, |\cdot|_0^q)$, $0 < q < 1$, if and only if $v \neq 0, v \neq 1$, while $(0, 1, v)$ is a triangle in $(K, |\cdot|_0)$ if and only if $v \notin K \cap \mathbf{R}$. Therefore, there are more triangles is $(K, |\cdot|_0^q)$ than in $(K, |\cdot|_0)$, the surplus being supplied by triangles of type $(0, 1, v)$, $v \in \mathbf{R}$. An argument similar to that given in Proposition 2.2 shows however that these extra triangles have no angles – basically the angles would have to equal 1, since $v = \overline{v}$. In conclusion, the triangles admitting angles are the same for both absolute values, and so is Morley's problem □

We are therefore left with analyzing the existence of solutions to Morley's problem for subfields K of $\mathbf{C}$, equipped with the standard absolute value $|\cdot|_0$. We will assume that K contains the complex cube roots of unity since otherwise there are no equilateral triangles.

Proposition 3.4. *If $(K, |\cdot|_0)$ is a valued field as above, then any element of $\{a \in K \mid |a|_0 = 1, a \neq 1\}$, is an angle in some triangle. In other words, the set of angles is as large as possible. Moreover, this set of angles is dense in the unit circle $\mathbf{S}^1$ of $\mathbf{C}$.*

Proof. Fix $\theta \in K$, $|\theta|_0 = 1$, $\theta \neq 1$. By Proposition 2.2, θ will be the first angle of a triangle $(0, 1, v)$ if a $v \in K \setminus \mathbf{R}$ can be found such that $\theta = \frac{v}{\bar{v}}$. To this end, if $\theta \neq -1$ we can choose $v = 1 + \theta$, and if $\theta = -1$ then $v = \frac{1}{2} + j$ will work, j a nontrivial cube root of unity.

A more general proof would be to choose τ arbitrarily in $\{a \in K \mid |a|_0 = 1, a \neq 1, a \neq \frac{1}{\theta},\}$ and to realize that, by (2.4), $\left(\theta, \tau, \frac{1}{\theta\tau}\right)$ is the set of angles of the triangle $\left(0, 1, \frac{\theta(1-\tau)}{1-\theta\tau}\right)$.

The density claim follows from the fact that for any nontrivial cube root of unity j, $\mathbf{Q}(j) := \{a + bj | a, b \in \mathbf{Q}\}$ is a subset of K dense in $\mathbf{C}$. Then the angles $\frac{v}{\bar{v}}$, $v \in \mathbf{Q}(j)$, are dense in $\mathbf{S}^1$. □

The above proposition, Theorem 2.3, and Corollary 2.4, show that a solution to Morley's problem in $(K, |\cdot|_0)$ amounts to the existence of functions $a_i(\theta)$, $a'_i(\theta)$, $i = 1, 2, 3$, with domain and codomain $\{a \in K \mid |a|_0 = 1, a \neq 1\}$ such that if $(\theta_1, \theta_2, \theta_3)$ are angles in a triangle T then $a'_i(\theta_i)a_{i+1}(\theta_{i+1}) \neq 1$ and Equations (2.7), (2.8), and (2.10) hold, if a_i, a'_i, $i = 1, 2, 3$ are replaced by $a_i(\theta_i)$, $a'_i(\theta_i)$.

Definition 3.1. Call a triangle T in $(K, |\cdot|_0)$ with angles

$$(\theta_1, \theta_2, \theta_3) = \left(\exp\left(\sqrt{-1}\arg(\theta_1)\right), \exp\left(\sqrt{-1}\arg(\theta_2)\right), \exp\left(\sqrt{-1}\arg(\theta_3)\right)\right),$$

$0 < \arg(\theta_i) < 2\pi$, $\quad i = 1, 2, 3$, *positively oriented* if $\sum_i \arg(\theta_i) = 2\pi$. Otherwise, T will be *negatively oriented*, and $\sum_i \arg(\theta_i) = 4\pi$.

We are now going to make two very reasonable assumptions, which guarantee that we are searching for solutions to Morley's problem which are most natural, in the sense that the tri-sectioning

a) Occurs always "inside the angles of triangles", as in Fig. 1, and

b) Varies continuously with the angles of the triangle.

Assumption 1. We assume that the functions $a_i(\theta)$, $a'_i(\theta)$, $i = 1, 2, 3$, in a solution to Morley's problem, which are necessarily of type

$$f\left(\exp\left(\sqrt{-1}s\right)\right) = \exp\left(\sqrt{-1}\alpha(s)\right),$$
$$0 < s, \alpha(s) < 2\pi, \quad \exp\left(\sqrt{-1}s\right), \exp\left(\sqrt{-1}\alpha(s)\right) \in K,$$

all satisfy the condition $\quad \alpha(s) \leq s \quad$ for every s.

One immediate consequence of this assumption is that the following "one sided limits" exist and equal 1,

$$\lim_{\theta \curvearrowright 1} a_i(\theta) = 1, \ \lim_{\theta \curvearrowright 1} a'_i(\theta) = 1, \quad i = 1, 2, 3. \tag{3.16}$$

where $\theta \curvearrowright 1$ is a suggestive way of stating that θ approaches 1 by elements of the unit circle $\{a \in K \mid |a|_0 = 1\}$, situated above 1, i.e., $\Im(\theta) > 0$.

Assumption 2. We also assume that a solution of Morley's problem in $(K, |\cdot|_0)$ has the property that the functions $a_i(\theta), a'_i(\theta), i = 1, 2, 3$ are *continuous* on $\{a \in K \mid |a| = 1, a \neq 1\}$ and that for positively, respectively negatively, oriented triangles j is the same in anyone of the Equations (2.7), (2.10), say $j = j_+$, respectively $j = j_-$.

It is easy to see that if $K = \mathbf{C}$ then Assumption 2 holds if solely the continuity of the tri-sectioning functions a_i, a'_i, is required. Indeed, the positively and negatively oriented triangles form two connected sets in the obvious topology induced by $|\cdot|_0$ on the set of triangles in $\mathbf{C}$. This is so because via affine equivalence the positively oriented triangles correspond to triangles $(0, 1, v)$ with $\Im(v) > 0$, while for the negatively oriented ones $\Im(v) < 0$. Equations (2.7) and (2.10) show then that the dependence of j on triangles (or their angles) is continuous.

We are now in a position to state and prove the main result of this paper.

Theorem 3.5. *Let $(K, |\cdot|_0)$ be a subfield of the field of complex numbers equipped with the induced standard absolute value. Then Morley's problem admits a solution satisfying the above Assumptions 1 and 2 if and only if the complex cube roots of all the elements of $\{a \in K \mid |a|_0 = 1\}$ belong to K. Moreover, the solution is unique, namely Morley's trisector solution.*

Proof. The "if" part is essentially contained in Morley's Theorem, discussed at the beginning of this section. Indeed, if one defines the trisector functions a_i, a'_i, $i = 1, 2, 3$, by the same formula

$$\exp(\sqrt{-1}s) \longmapsto \exp\left(\sqrt{-1}\frac{s}{3}\right), \quad 0 < s < 2\pi, \quad \exp(\sqrt{-1}s) \in K,$$

then these are clearly functions which satisfy the Assumptions 1 and 2 and all the hypotheses of Morley's Theorem. For instance, if $(\theta_1, \theta_2, \theta_3)$ are the angles of some triangle T in K,

$$\theta_i = \exp(\sqrt{-1}\arg(\theta_i)), \quad 0 < \arg(\theta_i) < 2\pi, \quad i = 1, 2, 3,$$

then the t_1, t_2, t_3 of Morley's Theorem become

$$t_i = \exp\left(\sqrt{-1}\frac{\arg(\theta_i)}{3}\right), \quad i = 1, 2, 3,$$

and so $t_1t_2t_3 = \exp\left(\sqrt{-1}\frac{2\pi}{3}\right) \neq 1$, for positively oriented triangles T, and $t_1t_2t_3 = \exp\left(\sqrt{-1}\frac{4\pi}{3}\right) \neq 1$, for negatively oriented triangles.

