

Vector Maximal Principles and Dependent Choice

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Abstract. The vector maximal principle in Goepfert, Tammer and Zălinescu [Nonlin. Anal., 39 (2000), 909-922] is equivalent with the Dependent Choices Principle; and as such, equivalent with Ekeland's Variational Principle [Bull. Amer. Math. Soc. (N. S.), 1 (1979), 443-474].

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1 Introduction

Let Y be a (real) separated *locally convex space*; and K , some (convex) cone of it ($\alpha K + \beta K \subseteq K$ for each $\alpha, \beta \geq 0$). The relation \leq_K on Y defined as

(a01) $(y_1, y_2 \in Y)$: $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$

is reflexive and transitive; hence a *quasi-order* [also denoted as (\leq) , when K is understood]; moreover, it is *compatible* with the linear structure of Y . Let H be another (convex) cone of Y with $K \subseteq H$; and pick some $k^0 \in K \setminus (-H)$. Further, take some complete metric space (X, d) . Define the quasi-order (\succeq) on $X \times Y$

(a02) $(x_1, y_1) \succeq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq y_1 - y_2$;

and take some nonempty part A of $X \times Y$. For both practical and theoretical reasons, it would be useful to determine sufficient conditions under which (A, \succeq) has points with certain *maximal* properties. A basic result in this direction, obtained by Goepfert, Tammer and Zălinescu [10, Theorem 1] (in short: GTZ) deals with the case $H = \text{cl}(K)$ (= the *closure* of K), under the conditions

(a03) $P_Y(A)$ is bounded below (modulo K) ($\exists \tilde{y} \in Y: P_Y(A) \subseteq \tilde{y} + K$)

(a04) if $((x_n, y_n)) \subseteq A$ is (\succeq) -ascending and $x_n \rightarrow x$ then $x \in P_X(A)$
and there exists $y \in A(x)$ such that $(x_n, y_n) \succeq (x, y)$, for all n .

[Here, P_X, P_Y are the *projection* operators from $X \times Y$ to X and Y respectively].

Theorem 1.1. *Let these assumptions hold. Then, for each $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ with **I**) $(x_0, y_0) \succeq (\bar{x}, \bar{y})$, **II**) $(\bar{x}, \bar{y}) \succeq (x', y') \in A \implies \bar{x} = x'$.*

The authors' (constructive) argument is based on the Cantor's intersection principle and a separation theorem involving the pair $(k^0, -\text{cl}(K))$. Note that, the former of these requires a denumerable version of (AC) (= Axiom of Choice); and the latter one is based on the Zorn Maximality Principle (equivalent with (AC)). On the other hand, Hamel [11, Ch 4, Sect 4.7] established that GTZ is reducible to the ordering principle in Brezis and Browder [1] (in short: BB). But, his argument (again based on separation techniques) is restricted to the normed case (modulo Y); so, we may ask whether this is removable. The answer is positive (cf. Section 4): a metrical version of (BB) obtainable (cf. Section 2) from the Tarski's Dependent Choices Principle [18] (in short: DC) will suffice for deducing an extended version of GTZ for arbitrary (non-topological) vector spaces Y .

A relevant application of GTZ is the couple of vector type variational principles obtained by the quoted authors [10, Corollary 2], [10, Corollary 3]. These statements include the maximality results in Isac [13] and Nemeth [15]; which, in turn, extend Ekeland's variational principle [7] (in short: EVP); so, it is natural to denote their union as (EVPv). [An extended variant of all these (also denoted in this way) is given in Section 4; and the main tool for establishing it is the concept of *gauge function*, discussed in Section 3]. Concerning the converse inclusion, a positive answer to this is highly expectable; because for a (related to (EVPv)) result in Tammer [17] such a reduction scheme works. A confirmation of this position is provided in Section 5; where it is proved that (EVP) \implies (DC). Further aspects will be delineated elsewhere.

2 (DC) \implies (BB) \implies ((BBm), (EVPm))

Let M be a nonempty set; and $\mathcal{R} \subseteq M \times M$ stand for a (nonempty) relation over it. For each $x \in M$, denote $M(x, \mathcal{R}) = \{y \in M; x\mathcal{R}y\}$. The following "Dependent Choices Principle" (in short: DC) is our starting point:

Proposition 2.1. *Suppose that*

(b01) $M(c, \mathcal{R})$ is nonempty, for each $c \in M$.

Then, for each $a \in M$ there exists $(x_n) \subseteq M$ with $x_0 = a$ and $x_n\mathcal{R}x_{n+1}$, for all n .

This principle, due to Tarski [18], is deductible from AC, but not conversely; cf. Wolk [21]. Moreover – as a substitute of (AC) – it seems to suffice for a large part of the "usual" mathematics; see Moore [14, Appendix 2, Table 4].

(A) Let M be a nonempty set. Take a *quasi-order* (\leq) over it, and a function $\varphi : M \rightarrow R \cup \{\infty\}$. Call $z \in M$, (\leq, φ) -*maximal* when φ is constant on $M(z, \leq)$.

Proposition 2.2. *Assume that*

(b02) φ is *inf-proper* ($\text{Dom}(\varphi) \neq \emptyset$ and $\inf[\varphi(M)] > -\infty$)

(b03) φ is (\leq) -*decreasing* ($x \leq y \implies \varphi(x) \geq \varphi(y)$)

(b04) each ascending sequence in $\text{Dom}(\varphi)$ has an upper bound (in M).

Then, for each $u \in \text{Dom}(\varphi)$ there exists a (\leq, φ) -maximal $v \in M$ with $u \leq v$.

Proof. Define a function $\beta : M \rightarrow R \cup \{\infty\}$ as: $\beta(v) := \inf[\varphi(M(v, \leq))]$, $v \in M$. Clearly, β is increasing; and $\varphi(v) \geq \beta(v)$, for all $v \in M$. Moreover, (b03) gives at once a characterization like: $v \in \text{Dom}(\varphi)$ is (\leq, φ) -maximal iff $\varphi(v) = \beta(v)$. Now, assume by contradiction that the conclusion in this statement is false; i.e. [by the remark above] there must be some $u \in \text{Dom}(\varphi)$ such that:

(b05) for each $v \in M_u := M(u, \leq)$, one has $\varphi(v) > \beta(v)$.

