LIBERTAS MATHEMATICA, vol XXXI (2011)

Ran-Reurings Fixed Point Results in Ordered Metric Spaces

Mihai TURINICI

Abstract. The fixed point result due to Ran and Reurings [Proc. Amer. Math. Soc., 132 (2004), 1435-1443] is nothing but a particular case of Maia's [Rend. Sem. Mat. Univ. Padova, 40 (1968), 139-143].

Keywords: Partially ordered metric space, monotone operator, fixed point, contraction, reflexive/triangular pseudometric, comparison function.

Mathematics Subject Classification (2010): 47H10 (Primary), 54H25 (Secondary).

1 Introduction

Let X be a nonempty set. Take a metric $d(.,.)$ over it; as well as a selfmap $T : X \to X$. We say that $x \in X$ is a *Picard point* (modulo (d, T)) if **i)** $(T^n x; n \geq 0)$ (=the *orbit* of x) is d-convergent, **ii)** $z := \lim_n T^n x$ is in Fix(T) (i.e., $z = Tz$). If this happens for each $x \in X$ and **iii)** Fix(T) is a singleton, then T is referred to as a *Picard operator* (modulo d); cf. Rus [10, Ch 2, Sect 2.2]. For example, such a property holds whenever d is *complete* and T is d-*contractive*; cf. (b04). A structural extension of this fact – when an *order* $(\leq)$ on X is being added – was obtained in 2004 by Ran and Reurings [9]. For each $x, y \in X$, denote

(a01) $x <> y$ iff either $x \leq y$ or $y \leq x$ (i.e.: x and y are comparable).

This relation is reflexive and symmetric; but not in general transitive. Given $\alpha > 0$, call T, $(d, \alpha, \leq)$-*contractive* if

(a02) $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall x, y \in X$, $x \leq y$.

If this holds for some $\alpha \in]0, 1[$, we say that T is $(d, \leq)$-*contractive*.

Theorem 1.1. *Let d be complete and T be d-continuous. In addition, assume that T is $(d, \leq)$-contractive and*

(a03) $X(T, <>) := \{x \in X; x <> Tx\}$ *is nonempty*

(a04) T *is monotone (increasing or decreasing)*

(a05) *for each* $x, y \in X$, $\{x, y\}$ *has lower and upper bounds.*

Then, T is a Picard operator (modulo d).

According to some authors – including Nieto and Rodriguez-Lopez [6] or (quite recently) O'Regan and Petruşel [7] – this result is the first extension of the classical 1922 Banach's fixed point theorem [2] to the realm of (partially) ordered metric spaces. However, the assertion is false. Some early statements of this type have been obtained in 1970 by Chandra and Fleishman [3, Sect 5]. Further, in 1986, a couple of fixed point results over such structures is being established by Turinici [11, Theorems 2–4], under the lines in Matkowski [5]. Note that, none of these is covered by the recent contributions in the area; we do not give details.

Despite these historical remarks, the Ran-Reurings fixed point result found some useful applications to matrix equations; see the quoted paper for details. So, it cannot be surprising that, soon after, many extensions of Theorem 1.1 were provided. The most recent one (constructed under the lines above) was established in 2008 by O'Regan and Petruşel [7]. But (as we shall see in Section 4), their statement cannot extend the Ran-Reurings'; which, until now, remains the "absolute" model for such developments. Having these precise, it is our aim in the following to discuss the "ontological" status of Theorem 1.1. The conclusion to be derived reads (cf. Section 2): the Ran-Reurings theorem is but a particular case of the beautiful 1968 fixed point statement in Maia [4, Theorem 1]. Further, in Section 3, an extension is given for this last result. Some other aspects will be delineated elsewhere.

2 Main result

(A) Let again $(X, \leq, d)$ be an ordered metric space; and $T : X \to X$, a selfmap of X. Given $x, y \in X$, any subset $\{z_1, ..., z_k\}$ (for $k \geq 2$) in X with $z_1 = x$, $z_k = y$, and $[z_i <> z_{i+1}$, $i \in \{1, ..., k-1\}]$ will be referred to as a $<>$-*chain* between x and y; the class of all these will be denoted as $C(x, y; <>)$. Let $\sim$ stand for the relation over X attached to $<>$ as

(b01) $x \sim y$ iff $C(x, y; <>)$ is nonempty.