The proof of the far more complicated "only if" part will occupy a good part of the rest of the paper, and will consist in a sequence of lemmas showcasing a very delicate infinitesimal analysis of the situation.

In the following Lemmas, 3.6 through 3.15, we assume that a_i, a'_i, $i = 1, 2, 3$, is a solution to the Morley's problem on $(K, |\cdot|_0)$ satisfying the Assumptions 1 and 2 above.

Lemma 3.6. *There is a sequence $\{\omega_n\}_n$, $\omega_n \in \{a \in K | \ |a|_0 = 1, a \neq 1\}$, $\omega_n \curvearrowright 1$ as $n \to \infty$, such that the following limits exist:*

$$\lim_{n\to\infty} a_i\left(\omega_n^{-1}\right) := l_i, \quad \lim_{n\to\infty} a'_i\left(\omega_n^{-1}\right) := l'_i, \quad i = 1, 2, 3, \tag{3.17}$$

$$\lim_{n\to\infty} \frac{a_i(\omega_n) - 1}{\omega_n - 1} := d_i, \quad \lim_{n\to\infty} \frac{a'_i(\omega_n) - 1}{\omega_n - 1} := d'_i, \quad i = 1, 2, 3. \tag{3.18}$$

Proof. It suffices to show that all the quantities

$$a_i\left(\omega^{-1}\right), a'_i\left(\omega^{-1}\right), \frac{a_i(\omega) - 1}{\omega - 1}, \frac{a'_i(\omega) - 1}{\omega - 1}, \quad i = 1, 2, 3,$$

are bounded as $\omega \curvearrowright 1$. The statement is obvious for $a_i(\omega^{-1}), a_i'(\omega^{-1})$, since these functions take values in $\{a \in K|\ |a|_0 = 1\}$. For the remaining derivative-like quantities we have, for instance

$$\frac{a_i(\omega)-1}{\omega-1} = \frac{\exp\left(\sqrt{-1}\alpha(s)\right)-1}{\exp\left(\sqrt{-1}s\right)-1} = \frac{\exp\left(\sqrt{-1}\alpha(s)\right)-1}{\sqrt{-1}\alpha(s)} \frac{\sqrt{-1}s}{\exp\left(\sqrt{-1}s\right)-1} \frac{\alpha(s)}{s},$$

which is clearly bounded as $s \to 0^+$, since according to Assumption 1, $\alpha(s) \leq s$. □

Remark 3.1. The above limits are obviously a first step in trying to establish that the functions $a_i(\theta)$, $a_i'(\theta)$ admit limits also when $\theta \circlearrowleft 1$ and that they admit one-sided derivatives when $\theta \curvearrowright 1$. Notice that the quantities d_i, d_i', are *real*, and belong to the interval $[0,1]$.

Lemma 3.7. *For any $\theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}$ the following functional equations hold true, together with similar ones obtained by permuting circularly the indices* $1,2,3$*:*

$$\begin{aligned} &\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right]\left[1 - j_- a_2'\left(\theta^{-1}\right) a_1(\theta)\right](l_3 - 1) = 0, \\ &\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right]\left[1 - j_- a_2'\left(\theta^{-1}\right) a_1(\theta)\right](l_3' - 1) = 0. \end{aligned} \tag{3.19}$$

$$\begin{aligned} &-\theta\left[1 - a_2'(\theta)\right]\left[1 - j_+ a_2(\theta)\right]\left[1 - a_1\left(\theta^{-1}\right)\right] + (1-\theta)\left[a_2'(\theta) a_2(\theta) - j_+\theta\right] \\ &\quad \cdot \left[1 - a_1\left(\theta^{-1}\right)\right] d_3 + (1-\theta)\left[j_+ a_2(\theta) - \theta a_1\left(\theta^{-1}\right)\right]\left[1 - a_2'(\theta)\right] d_3' = 0. \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\left[1 - a_2'(\theta)\right]\left[1 + j_+ - j_+ a_2(\theta) - \theta a_1'\left(\theta^{-1}\right) - j_+\theta - a_1'\left(\theta^{-1}\right) a_2(\theta)\right. \\ &\quad \left. + \theta a_1'\left(\theta^{-1}\right) a_2(\theta) + j_+\theta a_1'\left(\theta^{-1}\right) a_2(\theta)\right] - (1-\theta)\left[1 - a_1'\left(\theta^{-1}\right) a_2(\theta)\right] \\ &\quad \cdot \left[1 - j_+ a_2'(\theta)\right] d_3 - (1+j_+)(1-\theta)\left[1 - a_1'\left(\theta^{-1}\right) a_2(\theta)\right]\left[1 - a_2'(\theta)\right] d_3' = 0. \end{aligned} \tag{3.21}$$

$$\begin{aligned} &-\theta\left[1 - a_1(\theta)\right]\left[1 - j_+ a_1'(\theta)\right]\left[1 - a_2'\left(\theta^{-1}\right)\right] + (1-\theta)\left[1 - a_1(\theta)\right] \\ &\quad \cdot \left[j_+ a_1'(\theta) - \theta a_2'\left(\theta^{-1}\right)\right] d_3 + (1-\theta)\left[a_1'(\theta) a_1(\theta) - j_+\theta\right]\left[1 - a_2'\left(\theta^{-1}\right)\right] d_3' = 0. \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\left[1 - a_1(\theta)\right]\left[1 + j_+ - j_+\theta - a_1'(\theta) a_2\left(\theta^{-1}\right) - j_+ a_1'(\theta) - \theta a_2\left(\theta^{-1}\right)\right. \\ &\quad \left. + \theta a_1'(\theta) a_2\left(\theta^{-1}\right) + j_+\theta a_1'(\theta) a_2\left(\theta^{-1}\right)\right] - (j_+ + 1)(1-\theta)\left[1 - a_1(\theta)\right] \\ &\quad \cdot \left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right] d_3 - (1-\theta)\left[1 - j_+ a_1(\theta)\right]\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right] d_3' = 0. \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\left[1 - a_2'\left(\theta^{-1}\right)\right]\left[\theta - j_+ a_1(\theta) + j_+\theta - a_1'(\theta) a_1(\theta) + a_1'(\theta) a_1(\theta) a_2\left(\theta^{-1}\right)\right. \\ &\quad \left. - \theta a_1'(\theta) a_2(\theta^{-1}) - j_+\theta a_2\left(\theta^{-1}\right) + j_+ a_1'(\theta) a_1(\theta) a_2\left(\theta^{-1}\right)\right] \\ &\quad + (1-\theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right]\left[1 - j_+ a_2'\left(\theta^{-1}\right) a_1(\theta)\right] d_3 = 0. \end{aligned} \tag{3.24}$$

$$\begin{aligned} &\left[1 - a_1(\theta)\right]\left[1 + j_+ - j_+ a_1'(\theta) - a_1'(\theta) a_2\left(\theta^{-1}\right) - \theta a_2'\left(\theta^{-1}\right) a_2\left(\theta^{-1}\right)\right. \\ &\quad \left. - j_+\theta a_2'\left(\theta^{-1}\right) + \theta a_1'(\theta) a_2'\left(\theta^{-1}\right) a_2\left(\theta^{-1}\right) + j_+\theta a_1'(\theta) a_2'\left(\theta^{-1}\right) a_2\left(\theta^{-1}\right)\right] \\ &\quad - (1-\theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right]\left[1 - j_+ a_2'\left(\theta^{-1}\right) a_1(\theta)\right] d_3' = 0. \end{aligned} \tag{3.25}$$

Proof. Fix an element $\theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}$. For n large enough the triple, $\left(\theta\omega_n, \theta^{-1}, \omega_n^{-1}\right)$, ω_n as in Lemma 3.6, is a set of angles for some triangle, by Proposition

2.6. Moreover, this triangle is negatively oriented. Set now $\theta_1 = \theta\omega_n$, $\theta_2 = \theta^{-1}$, and $j = j_-$ in the two equations (2.10). By continuity, as $n \longrightarrow \infty$, these equations yield

$$\begin{aligned}(l_3-1)P\left(\theta,\theta^{-1},a_1(\theta),a_2\left(\theta^{-1}\right),a_1'(\theta),a_2'\left(\theta^{-1}\right),j_-\right)=0,\\(l_3'-1)P\left(\theta^{-1},\theta,a_2'\left(\theta^{-1}\right),a_1'(\theta),a_2\left(\theta^{-1}\right),a_1(\theta),j_-\right)=0.\end{aligned}\tag{3.26}$$