Consequently (for all such v), $\varphi(v) > (1/2)(\varphi(v) + \beta(v)) > \beta(v)$; hence

$$v \leq w \text{ and } (1/2)(\varphi(v) + \beta(v)) > \varphi(w), \quad (2.1)$$

for at least one w (belonging to M_u). The relation \mathcal{R} over M_u introduced via (2.1) fulfills $M_u(v, \mathcal{R}) \neq \emptyset$, for all $v \in M_u$. So, by (DC), there must be a sequence (u_n) in M_u with $u_0 = u$ and

$$u_n \leq u_{n+1}, (1/2)(\varphi(u_n) + \beta(u_n)) > \varphi(u_{n+1}), \text{ for all } n. \quad (2.2)$$

We have thus constructed an ascending sequence (u_n) in $M_u \subseteq \text{Dom}(\varphi)$ for which $(\varphi(u_n))$ is (strictly) descending and bounded below; hence $\lambda := \lim_n \varphi(u_n)$ exists in R . By (b04), there exists $v \in \text{Dom}(\varphi)$ such that $u_n \leq v$, for all n . From (b02) (and the properties of β) $\varphi(u_n) \geq \varphi(v)$ and $\varphi(v) \geq \beta(v) \geq \beta(u_n)$, $\forall n$. The former of these gives $\lambda \geq \varphi(v)$; and the latter one yields (via (2.2)) $(1/2)(\varphi(u_n) + \beta(v)) > \varphi(u_{n+1})$, for all $n \in N$. Passing to limit as $n \rightarrow \infty$, we have $(\varphi(v) \geq) \beta(v) \geq \lambda$; so, combining with the preceding one, $\varphi(v) = \beta(v)(= \lambda)$, contradiction. \square

In particular, when $\text{Dom}(\varphi) = M$, Proposition 2.2 is identical with (BB); for simplicity reasons, it will be also denoted in this way.

(B) A useful application of these developments is to be given under the lines below. Let M be a nonempty set; and (\leq) , some *quasi-order* over it. By a *pseudometric* over M we shall mean any map $d : M \times M \rightarrow R_+$. If, in addition, d is *reflexive* [$d(x, x) = 0, \forall x \in M$], *triangular* [$d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in M$] and *symmetric* [$d(x, y) = d(y, x), \forall x, y \in M$], we say that it is a *semimetric* (on M). Suppose that we fixed such an object. Call the sequence (x_n) (in M), *d-Cauchy* when $[\forall \varepsilon > 0, \exists n(\varepsilon): n(\varepsilon) \leq p \leq q \implies d(x_p, x_q) \leq \varepsilon]$. Further, let us say that (x_n) , *d-converges* to $x \in M$ when $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. In this case, x is called the *d-limit* of (x_n) ; and if such elements exist, we say that (x_n) is *d-convergent* (in M). Note that (for any sequence) *d-convergent* \implies *d-Cauchy*; the reciprocal of this is not in general true.

Now, let the function $\varphi : M \rightarrow R \cup \{\infty\}$ be as in (b02)+(b03). Denote by $(\leq_{(d, \varphi)})$ the Brøndsted quasi-order [2] attached to (d, φ) [$x \leq_{(d, \varphi)} y$ iff $d(x, y) + \varphi(y) \leq \varphi(x)$]. The natural condition (with a practical finality) to be imposed is

(b06) (\leq) is coarser than $(\leq_{(d, \varphi)})$ over $\text{Dom}(\varphi)$:
 $x, y \in \text{Dom}(\varphi), x \leq y$ imply $d(x, y) \leq \varphi(x) - \varphi(y)$.

Proposition 2.3. *Assume that (in addition)*

(b07) *each ascending d -Cauchy sequence in $\text{Dom}(\varphi)$ is bounded above (modulo (\leq)).*

*Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with **k1** $u \leq v$, **k2** $v \leq x \in M$ imply $d(v, x) = 0$, $\varphi(v) = \varphi(x)$.*

Proof. We claim that (BB) applies to (M_u, \leq) and φ ; where $M_u := M(u, \leq)$. Let (x_n) be an ascending (modulo (\leq)) sequence in M_u :

(b08) $x_n \leq x_m$ (hence $d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$), if $n \leq m$.

The (real) sequence $(\varphi(x_n))$ is descending and bounded; hence a Cauchy one. This tells us that (x_n) is an ascending (modulo (\leq)) d -Cauchy sequence in $M_u \subseteq \text{Dom}(\varphi)$. So, from (b07), $x_n \leq x$, for all n and some $x \in M$ (fulfilling $x \in M_u$); hence the claim. By the quoted principle it then follows that, for the starting $u \in M_u$ there is another $v \in M_u$ with **h1** $u \leq v$, **h2** $v \leq x \in M_u$ implies $\varphi(v) = \varphi(x)$. The obtained point has, via (b06), all properties we need. \square

Call Proposition 2.3, the (*semi-*) *metrical* version of (BB) (in short: (BBm)). It is natural to ask what happens with the above conclusions when (b06) is no longer accepted. A positive answer to this is available, with respect to the (coarser) "product" quasi-order $(\preceq) := (\leq) \cap (\leq_{(d, \varphi)})$.

Proposition 2.4. *Assume that (in addition to (b02)+(b03))*

(b09) *each (\leq) -ascending d -Cauchy sequence in $\text{Dom}(\varphi)$ is d -convergent*

(b10) *the d -limit of each ascending d -convergent sequence in $\text{Dom}(\varphi)$ is an upper bound of it (modulo (\leq)).*

*Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with **k3** $u \preceq v$, **k4** $v \preceq x \in M$ imply $d(v, x) = 0$, $\varphi(v) = \varphi(x)$.*

Proof. As before, we claim that (BB) applies to (M_u, \preceq) and φ ; where $M_u := M(u, \preceq)$. Let (x_n) be an ascending (modulo (\preceq)) sequence in M_u

(b11) $x_n \leq x_m$, $d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, if $n \leq m$.

The (real) sequence $(\varphi(x_n))$ is descending and bounded; hence a Cauchy one. This tells us that (x_n) is an ascending (modulo (\leq)) d -Cauchy sequence in M_u ; so, from (b09), $x_n \rightarrow x$ as $n \rightarrow \infty$, for some $x \in M$. Taking (b10) into account, $x_n \leq x$ (hence $\varphi(x_n) \geq \varphi(x)$), for all n ; note that, as a consequence, $x \in \text{Dom}(\varphi)$. On the other hand, fix some rank n . From (b11), $d(x_n, x) \leq d(x_n, x_m) + d(x_m, x) \leq \varphi(x_n) - \varphi(x) + d(x_m, x)$, $\forall m \geq n$. Passing to limit as $m \rightarrow \infty$ yields (combining with the above) $x_n \preceq x$, $\forall n$; and the claim follows. By the quoted principle it then results that, for the starting $u \in M_u$ there is another $v \in M_u$ with **h3** $u \preceq v$, **h4** $v \preceq x \in M_u$ implies $\varphi(v) = \varphi(x)$. This point has all properties we need. \square

This result is (cf. Turinici [19]) the monotone version of EVP (in short: (EVPm)). Note that, if (b06) holds, (\leq) is identical with (\preceq) on $\text{Dom}(\varphi)$; and conclusions of Proposition 2.4 are identical with the ones of Proposition 2.3.