Since,

$$\begin{aligned}&P\left(\theta,\theta^{-1},a_1(\theta),a_2\left(\theta^{-1}\right),a_1'(\theta),a_2'\left(\theta^{-1}\right),j_-\right)\\&\qquad=(1-\theta)\left[1-a_1'(\theta)a_2\left(\theta^{-1}\right)\right]\cdot\left[1-j_-a_2'\left(\theta^{-1}\right)a_1(\theta)\right],\\&P\left(\theta^{-1},\theta,a_2'\left(\theta^{-1}\right),a_1'(\theta),a_2\left(\theta^{-1}\right),a_1(\theta),j_-\right)\\&\qquad=-\theta^{-1}(1-\theta)\left[1-a_1'(\theta)a_2\left(\theta^{-1}\right)\right]\left[1-j_-a_2'\left(\theta^{-1}\right)a_1(\theta)\right],\end{aligned}$$

the equations (3.26) are equivalent to the equations (3.19).

In order to prove (3.20) we start from

$$(a_1-1)P(\theta_2,\theta_3,a_2,a_3,a_2',a_3',j)-Q(\theta_2,\theta_3,a_2,a_3,a_2',a_3',j)=0,\tag{3.27}$$

an equation similar to the first equation (2.10). For n large enough $(\theta_1,\theta_2,\theta_3) := (\theta^{-1}\omega_n^{-1},\theta,\omega_n)$ form a set of angles for a positively oriented triangle. Set $\theta_2=\theta$, $\theta_3=\omega_n$, $j=j_+$, in (3.27). Equations (3.18) show that (3.27) admits the following first order Taylor expansion as $\omega_n \curvearrowright 1$, $a_3(\omega_n)\to 1$, $a_3'(\omega_n)\to 1$:

$$\begin{aligned}&-\theta\left[1-a_2'(\theta)\right]\left[1-j_+a_2(\theta)\right]\left[1-a_1\left(\theta^{-1}\omega_n^{-1}\right)\right](\omega_n-1)\\&\quad+(1-\theta)\left[a_2'(\theta)a_2(\theta)-j_+\theta\right]\left[1-a_1\left(\theta^{-1}\omega_n^{-1}\right)\right](a_3(\omega_n)-1)\\&\quad+(1-\theta)\left[j_+a_2(\theta)-\theta a_1\left(\theta^{-1}\omega_n^{-1}\right)\right]\left[1-a_2'(\theta)\right](a_3'(\omega_n)-1)+O\left((\omega_n-1)^2\right)=0.\end{aligned}$$

Dividing the above equation by ω_n-1 and letting $n\to\infty$ proves (3.20).

In order to prove (3.21) we repeat the proof of (3.20) for the companion equation of (3.27),

$$(c_1-1)P(\theta_3,\theta_2,a_3',a_2',a_3,a_2,j)-Q(\theta_3,\theta_2,a_3',a_2',a_3,a_2,j)=0.$$

Similarly, for the proof of (3.22), respectively (3.23), we use a first order Taylor expansion as $\theta_3\curvearrowright 1$, $a_3(\theta_3)\to 1$, $a_3'(\theta_3)\to 1$ of the equation

$$(c_2-1)P(\theta_1,\theta_3,a_1',a_3',a_1,a_3,j)-Q(\theta_1,\theta_3,a_1',a_3',a_1,a_3,j)=0,$$

respectively

$$(a_2-1)P(\theta_3,\theta_1,a_3,a_1,a_3',a_1',j)-Q(\theta_3,\theta_1,a_3,a_1,a_3',a_1',j)=0,$$

and the angle assignment $(\theta_1,\theta_2,\theta_3) := (\theta,\theta^{-1}\omega_n^{-1},\omega_n)$, corresponding to a positively oriented triangle.

Finally, the proof of (3.24) requires a first order Taylor expansion as $\theta_3\curvearrowright 1$, $a_3(\theta_3)\to 1$, of the first equation (2.10) and angles $(\theta,\theta^{-1}\omega_n^{-1},\omega_n)$, $j=j_+$, while the proof of (3.25) involves a Taylor expansion as $\theta_3\curvearrowright 1$, $a_3'(\theta_3)\to 1$ of the second equation (2.10) and the same angles. □

Lemma 3.8. *For every angle* $\theta\in\{a\in K|\ |a|_0=1, a\neq 1\}$, $a_i'(\theta)a_{i+1}\left(\theta^{-1}\right)\neq 1$, $i=1,2,3$.

Proof. Without loss of generality, assume there is $\theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}$ such that $a_1'(\theta)a_2\left(\theta^{-1}\right) = 1$. Then Equation (3.21) of Lemma 3.7, with θ replaced by θ^{-1}, implies

$$j_+\left[1 - a_2\left(\theta^{-1}\right)\right] + \theta^{-1}\left[1 - a_1'(\theta)\right] = 0, \tag{3.28}$$

while Equation (3.23) implies

$$j_+\left[1 - a_1'(\theta)\right] + \theta\left[1 - a_2\left(\theta^{-1}\right)\right] = 0. \tag{3.29}$$

Clearly, (3.28) and (3.29) are contradictory. □

Remark 3.2. In general, $a_i'(\theta_i)a_{i+1}(\theta_{i+1}) \neq 1, i = 1, 2, 3$, if $(\theta_1, \theta_2, \theta_3)$ are angles of some triangle, by the very definition of a solution to Morley's problem. Lemma 3.8 states that these inequalities still hold when one of the angles degenerates to 1.

Lemma 3.9. *There is no* $i \in \{1, 2, 3\}$ *such that* $d_i = d_i' = 0$.

Proof. Assume, for example, that $d_3 = d_3' = 0$. Then (3.22) becomes

$$-\theta\left[1 - a_1(\theta)\right]\left[1 - j_+a_1'(\theta)\right]\left[1 - a_2'\left(\theta^{-1}\right)\right] = 0, \quad \theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}$$

Clearly, the above equation cannot be satisfied for θ close to 1, $\Im\theta > 0$. □

Lemma 3.10. *Assume that at least one of the limits* l_i, l_i', $i = 1, 2, 3$, *is not equal to* 1. *Then they all equal* $1/j_-$, *and in fact full limits as* $\theta \circlearrowleft 1$ *exist and are equal:*

$$\lim_{\theta \circlearrowleft 1} a_i(\theta) = \lim_{\theta \circlearrowleft 1} a_i'(\theta) = \frac{1}{j_-}, \quad i = 1, 2, 3. \tag{3.30}$$

Moreover,

$$a_{i+1}'\left(\theta^{-1}\right) = \frac{1}{j_-a_i(\theta)}, \quad i = 1, 2, 3, \quad \theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}. \tag{3.31}$$

Proof. We will prove (3.30) and (3.31) simultaneously. Without loss of generality, assume $l_3 \neq 1$ or $l_3' \neq 1$. In either case, Lemma 3.8 applied to the appropriate equation (3.19) gives

$$a_2'\left(\theta^{-1}\right) = \frac{1}{j_-a_1(\theta)}, \quad \theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}.$$

As a result,

$$\lim_{\theta \circlearrowleft 1} a_2'(\theta) = \lim_{\theta \circlearrowleft 1} a_1(\theta) = \frac{1}{j_-}.$$

In particular, $l_1 \neq 1$ and $l_2' \neq 1$. We can repeat the above argument all over, by making use of equations similar to (3.19), like

$$\begin{aligned}
\left[1 - a_2'(\theta)a_3\left(\theta^{-1}\right)\right]\left[1 - j_-a_3'\left(\theta^{-1}\right)a_2(\theta)\right](l_1 - 1) = 0, \\
\left[1 - a_3'(\theta)a_1\left(\theta^{-1}\right)\right]\left[1 - j_-a_1'\left(\theta^{-1}\right)a_3(\theta)\right](l_2' - 1) = 0,
\end{aligned}$$

to conclude the proof. □

Lemma 3.11. *If at least one of the limits* l_i, l_i', $i = 1, 2, 3$, *is not equal to* 1, *then* $j_+ \neq j_-$ *and all the derivatives* d_i, d_i', $i = 1, 2, 3$, *equal* 1/3.

Proof. The lemma will be an easy consequence of the following three groups of equations involving the derivatives $d_i, d_i',\ i = 1, 2, 3$, which hold true under the given hypothesis:

$$d_i' = d_{i+1}, \quad i = 1, 2, 3. \tag{3.32}$$

$$(j_+ - j_-)(d_i + d_{i+1}) + (1 + j_+)(1 - j_-)(d_{i+2} - 1) = 0, \quad i = 1, 2, 3. \tag{3.33}$$

$$\begin{aligned} d_i[(j_+ + 2)(d_i - 1) + (2j_+ + 1)d_i' - (j_+ + j_- + 1)d_{i+2}] \\ = (j_- + 2)d_i(d_i - 1) - (2j_- + 1)d_i'(2d_i - 1), \quad i = 1, 2, 3. \end{aligned} \tag{3.34}$$

In order to prove (3.32) notice that if in (3.22) $a_2'\left(\theta^{-1}\right)$ is replaced by $1/[j_- a_1(\theta)]$ we obtain

$$\begin{aligned} &\theta\,[1 - a_1(\theta)]\,[1 - j_+ a_1'(\theta)]\,[1 - j_- a_1(\theta)] - (1 - \theta)\,[1 - a_1(\theta)] \\ &\quad\cdot [\theta - j_+ j_- a_1'(\theta) a_1(\theta)]\,d_3 - (1 - \theta)\,[a_1'(\theta) a_1(\theta) - j_+\theta]\,[1 - j_- a_1(\theta)]\,d_3' = 0. \end{aligned} \tag{3.35}$$

A first order Taylor expansion of the above equation as $\theta \curvearrowright 1,\ a_1(\theta) \to 1,\ a_1'(\theta) \to 1$ yields

$$-(1 - j_+)(1 - j_-)[a_1(\theta) - 1 - d_3'(\theta - 1)] + O\left((\theta - 1)^2\right) = 0.$$

After dividing by $\theta - 1$ and taking the limit as $\theta \curvearrowright 1$ we conclude that the one-sided derivative,

$$\lim_{\theta \curvearrowright 1} \frac{a_1(\theta) - 1}{\theta - 1}, \quad \text{exists and equals } d_3'.$$

In particular, $d_1 = d_3'$, which is one of the three equations (3.32). As usual, the other two follow by taking circular permutations of the indices $1, 2, 3$.

For the purpose of proving (3.34) we also notice that the following "second derivative limit" exists and takes the indicated value:

$$\lim_{\theta \curvearrowright 1} \frac{a_1(\theta) - 1 - d_1(\theta - 1)}{(\theta - 1)^2} = \frac{1}{3} d_1[(j_+ + 2)(d_1 - 1) + (2j_+ + 1)d_1' - (j_+ + j_- + 1)d_3]. \tag{3.36}$$

Indeed, (3.36) follows from a second order Taylor expansion of (3.35), which as $\theta \curvearrowright 1$, $a_1(\theta) \to 1,\ a_1'(\theta) \to 1$ reads as

$$\begin{aligned} (1 - j_+)(1 - j_-)[a_1(\theta) - 1 - d_3'(\theta - 1)] &= [j_+(j_- - 1)]d_3'(\theta - 1)^2 \\ &+ [(j_+ - j_-) + (j_+ j_- - 1)d_3 + (j_+ j_- - 2j_- + 1)(d_3' - 1)](\theta - 1)[a_1(\theta) - 1] \\ &+ (1 - j_-)d_3'(\theta - 1)[a_1'(\theta) - 1] + [j_-(1 - j_+)][a_1(\theta) - 1]^2 \\ &+ [j_+(1 - j_-)][a_1'(\theta) - 1][a_1(\theta) - 1] + O\left((\theta - 1)^3\right). \end{aligned}$$

After dividing the above equation by $(\theta - 1)^2$, taking the limit as $\theta \curvearrowright 1$, replacing d_3' by d_1, and making the necessary simplifications we do obtain (3.36).

(3.33) for $i = 2$ follows by applying the above procedure, and the result (3.32), to Equation 3.24, after replacing $a_2\left(\theta^{-1}\right)$ by $1/[j_- a_3'(\theta)]$, $a_2'\left(\theta^{-1}\right)$ by $1/[j_- a_1(\theta)]$, and after clearing the denominators. For the record, the first order Taylor expansion as $\theta \curvearrowright 1$, $a_1(\theta) \to 1$, $a_1'(\theta) \to 1$, and $a_3'(\theta) \to 1$, of this modified equation (3.24) is

$$\begin{aligned} &(1 - j_-)[(j_- - j_+)d_3 + (1 + j_+)(1 - j_-)](\theta - 1) \\ &\quad - (1 + j_+)(1 - j_-)^2[a_1(\theta) - 1] + (1 - j_-)(j_- - j_+)[a_1'(\theta) - 1] + O\left((\theta - 1)^2\right) = 0. \end{aligned}$$

which yields

$$(j_+ - j_-)(d_2 + d_3) + (1 + j_+)(1 - j_-)(d_1 - 1) = 0.$$

Finally, (3.34) follows from a different evaluation of the limit (3.36). If $\theta_i,\ i = 2, 3$ are angles near 1, $\Im(\theta_i) > 0$, we have

$$\begin{aligned}[a_1(\theta_2\theta_3) - 1]P(\theta_2^{-1}, \theta_3^{-1}, a_2(\theta_2^{-1}), a_3(\theta_3^{-1}), a_2'(\theta_2^{-1}), a_3'(\theta_3^{-1}), j_-)\\ - Q(\theta_2^{-1}, \theta_3^{-1}, a_2(\theta_2^{-1}), a_3(\theta_3^{-1}), a_2'(\theta_2^{-1}), a_3'(\theta_3^{-1}), j_-) = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}[a_1(\theta_2\theta_3) - 1]P(\theta_2^{-1}, \theta_3^{-1}, 1/[j_-a_3'(\theta_2)], 1/[j_-a_1'(\theta_3)], 1/[j_-a_1(\theta_2)], 1/[j_-a_2(\theta_3)], j_-)\\ - Q(\theta_2^{-1}, \theta_3^{-1}, 1/[j_-a_3'(\theta_2)], 1/[j_-a_1'(\theta_3)], 1/[j_-a_1(\theta_2)], 1/[j_-a_2(\theta_3)], j_-) = 0.\end{aligned}$$

After clearing the denominators in the equation above, a first order Taylor expansion as $\theta_2 \curvearrowright 1$, $a_1(\theta_2) \to 1$, $a_3'(\theta_2) \to 1$, gives

$$\begin{aligned}\{[-1 + \theta_3 - j_- - a_2(\theta_3) + a_1'(\theta_3) + j_-a_1'(\theta_3) - j_-a_2(\theta_3) - j_-\theta_3 a_1'(\theta_3)\\ - j_-a_1'(\theta_3)a_2(\theta_3)]a_1(\theta_2\theta_3) + j_-\theta_3 + j_-a_2(\theta_3) + \theta_3 a_2(\theta_3) - \theta_3 a_1'(\theta_3)\\ + a_1'(\theta_3)a_2(\theta_3) + j_-a_1'(\theta_3)a_2(\theta_3) - \theta_3 a_1'(\theta_3)a_2(\theta_3)\}(\theta_2 - 1)\\ + \{(1 - \theta_3)[1 - j_-a_1'(\theta_3)]a_1(\theta_2\theta_3) - (1 - \theta_3)a_2(\theta_3)[j_- + a_1'(\theta_3)\\ + j_-a_1'(\theta_3)]\}[a_1(\theta_2) - 1] + \{(j_- + 1)(1 - \theta_3)[1 - a_1'(\theta_3)]a_1(\theta_2\theta_3)\\ + (1 - \theta_3)a_2(\theta_3)[1 - a_1'(\theta_3)]\}[a_3'(\theta_2) - 1] + O\left((\theta_2 - 1)^2\right) = 0.\end{aligned}$$