In particular, when (\leq) is just $(\leq_{(d,\varphi)})$, (b10) holds under

(b12) φ is d -lsc over M : $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \rightarrow x$;

this (via (b09)+(b10) \implies (b07)) tells us that either of the above statements gives

Proposition 2.5. *Assume that (M, d) is complete and (b02)+(b12) hold. Then, for each $u \in \text{Dom}(\varphi)$ there exists $v \in \text{Dom}(\varphi)$ with **k5** $d(u, v) \leq \varphi(u) - \varphi(v)$, **k6** $x \in M$, $d(v, x) \leq \varphi(v) - \varphi(x)$ imply $d(v, x) = 0$, $\varphi(v) = \varphi(x)$.*

This is just EVP (the semimetric variant). For pseudometric extensions, we refer to Hyers, Isac and Rassias [12, Ch 5]; see also Turinici [20].

Summing up, (DC) \implies (BB) \implies (BBm) \implies (EVP), (DC) \implies (BB) \implies (EVPm) \implies (EVP). In fact, the list of these "intermediary" maximal/variational principles is much more comprehensive; we do not give details.

3 Gauge functions

Let Y be a (real) *vector space*. Take a convex cone H of Y ; and let (\leq) stand for its induced quasi-order. Pick some $k^0 \in H \setminus (-H)$; and put (for $y \in Y$)

$$(c01) \quad \Gamma(H; k^0; y) = \{s \in R_+; k^0 s \leq y\}, \quad \gamma(H; k^0; y) = \sup \Gamma(H; k^0; y).$$

(By convention, $\sup(\emptyset) = -\infty$). We therefore defined a couple of functions $\Gamma(\cdot) := \Gamma(H; k^0; \cdot)$ and $\gamma(\cdot) := \gamma(H; k^0; \cdot)$ from Y to $\mathcal{P}(R_+)$ and $R_+ \cup \{-\infty, \infty\}$ respectively; the latter of these will be referred to as the *gauge* function attached to $(H; k^0)$. Such objects were introduced (in the normed context) by Gerth (Tammer) and Weidner [8]. Some refinements of these (to the locally convex setting) were provided by Goepfert, Tammer and Zalinescu [10]. The present developments may be viewed as a non-topological extension of the preceding ones.

i) We start by noting that, for each $y \in Y$, $\Gamma(y)$ is hereditary ($s \in \Gamma(y) \implies [0, s] \subseteq \Gamma(y)$). In addition, we have (by definition)

$$y \notin H \Leftrightarrow \Gamma(y) = \emptyset \Leftrightarrow \gamma(y) = -\infty; \quad y \in H \Leftrightarrow \Gamma(y) \neq \emptyset \Leftrightarrow \gamma(y) \in [0, \infty].$$

ii) Further, Γ and γ are *positively homogeneous*: $\Gamma(ty) = t\Gamma(y)$, $\gamma(ty) = t\gamma(y)$, $\forall t > 0$, $\forall y \in Y$. (Here, $t\emptyset = \emptyset$, $\forall t > 0$). Moreover (as $\gamma(0) = 0$, $\gamma(k^0) = 1$), γ is *H -proper*: $\text{Dom}_H(\gamma) := \text{Dom}(\gamma) \cap H$ is nonempty.

iii) An interesting question is that of γ being *totally H -proper*: $\text{Dom}_H(\gamma) = H$. After Cristescu [6, Ch 5, Sect 1], we say that H is *Archimedean*, provided

$$(c02) \quad [h \in Y, v \in H \text{ and } \Gamma(H; h; v) \supseteq R_+^0] \text{ imply } h \in -H.$$

Lemma 3.1. *Assume that H is Archimedean. Then, **a**) $\infty \notin \gamma(H)$ (hence γ is totally H -proper); **b**) $\Gamma(y)$ is (bounded) closed (hence $\gamma(y) \in \Gamma(y)$), for each $y \in H$.*

Proof. The first part is clear, by the choice of k^0 (and the accepted hypotheses). For the second one, let $y \in H$ be arbitrary fixed. By the hereditary property, $k^0 \gamma(y) - y \leq k^0 t$, for all $t > 0$; wherefrom $\Gamma(H; k^0 \gamma(y) - y; k^0) \supseteq R_+^0$. This, and the Archimedean property of H , give the desired fact. \square

iv) Returning to the general case, note that γ is increasing: $y_1, y_2 \in Y$, $y_1 \leq y_2$ implies $\gamma(y_1) \leq \gamma(y_2)$. As a consequence, $\gamma(y) \leq t$, whenever $y \leq k^0 t$; so, $\text{Dom}_H(\gamma)$ is "large" enough. Some other useful properties are formulated in

Lemma 3.2. *The gauge function γ is super-additive: $\gamma(y_1 + y_2) \geq \gamma(y_1) + \gamma(y_2)$, whenever the right member exists. As a consequence, γ is subtractive: $\gamma(y_1 - y_2) \leq \gamma(y_1) - \gamma(y_2)$, whenever $y_1 \geq y_2$ and $\gamma(y_2) < \infty$.*

Proof. Without loss, one may assume that $\gamma(y_1) > 0$, $\gamma(y_2) > 0$. By the hereditary property, $y_1 \geq k^0 t_1, y_2 \geq k^0 t_2$, whenever $0 \leq t_1 < \gamma(y_1), 0 \leq t_2 < \gamma(y_2)$; and this yields (for all such (t_1, t_2)) $y_1 + y_2 \geq k^0 [t_1 + t_2]$ (that is, $\gamma(y_1 + y_2) \geq t_1 + t_2$). Combining with the arbitrariness of the precise couple gives the first part. The second part is directly obtainable from this one, in a standard way. \square

4 (BBm) \implies (GTZ) \implies (EVPv)

With this information at hand, we may now return to the question of the introductory part. Let Y be a (real) vector space; and K , some (convex) cone of it. Denote by (\leq_K) the induced quasi-order; also expressed as (\leq) , when K is understood. Let H be another (convex) cone of Y with $K \subseteq H$ and H is Archimedean (cf. (c02)). Further, take some $k^0 \in K \setminus (-H)$; and let $\gamma(\cdot) = \gamma(H; k^0; \cdot)$ stand for the gauge function attached to (H, k^0) . Note that (by the developments above)

$$0 \leq \gamma(y) < \infty, \text{ for all } y \in H \text{ (hence } \text{Dom}_H(\gamma) = H) \quad (4.1)$$

$$y_1, y_2 \in Y, y_1 \leq_H y_2 \implies \gamma(y_1) \leq \gamma(y_2). \quad (4.2)$$

Finally, take a semi-metric space (X, d) (cf. Section 2).