Dividing by $\theta_2 - 1$ and taking the limit as $\theta_2 \curvearrowright 1$ shows that, since $d_3' = d_1$ and $a_1(\theta_2\theta_3) \longrightarrow a_1(\theta_3)$,

$$\begin{aligned}(\theta_3 - 1)\{[-a_2(\theta_3) - 2a_1(\theta_3) - j_-a_1(\theta_3) + j_-a_2(\theta_3) + 2a_1'(\theta_3)a_2(\theta_3)\\ + a_1'(\theta_3)a_1(\theta_3) + j_-a_1'(\theta_3)a_2(\theta_3) + 2j_-a_1'(\theta_3)a_1(\theta_3)]d_1\\ + j_- + a_1(\theta_3) + a_2(\theta_3) - a_1'(\theta_3) - a_1'(\theta_3)a_2(\theta_3) - j_-a_1'(\theta_3)a_1(\theta_3)\}\\ - [a_1(\theta_3) - 1][j_- - a_1'(\theta_3) + a_2(\theta_3) + j_-a_2(\theta_3) + j_-a_1'(\theta_3)a_2(\theta_3)] = 0.\end{aligned} \tag{3.37}$$

Now, the second order Taylor expansion of (3.37), as $\theta_3 \curvearrowright 1$, $a_1(\theta_3) \to 1$, $a_2(\theta_3) \to 1$, $a_1'(\theta_3) \to 1$, yields

$$\begin{aligned}-3j_-[a_1(\theta_3) - 1 - d_1(\theta_3 - 1)] + (1 + 2j_-)d_1(\theta_3 - 1)[a_2(\theta_3) - 1]\\ + (1 - j_-)(1 - d_1)(\theta_3 - 1)[a_1(\theta_3) - 1] + [3(1 + j_-)d_1 - (2 + j_-)](\theta_3 - 1)[a_1'(\theta_3) - 1]\\ - (1 + 2j_-)[a_1(\theta_3) - 1][a_2(\theta_3) - 1] + (1 - j_-)[a_1(\theta_3) - 1][a_1'(\theta_3) - 1]\\ + O\left((\theta_2 - 1)^3\right) = 0.\end{aligned}$$

It follows that

$$\lim_{\theta \curvearrowright 1} \frac{a_1(\theta) - 1 - d_1(\theta - 1)}{(\theta - 1)^2} = \frac{1}{3}[(2 + j_-)d_1(d_1 - 1) - (1 + 2j_-)d_1'(2d_1 - 1)]. \tag{3.38}$$

Comparing now (3.36) and (3.38) gives us (3.34) for $i = 1$.

To conclude the proof, assume by contradiction that $j_+ = j_-$. Equations 3.32 and 3.33 imply immediately that $d_i = d_i' = 1$, $i = 1, 2, 3$. However, this contradicts Equation 3.34. Therefore, $j_- = 1/j_+$. Then (3.33) becomes

$$d_1 + d_2 + d_3 = 1, \tag{3.39}$$

while (3.34) becomes

$$(d_i - 1)(d_i - d_i') = 0, \quad i = 1, 2, 3.$$

But $d_i \neq 1, i = 1, 2, 3$. Indeed, if say $d_1 = 1$, then $d_2 = d_3 = 0$, since $d_i \geq 0$ satisfy (3.39). By (3.32), $d_2 = d_2' = 0$, which contradicts Lemma 3.9. Thus, $d_i = d_i'$, $i = 1, 2, 3$, and from this and (3.32), (3.39) we get $d_i = d_i' = 1/3$, $i = 1, 2, 3$. □

Remark 3.3. One consequence of the proof of the above lemma is that at least when some l_i, l_i' does not equal 1, the one-sided derivatives,

$$\lim_{\theta \curvearrowright 1} \frac{a_i(\theta) - 1}{\theta - 1} \quad \text{and} \quad \lim_{\theta \curvearrowright 1} \frac{a_i'(\theta) - 1}{\theta - 1}, \quad i = 1, 2, 3,$$

exist.

Lemma 3.12. *If at least one of l_i, l_i', is not equal to* 1*, then* $a_i^3(\theta) = (a_i'(\theta))^3 = \theta$, $i = 1, 2, 3$, *for θ near* 1, $\Im\theta > 0$.

Proof. According to Lemmas 3.10 and 3.11 we can set $a_2'\left(\theta^{-1}\right) = j_+/a_1(\theta)$, $d_3 = d_3' = 1/3$ in Equation 3.22. It becomes

$$\begin{aligned}[3\theta + a_1'(\theta) - j_+a_1'(\theta) - \theta a_1'(\theta) - 2j_+\theta a_1'(\theta)]a_1^2(\theta) - 3\theta[1 + j_+ + a_1'(\theta)]a_1(\theta) \\ + \theta[-1 + j_+ + \theta + 3a_1'(\theta) + 2j_+\theta + 3j_+a_1'(\theta)] = 0.\end{aligned} \tag{3.40}$$

Also, by setting $d_2 = d_2' = 1/3$ and $a_3\left(\theta^{-1}\right) = j_+/a_1'(\theta))$ in the following equation of type (3.20),

$$\begin{aligned}-\theta\left[1 - a_1'(\theta)\right]\left[1 - j_+a_1(\theta)\right]\left[1 - a_3\left(\theta^{-1}\right)\right] + (1 - \theta)\left[a_1'(\theta)a_1(\theta) - j_+\theta\right] \\ \cdot \left[1 - a_3\left(\theta^{-1}\right)\right] d_2 + (1 - \theta)\left[j_+a_1(\theta) - \theta a_3\left(\theta^{-1}\right)\right]\left[1 - a_1'(\theta)\right] d_2' = 0.\end{aligned}$$

we get

$$\begin{aligned}[3\theta + a_1(\theta) - j_+a_1(\theta) - \theta a_1(\theta) - 2j_+\theta a_1(\theta)]\left(a_1'(\theta)\right)^2 - 3\theta[1 + j_+ + a_1(\theta)]a_1'(\theta) \\ + \theta[-1 + j_+ + \theta + 3a_1(\theta) + 2j_+\theta + 3j_+a_1(\theta)] = 0,\end{aligned} \tag{3.41}$$

which is just (3.40) with $a_1(\theta)$ and $a_1'(\theta)$ interchanged.

Subtracting now Equation 3.40 from Equation 3.41 gives us

$$\begin{aligned}[a_1'(\theta) - a_1(\theta)][6\theta + 6j_+\theta - a_1'(\theta)a_1(\theta) - 3\theta a_1(\theta) - 3\theta a_1'(\theta) \\ + j_+a_1'(\theta)a_1(\theta) + \theta a_1'(\theta)a_1(\theta) + 2j\theta a_1(\theta)a_1'(\theta)] = 0.\end{aligned}$$

Since in the above equation the second factor approaches the non-zero quantity $9j_+$ as $\theta \curvearrowright 1$ we conclude that at least for θ near 1, $\Im\theta > 0$, we have $a_1'(\theta) = a_1(\theta)$. By circular symmetry, we also have $a_2'(\theta) = a_2(\theta)$, $a_3'(\theta) = a_3(\theta)$, near 1. So (3.40) becomes

$$[a_1^3(\theta) - \theta](1 - j_+ - \theta - 2j_+\theta) = 0, \quad \theta \text{ near } 1, \Im\theta > 0.$$

Thus, $a_1^3(\theta) = \theta$, near $\theta = 1$. Identical equations hold also for $a_2(\theta)$ and $a_3(\theta)$, and this proves the lemma. □

Now we turn to the possibility of having solutions to Morley's problem for which $l_i = l_i' = 1$, $i = 1, 2, 3$.