(A) Let (\succeq) stand for the quasi-order (a02) on $X \times Y$; and A be some part of $X \times Y$. As in Section 1, we are interested to determine sufficient conditions under which (A, \succeq) has points with certain maximality properties. These are as follows:

(d01) $P_Y(A)$ is bounded below (modulo H) $[\exists \tilde{y} \in Y: P_Y(A) \subseteq \tilde{y} + H]$

(d02) each ascending e -Cauchy sequence $((x_n, y_n)) \subseteq A$ is bounded above in A (modulo (\succeq)).

Here, e stands for the semi-metric on $X \times Y$ introduced as

(d03) $e((x_1, y_1), (x_2, y_2)) = d(x_1, x_2)$, $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Theorem 4.1. *Let the precise assumptions be in force. Then, for each starting $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ with the properties **III**) $(x_0, y_0) \succeq (\bar{x}, \bar{y})$, **IV**) $(\bar{x}, \bar{y}) \succeq (x', y') \in A \implies d(\bar{x}, x') = 0', \gamma(\bar{y} - \tilde{y}) = \gamma(y' - \tilde{y})$ (hence $\gamma(\bar{y} - y') = 0$).*

Proof. Without loss, one may assume that (d01) may be written as

(d04) $P_Y(A) \subseteq H$ (i.e.: $\tilde{y} = 0$ in that condition).

For otherwise, passing to the subset $\tilde{A} = \{(x, y) \in X \times Y; (x, y + \tilde{y}) \in A\}$, this requirement is fulfilled, in view of $P_Y(\tilde{A}) = P_Y(A) - \tilde{y}$; as well as (d02). And from the conclusion involving \tilde{A} it clearly follows the one relative to A . Having these precise, let us introduce the function (from $X \times Y$ to $R_+ \cup \{-\infty, \infty\}$) $\Phi(x, y) = \gamma(y)$, $(x, y) \in X \times Y$ (i.e.: $\Phi = \gamma \circ P_Y$); its restriction to A will be also denoted in this way. We claim that (BBm) is applicable to $(A, \succeq; e)$ and Φ . In fact, Lemma 3.1 gives us [via (d04)] $0 \leq \Phi(x, y) = \gamma(y) < \infty$, for each $(x, y) \in A$; wherefrom, Φ (restricted to A) is bounded from below on A and proper ($\text{Dom}(\Phi) = A$). In addition, by the monotonicity of the gauge function, $(x_1, y_1) \succeq (x_2, y_2) \implies y_1 \geq y_2 \implies \gamma(y_1) \geq \gamma(y_2) \implies \Phi(x_1, y_1) \geq \Phi(x_2, y_2)$; which tells us that Φ (restricted to A) is (\succeq) -decreasing. Further, let $(x_1, y_1), (x_2, y_2)$ in A be such that $(x_1, y_1) \succeq (x_2, y_2)$ [that is: $k^0 d(x_1, x_2) \leq y_1 - y_2$]. By Lemma 3.2, this yields $d(x_1, x_2) \leq \gamma(y_1 - y_2) \leq \gamma(y_1) - \gamma(y_2)$; whence

$$(x_1, y_1) \succeq (x_2, y_2) \implies e((x_1, y_1), (x_2, y_2)) \leq \Phi(x_1, y_1) - \Phi(x_2, y_2);$$

and (b06) holds. Finally, (b07) is identical with (d02); and our claim is proved. By (BBm) it then follows that, for $a_0 = (x_0, y_0) \in A$ there exists $\bar{a} = (\bar{x}, \bar{y}) \in A$ with the properties **h5**) $a_0 \succeq \bar{a}$ and **h6**) if $a' = (x', y') \in A$ fulfills $\bar{a} \succeq a'$ then $e(\bar{a}, a') = 0$, $\Phi(\bar{a}) = \Phi(a')$. This gives all conclusions in the statement. \square

In particular, when Y is a (real) separated locally convex space, the choice $H = \text{cl}(K)$ is allowed in (c02); and Theorem 4.1 reduces to (GTZ). This inclusion is technically strict; because the last relation in **IV**) cannot be deduced from the one in **II**). However, for simplicity reasons, this statement will be still denoted in this way. Further aspects may be found in Turinici [20]; see also Goepfert, Riahi, Tammer and Zălinescu [9, Ch 3, Sect 10].

(B) Passing to the initial setting [involving the (convex) cones K and H], let us adjoin to Y an element $\infty \notin Y$ taken as

$$(d05) \quad (\forall b \in Y, \forall \lambda > 0): \infty = b + \infty = \infty + b; \infty = \lambda \infty; b \leq \infty \text{ and } \infty \not\leq b.$$

Further, take a function $F : X \rightarrow Y \cup \{\infty\}$ according to

$$(d06) \quad F \text{ is inf-proper (modulo } H): \text{Dom}(F) \neq \emptyset \text{ and} \\ \text{there exists } \tilde{y} \in Y \text{ such that } F(\text{Dom}(F)) \subseteq \tilde{y} + H.$$

It is our aim in the following to formulate a vector Ekeland type variational principle involving these data. To this end, let (\geq) stand for the quasi-order on X

$$(d07) \quad (x_1, x_2 \in X): x_1 \geq x_2 \text{ iff } k^0 d(x_1, x_2) + F(x_2) \leq F(x_1).$$

Note that, as a direct consequence, one has (via (a02))

$$(x_1, x_2 \in \text{Dom}(F)): x_1 \geq x_2 \text{ iff } (x_1, F(x_1)) \succeq (x_2, F(x_2)). \quad (4.3)$$

Theorem 4.2. *Assume that (in addition), (X, d) is complete and*

$$(d08) \quad \text{the } d\text{-limit of each ascending sequence in } \text{Dom}(F) \\ \text{is an upper bound of it (modulo } (\geq)).$$

Then, for each $x_0 \in \text{Dom}(F)$ there exists $\bar{x} \in \text{Dom}(F)$ with **V**) $x_0 \geq \bar{x}$, **VI**) $\bar{x} \geq x' \in X \implies d(\bar{x}, x') = 0$, $\gamma(F(\bar{x}) - \tilde{y}) = \gamma(F(x') - \tilde{y})$ (hence $\gamma(F(\bar{x}) - F(x')) = 0$).