Lemma 3.13. *If* $\lim_{\theta \circlearrowleft 1} a_i(\theta) = \lim_{\theta \circlearrowleft 1} a_i(\theta) = 1$, $i = 1, 2, 3$, *then the one-sided derivatives*

$$\delta_i := \lim_{\theta \circlearrowleft 1} \frac{a_i(\theta) - 1}{\theta - 1}, \quad \delta_i' := \lim_{\theta \circlearrowleft 1} \frac{a_i'(\theta) - 1}{\theta - 1}, \quad i = 1, 2, 3,$$

exist and the following relationships hold true:

$$\delta_i' d_{i+1}' + d_{i+1} d_{i+2} = d_{i+2} \delta_i' \tag{3.42}$$
$$\delta_i d_{i+2} + d_{i+2}' d_{i+1}' = d_{i+1}' \delta_i, \quad i = 1, 2, 3. \tag{3.43}$$
$$d_i' \delta_{i+1}' + \delta_{i+1} \delta_{i+2} = \delta_{i+2} d_i' \tag{3.44}$$

Proof. The proof is of course similar to those of Lemmas 3.7 and 3.11. Fix an angle θ and choose another angle ω close to 1, $\Im\omega < 0$. Then the triple $(\theta_1, \theta_2, \theta_3) := \left(\theta, \theta^{-1}\omega^{-1}, \omega\right)$ is a set of angles for a negatively oriented triangle. Consequently, the Equations (2.10) hold true, with $j = j_-$. A first order Taylor expansion of the first equation (2.10), as $\omega \circlearrowleft 1$, $a_3(\omega) \to 1$, gives

$$\begin{aligned} &\left[1 - a_2'\left(\theta^{-1}\omega^{-1}\right)\right]\left[\theta - j_- a_1(\theta) + j_-\theta - a_1'(\theta) a_1(\theta) - j_-\theta a_2\left(\theta^{-1}\omega^{-1}\right)\right. \\ &\quad \left. - \theta a_1'(\theta) a_2(\theta^{-1}\omega^{-1}) + a_1'(\theta) a_1(\theta) a_2\left(\theta^{-1}\omega^{-1}\right) + j_- a_1'(\theta) a_1(\theta) a_2\left(\theta^{-1}\omega^{-1}\right)\right](\omega - 1) \\ &\quad + (1 - \theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\omega^{-1}\right)\right]\left[1 - j_- a_2'\left(\theta^{-1}\omega^{-1}\right) a_1(\theta)\right]\left[a_3(\omega) - 1\right] \\ &\quad + O\left((\omega - 1)^2\right) = 0. \end{aligned} \tag{3.45}$$

As $\omega \circlearrowleft 1$ the quantity $(1-\theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\omega^{-1}\right)\right]\left[1 - j_- a_2'\left(\theta^{-1}\omega^{-1}\right) a_1(\theta)\right]$ approaches $(1 - \theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right]\left[1 - j_- a_2'\left(\theta^{-1}\right) a_1(\theta)\right]$. This latter quantity does not vanish if θ is close enough to 1, $\Im\theta > 0$. Indeed, $1 - a_1'(\theta) a_2\left(\theta^{-1}\right)$ does not vanish by Lemma 3.8, while $1 - j_- a_2'\left(\theta^{-1}\right) a_1(\theta)$ is close to $1 - j_- \neq 0$ if θ is close to 1. So dividing Equation 3.45 by $(\omega - 1)(1 - \theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\omega^{-1}\right)\right] \cdot \left[1 - j_- a_2'\left(\theta^{-1}\omega^{-1}\right) a_1(\theta)\right]$ and then taking the limit as $\omega \circlearrowleft 1$ proves that δ_3 exists and (in fact for every angle θ) satisfies the equation

$$\begin{aligned} &\left[1 - a_2'\left(\theta^{-1}\right)\right]\left[\theta - j_- a_1(\theta) + j_-\theta - a_1'(\theta) a_1(\theta) + a_1'(\theta) a_1(\theta) a_2\left(\theta^{-1}\right)\right. \\ &\qquad \left. - \theta a_1'(\theta) a_2(\theta^{-1}) - j_-\theta a_2\left(\theta^{-1}\right) + j_- a_1'(\theta) a_1(\theta) a_2\left(\theta^{-1}\right)\right] \\ &\qquad + (1 - \theta)\left[1 - a_1'(\theta) a_2\left(\theta^{-1}\right)\right]\left[1 - j_- a_2'\left(\theta^{-1}\right) a_1(\theta)\right]\delta_3 = 0. \end{aligned} \tag{3.46}$$

Notice that the above equation can be obtained from (3.24) by replacing j_+ with j_- and d_3 with δ_3. The existence of δ_3' can be obtained in a similar way from the second equation (2.10) applied to the angles $\left(\theta, \theta^{-1}\omega^{-1}, \omega\right)$. Obviously, δ_3' satisfies an equation similar to (3.25). As usual, the existence and functional properties of the other $\delta_i, \delta_i', i = 1, 2$, follows by taking circular permutations.

To proceed, the second order Taylor expansion of (3.46) as $\theta \curvearrowright 1$, $a_1(\theta) \to 1$, $a_2\left(\theta^{-1}\right) \to 1$, $a_1'(\theta) \to 1$, $a_2'\left(\theta^{-1}\right) \to 1$ reads

$$\begin{aligned} &(j_- - 1)\{\delta_3(\theta - 1)\left[a_2(\theta^{-1}) - 1\right] + \delta_3(\theta - 1)[a_1'(\theta) - 1] \\ &\qquad + [a_1'(\theta) - 1]\left[a_2'(\theta^{-1}) - 1\right]\} + O\left((\theta - 1)^3\right) = 0. \end{aligned}$$

A division by $(\theta-1)^2$ followed by a passage to limit as $\theta \curvearrowright 1$ yields (3.44) for $i=1$. Similarly, we can obtain (3.42), respectively (3.43), for $i=1$, by taking the second order Taylor expansions of the equation (3.24), respectively (3.25). □

Lemma 3.14. *If* $\lim_{\theta \circlearrowleft 1} a_i(\theta) = \lim_{\theta \circlearrowleft 1} a_i'(\theta) = 1$, $i=1,2,3$, *then at least one of the six derivatives* d_i, d_i', $i=1,2,3$, *vanishes. If, for instance,* $d_3=0$, *then it is also true that* $d_1' = \delta_1' = 0$ *and* $d_1 = \delta_1 = d_3'$. *Similarly, if* $d_1'=0$, *then* $d_3=\delta_3=0$ *and* $d_3' = \delta_3' = d_1$.

Proof. Assume, by contradiction, that no $d_i, d_i', i=1,2,3$, vanishes. Then $d_{i+1}' \neq d_{i+2}$ in Equation 3.42 and therefore

$$\delta_i' = \frac{d_{i+1}d_{i+2}}{d_{i+2}-d_{i+1}'}, \quad i=1,2,3.$$

Similarly, from Equation 3.43 we obtain

$$\delta_i = -\frac{d_{i+2}'d_{i+1}'}{d_{i+2}-d_{i+1}'}, \quad i=1,2,3.$$

Substituting the above values of δ_i, δ_i', $i=1,2,3$, in (3.44) we have, after the necessary simplifications,

$$d_i'd_{i+1}' + d_{i+1}d_{i+2} = d_i'd_{i+2}, \quad i=1,2,3. \tag{3.47}$$

However, the equations corresponding to $i=1,2$ in (3.47) lead to a contradiction. Indeed, multiplying $d_1'd_2' + d_2d_3 = d_1'd_3$ by d_1 and $d_2'd_3' + d_3d_1 = d_2'd_1$ by d_1' imply that $d_1d_2d_3 + d_1'd_2'd_3' = 0$. This is impossible, since by assumption $d_i > 0, d_i' > 0, i=1,2,3$. Some d_i or d_i' must therefore vanish.

If, for example, $d_3=0$, then Equation 3.24 becomes

$$\theta - j_+a_1(\theta) + j_+\theta - a_1'(\theta)a_1(\theta) + a_1'(\theta)a_1(\theta)a_2\left(\theta^{-1}\right) - \theta a_1'(\theta)a_2(\theta^{-1}) - j_+\theta a_2\left(\theta^{-1}\right) + j_+a_1'(\theta)a_1(\theta)a_2\left(\theta^{-1}\right) = 0.$$

A first order Taylor expansion of the above equation as $\theta \to 1$, $a_1(\theta) \to 1$, $a_1'(\theta) \to 1$, and $a_2\left(\theta^{-1}\right) \to 1$ shows that

$$(j_+-1)\left[a_1'(\theta)-1\right] + O\left((\theta-1)^2\right) = 0. \tag{3.48}$$

After dividing (3.48) by $\theta-1$ we see that as $\theta \curvearrowright 1$, $d_1'=0$, and as $\theta \circlearrowleft 1$, $\delta_1'=0$.

By setting now $d_3=0$ in Equation 3.22 we obtain

$$-\theta\left[1-a_1(\theta)\right]\left[1-j_+a_1'(\theta)\right] + (1-\theta)\left[a_1'(\theta)a_1(\theta) - j_+\theta\right]d_3' = 0.$$

Consequently, for θ near 1,

$$\frac{a_1(\theta)-1}{\theta-1} = \frac{\left[a_1'(\theta)a_1(\theta)-j_+\theta\right]d_3'}{\theta\left[1-j_+a_1'(\theta)\right]},$$

and so, as $\theta \curvearrowright 1$, $d_1 = d_3'$, and as $\theta \circlearrowleft 1$, $\delta_1 = d_3'$.

Similar arguments prove the rest of the lemma in the case $d_1'=0$. □

Lemma 3.15. *There is no solution to Morley's problem satisfying Assumptions* 1 *and* 2 *for which*

$$\lim_{\theta \circlearrowright 1} a_i(\theta) = \lim_{\theta \circlearrowright 1} a_i'(\theta) = 1, \quad i = 1, 2, 3.$$

Proof. Assume, by contradiction, that there are $a_i, a_i',\ i = 1, 2, 3$, satisfying the hypotheses of the lemma. By Lemmas 3.9 and 3.14 we can assume then that $d_1' = 0$ and $d_1 = \delta_1 > 0$. Thus,

$$\lim_{\theta \curvearrowright 1} \frac{a_1(\theta) - 1}{\theta - 1} = \lim_{\theta \circlearrowright 1} \frac{a_1(\theta) - 1}{\theta - 1} = d_1 > 0. \tag{3.49}$$

There are no values of θ, $\theta \circlearrowright 1$ for which $a_1(\theta) \curvearrowright 1$. Otherwise, for such values we have $\arg(\theta) \to 2\pi$, $\arg(a_1(\theta)) \to 0$, and so

$$\lim_{\theta \circlearrowright 1} \frac{a_1(\theta) - 1}{\theta - 1} = \lim_{\arg(\theta) \to 2\pi} \frac{\arg(a_1(\theta))}{\arg(\theta)} = 0.$$

Thus, $\arg(\theta) \to 2\pi$ implies $\arg(a_1(\theta)) \to 2\pi$, and since by Assumption 1, $\arg(a_1(\theta)) \leq \arg(\theta)$, we conclude that

$$|a_1(\theta) - 1|_0 \geq |\theta - 1|_0, \quad \text{for } \ \theta \ \text{ near } 1,\ \Im\theta < 0.$$

Equation 3.49 gives then $d_1 \geq 1$, and since $d_1 \leq 1$ to start with, we must have $d_1 = 1$.

By setting now $d_1 = 1$ and $d_1' = 0$ in the following equation of type (3.23),

$$\begin{aligned}
&[1 - a_2(\theta)][1 + j_+ - j_+\theta - a_2'(\theta)a_3\left(\theta^{-1}\right) - j_+a_2'(\theta) - \theta a_3\left(\theta^{-1}\right) \\
&\quad + \theta a_2'(\theta)a_3\left(\theta^{-1}\right) + j_+\theta a_2'(\theta)a_3\left(\theta^{-1}\right)] - (j_+ + 1)(1 - \theta)\,[1 - a_2(\theta)] \\
&\quad \cdot \left[1 - a_2'(\theta)a_3\left(\theta^{-1}\right)\right] d_1 - (1 - \theta)\,[1 - j_+a_2(\theta)]\left[1 - a_2'(\theta)a_3\left(\theta^{-1}\right)\right] d_1' = 0.
\end{aligned}$$

we conclude that

$$[1 - a_3\left(\theta^{-1}\right)][\theta - j_+a_2'(\theta)] = 0, \quad \theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}. \tag{3.50}$$

However, (3.50) cannot hold for values of θ near 1, a contradiction. □

End of proof of Theorem 3.5. Let now $a_i, a_i',\ i = 1, 2, 3$, be a solution to Morley's problem satisfying Assumption 1 and Assumption 2. According to Lemma 3.15, at least one of the limits $\lim_{\theta \circlearrowright 1} a_i(\theta)$, $\lim_{\theta \circlearrowright 1} a_i'(\theta)$, does not equal 1. There exists then a sequence $\{\omega_n\}_n$ as in Lemma 3.6 such that at least one of the limits l_i, l_i' is different from 1. As a result, Lemmas 3.10, 3.11, and 3.12 apply, and in particular (Lemma 3.12),

$$a_i^3(\theta) = \left(a_i'(\theta)\right)^3 = \theta,\ i = 1, 2, 3, \quad \theta \text{ near } 1, \Im\theta > 0.$$

Since by Assumption 1, $\arg a_i(\theta)) \leq \arg(\theta)$, $\arg a_i'(\theta)) \leq \arg(\theta)$, $i = 1, 2, 3$, there exists $\epsilon > 0$, such that

$$a_i\left(\exp(\sqrt{-1}s)\right) = a_i'\left(\exp(\sqrt{-1}s)\right) = \exp\left(\sqrt{-1}\frac{s}{3}\right),\ i = 1, 2, 3, \quad s \in (0, \epsilon). \tag{3.51}$$

In particular, (3.51) implies that the cube roots of all the angles $\theta \in K$, $\arg(\theta) \in (0, \epsilon)$, belong to K. Since every angle θ is the product of finitely many angles with small arguments, we conclude that the cube roots of all the elements in $\{a \in K|\ |a|_0 = 1\}$ belong to K.

In order to conclude the proof of the theorem we must show that the equations in (3.51) hold for $s \in (0, 2\pi)$. To this end it suffices to show that if they hold in some interval $(0, \rho)$, $\rho \in (0, 2\pi)$, then they also hold in $(0, \min\{\rho + \epsilon, 2\pi\})$. We will prove this for a_1'.

If θ is an angle in K with $\arg(\theta) < \min\{\rho + \epsilon, 2\pi\})$ we can write

$$\theta = \exp(\sqrt{-1}s_1)\exp(\sqrt{-1}s_2) = \exp(\sqrt{-1}(s_1 + s_2)),$$

where $\exp(\sqrt{-1}s_1), \exp(\sqrt{-1}s_1) \in K$ and $0 < s_1 < \rho$, $0 < s_2 < \epsilon$. With the substitutions,

$$t_1 := \exp\left(\sqrt{-1}\frac{s_1}{3}\right),\ t_2 := \exp\left(\sqrt{-1}\frac{s_2}{3}\right),$$

the triple, $\left(t_1^3, t_2^3, 1/\theta\right)$ forms a set of angles for a positively oriented triangle. The first equation (2.10) applied to this set of angles reads, via (3.31),

$$\left(\frac{1}{j_- a_1'(\theta)} - 1\right) P\left(t_1^3, t_2^3, t_1, t_2, t_1, t_2, j_+\right) = Q\left(t_1^3, t_2^3, t_1, t_2, t_1, t_2, j_+\right) \tag{3.52}$$

Since by Lemma 3.11, $j_- = 1/j_+$, (3.52) factors as

$$\frac{[a_1'(\theta) - t_1t_2]\, t_1\,(1 - t_1)\,(1 - t_2)\,(1 - t_1t_2)\,\Omega}{a_1'(\theta)} = 0, \tag{3.53}$$

where

$$\Omega := j_+ + t_1 + j_+t_1 - t_1t_2^2 + j_+t_1^3t_1^2.$$

Given that j_+ is a nontrivial cube root of unity, with some care one can choose s_1 and s_2 such that $\Omega \neq 0$. It follows from (3.53) that $a_1'(\theta) = \exp\left(\sqrt{-1}\frac{\arg(\theta)}{3}\right)$, as desired. Thus,

$$a_i(\theta) = a_i'(\theta) = \exp\left(\sqrt{-1}\frac{\arg(\theta)}{3}\right),\ i = 1, 2, 3, \quad \theta \in \{a \in K|\ |a|_0 = 1, a \neq 1\}. \tag{3.54}$$

Notice that (3.54) and (3.31) yield $j_+ = \exp\left(\sqrt{-1}\frac{2\pi}{3}\right)$ and $j_- = \exp\left(\sqrt{-1}\frac{4\pi}{3}\right)$. The proof of Theorem 3.5 is now complete. □

Remark 3.4. It is worth asking whether Assumptions 1 and 2 can be relaxed with the same endresult. Quite a bit of extra work can show that it suffices to postulate Assumption 1 only for angles with small argument, and it also suffices to work only with positively oriented triangles. Assumptions 1 and 2 are instrumental in developing the lengthy infinitesimal analysis provided in the proof of Theorem 3.5, but ultimately we believe that neither should be necessary, and that, in fact, the following conjecture should hold true:

Conjecture. The only solutions to Morley's problem in $(\mathbf{C}, |\cdot|_0)$ are the 18 variants of Morley's cube root solution, as listed in Connes' paper [C].