Proof. Put $A = \{(x, F(x)); x \in \text{Dom}(F)\}$. We claim that Theorem 4.1 applies to $(A, \succeq; e)$ [where (\succeq) and e are the above ones]; wherefrom, all is clear. In fact, (d01) is retainable for these data, via (d06). Further, let $((x_n, F(x_n)))$ be an ascending (modulo (\succeq)) e -Cauchy sequence in A . By (4.3), (x_n) is an ascending (modulo (\geq)) d -Cauchy sequence in $\text{Dom}(F)$; so (by completeness) $x_n \rightarrow x$ as $n \rightarrow \infty$, for some $x \in X$. Combining with (d08) yields $x_n \geq x, \forall n$ (hence, in particular, $x \in \text{Dom}(F)$). This, again via (4.3), yields $(x_n, F(x_n)) \succeq (x, F(x)) \in A$, for all n ; and (d02) is retainable as well. The proof is thereby complete. \square

Note that, with the convention $\gamma(\infty) = \infty$, we may extend the gauge function γ attached to (H, k^0) up to some function $\gamma : Y \cup \{\infty\} \rightarrow R_+ \cup \{-\infty, \infty\}$ which also fulfills (4.1)+(4.2). Further, define the function (from X to $R \cup \{\infty\}$) $\varphi(x) = \gamma(F(x) - \tilde{y}), x \in X$. It is not without importance to emphasize that an alternate way of proving Theorem 4.2 is to establish that conditions of (EVPm) are fulfilled over $(X, \succeq; d)$ and φ . This is essentially based on the relations

$$\varphi(x) = \infty \text{ iff } F(x) = \infty \quad (\text{i.e.: } \text{Dom}(\varphi) = \text{Dom}(F)) \quad (4.4)$$

$$\varphi(x) \geq 0, \text{ for all } x \in X \quad (\text{hence } \inf[\varphi(X)] \geq 0); \quad (4.5)$$

we do not give details.

Now, a sufficient condition for (d08) above is evidently

(d09) $X(x, \geq)$ is closed, for all $x \in \text{Dom}(F)$.

Another circumstance for the same is the couple

(d10) $[\text{Dom}(F) \supseteq (x_n) = (\geq)\text{-ascending}, x_n \rightarrow x \in X] \implies F(x_n) \geq F(x), \forall n$

(d11) $\Gamma(K; k^0; y)$ is closed, for each $y \in K$.

In fact, let (x_n) be a (\geq) -ascending sequence in $\text{Dom}(F)$ with $x_n \rightarrow x$ (for some $x \in X$). By definition, $k^0 d(x_n, x_m) \leq F(x_n) - F(x_m)$, whenever $n \leq m$; and, from (d10), $F(x_n) \geq F(x)$, for all n (hence, in particular, $x \in \text{Dom}(F)$). As a consequence, $k^0 d(x_n, x_m) \leq F(x_n) - F(x)$, whenever $n \leq m$; and this, along with (d11), gives (passing to limit as $m \rightarrow \infty$) $k^0 d(x_n, x) \leq F(x_n) - F(x)$ (i.e.: $x_n \geq x$), for all n ; hence the claim. In particular, when Y is a (real) separated locally convex space, Theorem 4.2 includes (EVPv) (cf. Section 1). However, as before, the obtained result will be still denoted in this way. Further structural aspects may be found in Rozoveanu [16]; see also Chen and Huang [5].

5 (EVPdLc) \implies (DC)

By the developments above, we have the chain of implications: (DC) \Rightarrow (BB) \implies (BBm) \implies (EVP), (BB) \implies (EVPm) \implies (EVP), (BBm) \implies (GTZ) \implies (EVPv) \implies (EVP). So, it is natural asking whether these implications may be reversed. Clearly, the natural setting for solving this problem is (ZF)(=the standard Zermelo-Fraenkel system) without

(AC) (=the Axiom of Choice); referred to in the following as the *reduced* Zermelo-Fraenkel system.

Let X be a nonempty set; and (\leq) be an order on it. We say that (\leq) has the *inf-lattice* property, provided: $x \wedge y := \inf(x, y)$ exists, for all $x, y \in X$. Further, we say that $z \in X$ is a (\leq) -*maximal* element if $X(z, \leq) = \{z\}$; the class of all these points will be denoted as $\max(X, \leq)$. In this case, (\leq) is called a *Zorn order* when $\max(X, \leq)$ is nonempty and *cofinal* in X [for each $u \in X$ there exists a (\leq) -maximal $v \in X$ with $u \leq v$]. Further aspects are to be described in a metric setting. Let $d : X \times X \rightarrow R_+$ be a metric over X ; and $\varphi : X \rightarrow R_+$ be some function. Then, the natural choice for (\leq) above is the Brøndsted order $(\leq_{(d, \varphi)})$ taken as in Section 2. Denote $X(x, \rho) = \{u \in X; d(x, u) < \rho\}$, $x \in X$, $\rho > 0$ [the open sphere with center x and radius ρ]. Call the ambient metric space (X, d) , *discrete* when for each $x \in X$ there exists $\rho = \rho(x) > 0$ such that $X(x, \rho) = \{x\}$. Note that, under such an assumption, any function $\psi : X \rightarrow R$ is continuous over X . However, the *Lipschitz* property ($|\psi(x) - \psi(y)| \leq Ld(x, y)$, $x, y \in X$, for some $L > 0$) cannot be assured, in general.

Now, the statement below is a particular case of EVP:

Theorem 5.1. *Let the metric space (X, d) and the function $\varphi : X \rightarrow R_+$ satisfy*

- (e01) (X, d) is discrete bounded and complete
- (e02) $(\leq_{(d, \varphi)})$ has the inf-lattice property
- (e03) φ is d -nonexpansive and $\varphi(X)$ is countable.

Then, $(\leq_{(d, \varphi)})$ is a Zorn order.

We shall refer to it as: the discrete Lipschitz countable version of EVP (in short: (EVPdLc)). Clearly, (EVP) \implies (EVPdLc). The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

Proposition 5.1. *We have (in the reduced Zermelo-Fraenkel system) (EVPdLc) \implies (DC). So, the maximal/variational principles (BB), (BBm), (EVPm), (EVP), (GTZ) and (EVPv) are all equivalent with (DC); hence, mutually equivalent.*

Proof. Let M be some nonempty set; and \mathcal{R} stand for a relation over M with the property (b01). Fix in the following $a \in M$; as well as some $b \in M(a, \mathcal{R})$. For each $n \geq 2$ in N (= the set of natural numbers) let $N(n, >) := \{0, \dots, n-1\}$ stand for the initial segment determined by n ; and X_n denote the class of all finite sequences $x : N(n, >) \rightarrow M$ with: $x(0) = a$, $x(1) = b$ and $x(m)\mathcal{R}_m x(m+1)$ for $0 \leq m \leq n-2$. In this case, $N(n, >)$ is just $\text{Dom}(x)$ (the *domain* of x); and $n = \text{card}(N(n, >))$ will be referred to as the *order* of x ; [denoted as $\omega(x)$]. Put $X = \cup\{X_n; n \geq 2\}$. Let \preceq stand for the partial order (on X)

- (e04) $x \preceq y$ iff $\text{Dom}(x) \subseteq \text{Dom}(y)$ and $x = y|_{\text{Dom}(x)}$;

and \prec denote its associated strict order. All we have to prove is that (X, \preceq) has strictly ascending infinite sequences.

(A) Let $x, y \in X$ be arbitrary fixed. Denote

$$K(x, y) := \{n \in \text{Dom}(x) \cap \text{Dom}(y); x(n) \neq y(n)\}.$$

If x and y are *comparable* (i.e.: either $x \preceq y$ or $y \preceq x$; written as: $x \langle \rangle y$) then $K(x, y) = \emptyset$. Conversely, if $K(x, y) = \emptyset$, then $x \preceq y$ if $\text{Dom}(x) \subseteq \text{Dom}(y)$ and $y \preceq x$ if $\text{Dom}(y) \subseteq \text{Dom}(x)$; hence $x \langle \rangle y$. Summing up, we have

$$(x, y \in X): x \langle \rangle y \text{ if and only if } K(x, y) = \emptyset.$$

The negation of this property means: x and y are *not comparable* (denoted as: $x || y$). By the characterization above, it is equivalent with $K(x, y) \neq \emptyset$. Note that, in such a case, $k(x, y) := \min(K(x, y))$ is well defined; and $N(k(x, y), >)$ it is the largest initial interval of $\text{Dom}(x) \cap \text{Dom}(y)$ where x and y are identical.

Lemma 5.1. *The partial order (\preceq) has the inf-lattice property. Moreover, $x \mapsto \omega(x)$ is strictly increasing ($x \prec y$ implies $\omega(x) < \omega(y)$) and*

$$2 \leq \omega(x \wedge y) \leq \min\{\omega(x) - 1, \omega(y) - 1\}, \text{ whenever } x || y. \quad (5.1)$$

Proof. (of Lemma 5.1) i) Let $x, y \in X$ be arbitrary fixed. The case $x \langle \rangle y$ is clear; so, without loss, one may assume that $x || y$. Note that, by the remark above, $K(x, y) \neq \emptyset$ and $k := k(x, y)$ exists. Let the finite sequence $z \in X_k$ be introduced as $z = x|_{N(k, >)} = y|_{N(k, >)}$. For the moment $z \preceq x$ and $z \preceq y$. Suppose that $w \in X_h$ fulfills the same properties. Then, the restrictions of x and y to $N(h, >)$ are identical; wherefrom (see above) $h \leq k$ and $w \preceq z$. **ii)** Evident. **iii)** By the above notations, $\omega(x \wedge y) = k(x, y)$. This, and $k(x, y) \in \text{Dom}(x) \cap \text{Dom}(y)$, give the desired relation. \square

(B) Our next objective is to introduce a metrical structure as well as an associated objective function over X , which should have all required properties. To this end, put $\varphi(x) = 3^{-\omega(x)}$, $x \in X$; and note that $\varphi(X) = \{3^{-n}; n \geq 2\}$ (hence, φ has countable many strictly positive values). Then, define

$$(e05) \quad d(x, y) = |\varphi(x) - \varphi(y)|, \text{ if } x \langle \rangle y; \quad d(x, y) = \varphi(x \wedge y), \text{ whenever } x || y.$$

Lemma 5.2. *The mapping $(x, y) \mapsto d(x, y)$ is a metric on X (in the usual sense).*

Proof. (of Lemma 5.2) Clearly, d is reflexive and symmetric [$d(x, y) = d(y, x)$, $x, y \in X$]. On the other hand, d is sufficient. In fact, assume $d(x, y) = 0$. By a previous evaluation of $\varphi(X)$, it results that x and y are comparable and $\omega(x) = \omega(y)$; wherefrom, $x = y$. Finally, let us verify the triangular property: $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$. Without loss, one may assume that $\omega(x) \leq \omega(z)$; for, otherwise, we simply interchange x and z . Two alternatives are open before us.

a) The points x and z are comparable; that is, $x \preceq z$ (by the hypothesis above). We start from the obvious relation

$$|\varphi(s) - \varphi(t)| \leq \max\{\varphi(s), \varphi(t)\} \leq \varphi(s \wedge t), \quad s, t \in X.$$

Combining with $d(x, z) = |\varphi(x) - \varphi(z)| \leq |\varphi(x) - \varphi(y)| + |\varphi(y) - \varphi(z)|$ yields the desired fact, for all possible cases concerning (x, y) and (y, z) .

b) The points x and z are not comparable ($x||z$). Four sub-cases appear:

Sub-case b1): Suppose that $x \langle \rangle y$, $y \langle \rangle z$. The alternatives $[x \preceq y, y \preceq z]$ and $[y \preceq x, z \preceq y]$ give $x \langle \rangle z$; contradiction. So, it remains to discuss:

b11) $x \preceq y, z \preceq y$. Then, x and z are the restrictions of y to $\text{Dom}(x)$ and $\text{Dom}(z)$ respectively; wherefrom $x \preceq z$, contradiction.

b12) $y \preceq x, y \preceq z$. We start from the direct consequence of (5.1) above

$$3 \max\{\varphi(s), \varphi(t)\} \leq \varphi(s \wedge t) \leq 3^{-2}, \quad s, t \in X, s||t.$$

The relation to be checked becomes $\varphi(x \wedge z) \leq 2\varphi(y) - \varphi(x) - \varphi(z)$. By the imposed conditions, $y \preceq x \wedge z$; wherefrom $\varphi(y) \geq \varphi(x \wedge z)$. A sufficient condition for the desired relation to be true is then $\varphi(x \wedge z) \leq 2\varphi(x \wedge z) - \varphi(x) - \varphi(z)$; or, equivalently, $\varphi(x) + \varphi(z) \leq \varphi(x \wedge z)$; evident, by the precise consequence.