The methods developed in our paper do not appear to be enough for settling such a conjecture.

4 An Example of a Near Solution

We end the paper with a (near) solution to Morley's problem in $(\mathbf{C}, |\cdot|_0)$ which is different from any of the 18 variants of Morley's cube root solution. The tri-sectioning functions a_i, a'_i will be continuous but the required restrictions $a'_i a_{i+1} \neq 1$ will fail on a small set of triangles.

Theorem 4.1. *For $\theta \in \mathbf{S}^1 = \{a \in \mathbf{C} \mid |a|_0 = 1\}$ define tri-sectioning functions a_i, a'_i, $i = 1, 2, 3$, by the formulas,*

$$\begin{aligned} a_1(\theta) = a'_3(\theta) &= \frac{1 + 3\theta + 2j\theta}{3 + j + \theta + j\theta} \\ a_2(\theta) = a'_2(\theta) &= \theta \\ a_3(\theta) = a'_1(\theta) &= \frac{\theta(7 + 5j - \theta + j\theta)}{1 - j + 5\theta + 7j\theta}, \end{aligned} \tag{4.55}$$

where $j = \exp\left(\sqrt{-1}\frac{2\pi}{3}\right)$ is one of the two non-trivial cube roots of unity in $\mathbf{C}$. Then the assignment (4.55) represents a solution to Morley's problem which works for all triangles (v_1, v_2, v_3) in $\mathbf{C}$, except those similar to triangles $(0, 1, v)$, v on the circle whose equation in polar coordinates, $z = r\exp\left(\sqrt{-1}\alpha\right)$, is $r = \sqrt{3}/3 \sin\alpha$. One can argue that for such triangles the corresponding equilateral triangles have vertices at infinity.

Proof. It is easy to check that a_i, a'_i given by (4.55) are well defined smooth rational functions from $\mathbf{S}^1$ into itself. Moreover, they all take the value 1 if and only if $\theta = 1$.

No lesser miracle that Morley's cube root solution itself, a_i, a'_i satisfy Equation (2.7). In other words, for every set of angles $\left(\theta_1, \theta_2, \frac{1}{\theta_1\theta_2}\right)$ of some triangle, θ_1, θ_2, $a_1(\theta_1), a'_1(\theta_1)$, $a_2(\theta_2), a'_2(\theta_2)$, $a_3\left(\frac{1}{\theta_1\theta_2}\right), a'_3\left(\frac{1}{\theta_1\theta_2}\right)$, and $1/j$, satisfy (2.7). The verification is painful at human level but a breeze for a Computer Algebra System like Maple.

The last thing that remains to be checked is what happens to the constraints $a'_i a_{i+1} \neq 1$, $i = 1, 2, 3$, and (cf. (2.8))

$$a'_1(1 - a'_3 a_1)(1 - a_2)(1 - \theta_1\theta_2) - (1 - a'_1 a_2)(1 - a'_3)\theta_1(1 - \theta_2) \neq 0. \tag{4.56}$$

The first two, $a'_1 a_2 \neq 1$ and $a'_2 a_3 \neq 1$, are equivalent to

$$1 - j + 5\theta_1 + 7j\theta_1 - 7\theta_1\theta_2 - 5j\theta_1\theta_2 + \theta_1^2\theta_2 - j\theta_1^2\theta_2 \neq 0, \tag{4.57}$$

the third, $a'_3 a_1 \neq 1$, is equivalent to

$$j - 4\theta_1 - 3j\theta_1 + 4\theta_1\theta_2 + j\theta_1\theta_2 + j\theta_1^2\theta_2 \neq 0, \tag{4.58}$$

while the last, (4.56), is again equivalent to (4.57).

Now, any triangle (v_1, v_2, v_3) is similar (same angles) to a triangle $(0, 1, v)$, $v \notin \mathbf{R}$. Since the latter has angles

$$\theta_1 = \frac{v}{\overline{v}},\ \theta_2 = \frac{\overline{v} - 1}{v - 1},$$

(4.57) becomes equivalent to

$$(v - \overline{v})[(j - 1)(v - \overline{v}) + 6(j + 1)v\overline{v}] \neq 0, \tag{4.59}$$

while (4.58) becomes equivalent to

$$(v - \overline{v})[j(v - \overline{v}) + 2(j + 2)v\overline{v}] \neq 0 \tag{4.60}$$

Finally, both (4.59) and (4.60) are equivalent to v not being on the polar circle $r = \sqrt{3}/3 \sin \alpha$. It can be shown that for the assignment (4.55) the equilateral triangle (w_1, w_2, w_3) associated to the triangle (v_1, v_2, v_3) is never located inside (v_1, v_2, v_3).

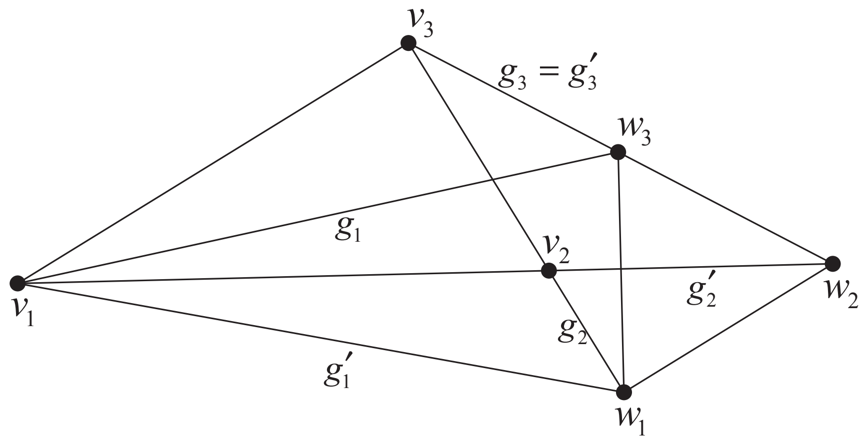

Fig. 2

Fig. 2 is showing the equilateral triangle (w_1, w_2, w_3) associated to a 30-60 right triangle (v_1, v_2, v_3) with angles $(-1/j, j, -1)$. We note that a triangle with angles $(j, -1/j, -1)$ belongs to the excepted category of triangles. □

Remark 4.1. Of course, one would want to know how the assignment (4.55) came about. The answer can be traced back to Lemma 3.14 where it was shown that a solution to Morley's problem satisfying Assumptions 1 and 2, and such that $\lim_{\theta \circlearrowright 1} a_i(\theta) = \lim_{\theta \circlearrowright 1} a_i'(\theta) = 1$, $i = 1, 2, 3$, must satisfy $d_3 = d_1' = 0$ and $d_1 = d_3' \neq 0$. Since $d_1 = 1$ cannot lead to a solution, the simplest instance when these requirements are hoped to be met is $d_1 = 1/2, d_2 = d_2' = 1$. It turns out that these derivative values, when used in the various equations given by Lemma 3.7 or its counterpart involving d_2 and d_2', determine uniquely the values of a_i, a_i' given by (4.55).

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