Sub-case b2): Suppose that $x||y, y \langle \rangle z$. Two logical possibilities occur:

b21) $x||y, y \preceq z$. We have to establish that: $\varphi(x \wedge z) \leq \varphi(x \wedge y) + \varphi(y) - \varphi(z)$. But, evidently, $x \wedge z \succeq x \wedge y$; wherefrom $\varphi(x \wedge z) \leq \varphi(x \wedge y)$; and then, all is clear.

b22) $x||y, z \preceq y$ (or, equivalently: $z \preceq y, y||x$). The desired relation becomes: $\varphi(x \wedge z) \leq \varphi(x \wedge y) + \varphi(z) - \varphi(y)$. For the moment, $x \wedge z \preceq x \wedge y$. If $x \wedge z \prec x \wedge y$, we must get $q := \omega(x \wedge z) < \omega(x \wedge y)$; so (by definition) $x(q) = y(q)$. As $z = y|_{\text{Dom}(z)}$ and $q \in \text{Dom}(x) \cap \text{Dom}(z)$, this yields $y(q) = z(q)$; hence $x(q) = z(q)$, contradiction. Consequently, $x \wedge z = x \wedge y$; and conclusion follows.

Sub-case b3): $x \langle \rangle y, y||z$. As before, two logical possibilities occur:

b31) $x \preceq y, y||z$. This is just the alternative **b22)**, with (x, y, z) in place of (z, y, x) .

b32) $y \preceq x, y||z$. The relation under consideration becomes $\varphi(x \wedge z) \leq \varphi(y) - \varphi(x) + \varphi(y \wedge z)$. But, from hypothesis, $x \wedge z \succeq y \wedge z$; wherefrom, all is clear.

Sub-case b4): $x||y, y||z$. We have to establish that: $\varphi(x \wedge z) \leq \varphi(x \wedge y) + \varphi(y \wedge z)$. As before, the alternative $[\omega(x \wedge z) \geq \omega(x \wedge y) \text{ or } \omega(x \wedge z) \geq \omega(y \wedge z)]$ gives the desired fact. On the other hand, the alternative $q := \omega(x \wedge z) < \min\{\omega(x \wedge y), \omega(y \wedge z)\}$ yields $x(q) = y(q)$, $y(q) = z(q)$; hence $x(q) = z(q)$, contradiction.

Having discussed all possible cases, the conclusion in the statement follows. \square

(C) Note that, by a previous remark involving $\varphi(X)$, one has $\text{diam}(X) \leq 3^{-2}$. Further properties of the triplet $(X, d; \varphi)$ are contained in

Lemma 5.3. *Under the notations above, one has (for each m)*

$$x, y \in X, \omega(x) \leq m, d(x, y) < 2 \cdot 3^{-m-1} \implies x = y; \quad (5.2)$$

so that, the metric space (X, d) is discrete.

Proof. (of Lemma 5.3) Assume that $x \neq y$. We show that this cannot be in agreement with the accepted hypothesis. Two cases are open before us:

i) Let x and y be comparable: either $x \prec y$ or $y \prec x$. If $x \prec y$, we have $\omega(x) + 1 \leq \omega(y)$; and then $(1/3)\varphi(x) \geq \varphi(y)$; hence (by definition) $d(x, y) = \varphi(x) - \varphi(y) \geq (2/3)\varphi(x) \geq$

$2 \cdot 3^{-m-1}$, contradiction. If $y \prec x$ then (by the same way as before) $d(x, y) \geq (2/3)\varphi(y) \geq (2/3)\varphi(x) \geq 2 \cdot 3^{-m-1}$, again a contradiction.

ii) Suppose that x and y are not comparable. Then (by definition) $d(x, y) = \varphi(x \wedge y) \geq \varphi(x) \geq 3 \cdot 3^{-m-1}$, contrary to the accepted hypothesis. \square

Lemma 5.4. *Under the same notations above,*

$$|\varphi(x) - \varphi(y)| \leq d(x, y), \quad \forall x, y \in X; \quad (5.3)$$

so, φ is d -nonexpansive (hence, all the more d -Lipschitz).

Proof. (of Lemma 5.4) If x and y are comparable, then $d(x, y) = |\varphi(x) - \varphi(y)|$; and we are done. If x and y are not comparable then, without loss, one may assume $\omega(x) \leq \omega(y)$; hence $\varphi(x) \geq \varphi(y)$. As $x \wedge y \preceq x$, we have $d(x, y) = \varphi(x \wedge y) \geq \varphi(x) \geq \varphi(x) - \varphi(y) = |\varphi(x) - \varphi(y)|$; and the conclusion follows. \square

(D) Given the couple (d, φ) as before, we may introduce the associated Brøndsted order $(\leq_{(d, \varphi)})$ on X ; also denoted as (\leq) , for simplicity. It is natural to ask which is the relationship between it and the initial order (\preceq) on X .

Lemma 5.5. *We necessarily have (under these conventions)*

$$x \preceq y \text{ if and only if } x \leq y. \quad (5.4)$$

That is: these partial orders coincide over X .

Proof. (of Lemma 5.5) Clearly, $x \preceq y$ gives $\omega(x) \leq \omega(y)$; wherefrom $d(x, y) = \varphi(x) - \varphi(y)$; i.e., $x \leq y$. Conversely, assume that $x \leq y$. For the moment, x and y are comparable; since, otherwise, the imposed condition gives $\varphi(x \wedge y) \leq \varphi(x) - \varphi(y) \leq \varphi(x)$ [hence $\omega(x \wedge y) \geq \omega(x)$]; in contradiction with (5.1). The alternative $y \preceq x$ yields (by the first part) $y \leq x$; wherefrom (as (\leq) is order) $x = y$. Hence, any way $x \preceq y$. \square

(E) We are now in position to complete the argument. As (b01) holds, we necessarily have $\max(X, \preceq) = \emptyset$; i.e.: for each $x \in X$ there exists $y \in X$ with $x \prec y$. This, along with (EVPdLc), tells us that (X, d) is not complete; i.e., there exists at least one d -Cauchy sequence (x_n) in X which is not convergent. Note that both these properties are transferable to all subsequences of (x_n) . This allows us to take our sequence in such a way that, for all m ,

$$(e06) \quad d(x_p, x_q) < 3^{-m-1}, \text{ whenever } p, q \geq m.$$

In fact, the d -Cauchy property assures us (with $\varepsilon = 3^{-m-1}$) that

$$C(m) := \{n \in N; d(x_p, x_q) < 3^{-m-1}, \text{ for all } p, q \geq n\} \neq \emptyset, \quad \forall m \in N.$$

In addition, $n \mapsto C(n)$ is (\subseteq) -decreasing; hence $n \mapsto g(n) := \min[C(n)]$ is (\leq) -increasing. This finally tells us that $n \mapsto h(n) := n + g(n)$ is strictly (\leq) -increasing; wherefrom $(y_n = x_{h(n)}; n \in N)$ is a subsequence of (x_n) fulfilling (e06), in view of: $p, q \geq m \implies h(p), h(q) \geq h(m) \geq g(m)$.

Now, by the divergence of (x_n) , we must have

$$B(k; m) := \{n \in N(k, <); \omega(x_n) > m\} \neq \emptyset, \text{ for all } k, m \in N. \quad (5.5)$$

For, otherwise, there exist $k, m \in N$ (with $k \geq m$) such that $\omega(x_n) \leq m, \forall n \geq k$; [wherefrom (by (e06)) $d(x_k, x_n) < 3^{-k-1} \leq 3^{-m-1}, \forall n \geq k$]; and this, by Lemma 5.3, leads us to $x_n = x_k, \forall n \geq k$; hence, (x_n) is convergent, contradiction. We now consider the following sequential type algorithm:

$$\begin{aligned} p(0) &= 1, q(0) = \omega(x_{p(0)}) \text{ (hence } q(0) \geq 1) \\ p(n+1) &= \min B(p(n); q(n)), q(n+1) = \omega(x_{p(n+1)}), n \geq 0. \end{aligned}$$

Note that, by the precise character of such conventions, no use of (DC) is necessary. The rank sequence $(p(n); n \in N)$ is strictly ascending [$1 = p(0) < p(1) < p(2) < \dots$]; hence, $p(n) > n$, for all n . It therefore generates a subsequence $(y_n := x_{p(n)}; n \in N)$ of (x_n) with the supplementary property (deductible from (e06))

$$d(y_k, y_h) < 3^{-k-1}, \text{ whenever } k, h \in N \text{ satisfy } k < h. \quad (5.6)$$

In addition, $(q(n) := \omega(y_n); n \in N)$ is strictly ascending too [$1 \leq q(0) < q(1) < q(2) < \dots$]; hence, $q(n) > n$, for all n . The sequence $(z_n := y_n|_{N(n, \geq)}; n \in N)$ is therefore well defined in X . We claim that

$$(z_n) \text{ is strictly ascending: } z_k \prec z_h, \text{ for } k < h. \quad (5.7)$$

In fact, if $y_k \ll y_h$ we must have $y_k \prec y_h$ (as $\omega(y_k) < \omega(y_h)$); and, from this, $z_k \prec z_h$. And, if $y_k || y_h$ we have (by (5.6)) $\omega(y_k \wedge y_h) > k$; which yields

$$z_k = y_k|_{N(k, \geq)} = y_h|_{N(k, \geq)} = z_h|_{N(k, \geq)};$$

wherefrom $z_k \prec z_h$; hence the claim. But then, the sequence $(c_n = z_n(n); n \in N)$ is well defined in M ; and, moreover, $c_0 = a, c_n \mathcal{R} c_{n+1}$, for all n . This gives us the desired conclusion. \square

In particular, when the specific assumptions (e02) and (e03) are ignored in Theorem 5.1, Proposition 5.1 above reduces to the result in Brunner [3]; but, the lattice type methods used here seem to be new.

References

- [1] H. Brezis and F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math., 21 (1976), 355-364.
- [2] A. Brøndsted, *Fixed points and partial orders*, Proc. Amer. Math. Soc., 60 (1976), 365-366.
- [3] N. Brunner, *Topologische Maximalprinzipien*, Zeitschr. Math. Logik und Grundle. Math., 33 (1987), 135-139.
- [4] O. Cârjă, M. Necula and I. I. Vrabie, *Viability, Invariance and Applications*, North Holland Mathematics Studies vol. 207, Elsevier B. V., Amsterdam, 2007.

- [5] G. Y. Chen and X. X. Huang, *A unified approach to the existing three types of variational principles for vector valued functions*, Math. Meth. Oper. Res., 48 (1998), 349-357.
- [6] R. Cristescu, *Topological Vector Spaces*, Noordhoff Intl. Publishers, Leyden (The Netherlands), 1977.
- [7] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New Series), 1 (1979), 443-474.
- [8] C. Gerth (Tammer) and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl., 67 (1990), 297-320.
- [9] A. Goepfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Canad. Math. Soc. Books in Math. vol. 17, Springer, New York, 2003.
- [10] A. Goepfert, C. Tammer and C. Zălinescu, *On the vectorial Ekeland's variational principle and minimal points in product spaces*, Nonlinear Analysis, 39 (2000), 909-922.
- [11] A. Hamel, *Variational Principles on Metric and Uniform Spaces*, (Habilitation Thesis), Martin-Luther University, Halle-Wittenberg (Germany), 2005.
- [12] D. H. Hyers, G. Isac and T. M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Sci. Publ., Singapore, 1997.
- [13] G. Isac, *The Ekeland's principle and Pareto ε -efficiency*, in "Multi-Objective Programming and Goal Programming" (M. Tamiz ed.), pp. 148-163, L. Notes Econ. Math. Systems vol. 432, Springer, Berlin, 1996.
- [14] G. H. Moore, *Zermelo's Axiom of Choice: its Origin, Development and Influence*, Springer, New York, 1982.
- [15] A. B. Nemeth, *A nonconvex vector minimization problem*, Nonlinear Analysis, 10 (1986), 669-678.
- [16] P. Rozoveanu, *Ekeland's variational principle for vector valued functions*, Math. Reports [St. Cerc. Mat.], 2(52) (2000), 351-366.
- [17] C. Tammer, *A generalization of Ekeland's variational principle*, Optimization, 25 (1992), 129-141.
- [18] A. Tarski, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, Fund. Math., 35 (1948), 79-104.
- [19] M. Turinici, *A monotone version of the variational Ekeland's principle*, An. St. Univ. "A. I. Cuza" Iasi (S. I-a, Mat.) 36 (1990), 329-352.
- [20] M. Turinici, *Minimal points in product spaces*, An. Șt. Univ. "Ovidius" Constanța (Ser. Math.), 10 (2002), 109-122.
- [21] E. S. Wolk, *On the principle of dependent choices and some forms of Zorn's lemma*, Canad. Math. Bull., 26 (1983), 365-367.