Clearly, $(\sim)$ is reflexive and symmetric; because so is $<>$. Moreover, $(\sim)$ is transitive; hence, it is an equivalence over X.

The following variant of Theorem 1.1 is our starting point.

Theorem 2.1. *Let d be complete and T be d-continuous. In addition, assume that T is $(d, \leq)$-contractive,* (a03) *holds, and*

(b02) *T is $<>$-increasing [$x <> y$ implies $Tz <> Ty$]*

(b03) *$(\sim) = X \times X$ [$C(x, y; <>)$ is nonempty, for each $x, y \in X$].*

Then, T is a Picard operator (modulo d).

This result is a weaker form of Theorem 1.1; because (a04) $\Longrightarrow$ (b02), (a05) $\Longrightarrow$ (b03). [In fact, given $x, y \in X$, there exist, by (a05), some $u, v \in X$ with $u \leq x \leq v$, $u \leq y \leq v$. This yields $x <> u$, $u <> y$; wherefrom, $x \sim y$; i.e., (b03) holds]. The remarkable fact to be noted is that Theorem 2.1 (hence the Ran-Reurings statement as well) is deductible from the 1968 Maia's fixed point statement [4, Theorem 1]. Let $e(.,.)$ be another metric over X. Call the selfmap $T : X \to X$, (e, α)-*contractive* (for some $\alpha > 0$) when (b04) $e(Tx, Ty) \leq \alpha e(x, y)$, $\forall x, y \in X$;

if this holds for some $\alpha \in]0, 1[$, the resulting convention will read as: T is *e-contractive*. Further, let us say that d is *subordinated* to e when $d \leq e$ (i.e.: $d(x, y) \leq e(x, y)$, $\forall x, y \in X$). The announced Maia's result is:

Theorem 2.2. *Assume that d is complete, T is d-continuous and e-contractive, and d is subordinated to e. Then, T is a Picard operator (modulo d).*

In particular, when $d = e$, Theorem 2.2 is just the Banach contraction principle [2]. However, its potential is much more spectacular; as certified by

Proposition 2.1. *Under these conventions, we have Theorem 2.2 $\Longrightarrow$ Theorem 2.1; hence (by the above) Maia's fixed point result implies Ran-Reurings'.*

Proof. Let the conditions of Theorem 2.1 hold; and $(\sim)$ stand for the equivalence relation (b01). We introduce a mapping $e : X \times X \to R_+$ as: for each $x, y \in X$,

(b05) $e(x, y) = \inf[d(z_1, z_2) + ... + d(z_{k-1}, z_k)]$,
where $\{z_1, ..., z_k\}$ (for $k \geq 2$) is a <>-chain between x and y.

Clearly, e is reflexive $[e(x, x) = 0, \forall x \in X]$, symmetric $[e(y, y) = e(y, x), \forall x, y \in X]$ and triangular $[e(x, z) \leq e(x, y) + e(y, z), \forall x, y, z \in X]$. In addition, the triangular property of d gives $d(x, y) \leq d(z_1, z_2) + ... + d(z_{k-1}, z_k)$, for any <>-chain $\{z_1, ..., z_k\}$ (where $k \geq 2$) between x and y. So, passing to infimum, yields $d(x, y) \leq e(x, y)$, $\forall x, y \in X$; i.e.: d is subordinated to e. Note that e is sufficient in such a case $[e(x, y) = 0 \Longrightarrow x = y]$; hence, it is a (standard) metric on X. Finally, let $\alpha \in]0, 1[$ be the number appearing in (a02). Given $x, y \in X$, let $\{z_1, ..., z_k\}$ (for $k \geq 2$) be a <>-chain connecting them (existing by (b03)). From (b02), $\{Tz_1, ..., Tz_k\}$ is a <>-chain between Tx and Ty. So, combining with the contractive condition,

$$e(Tx, Ty) \leq \sum_{i=1}^{k-1} d(Tz_i, Tz_{i+1}) \leq \alpha \sum_{i=1}^{k-1} d(z_i, z_{i+1}),$$

for all such <>-chains; wherefrom, passing to infimum, $e(Tx, Ty) \leq \alpha e(x, y)$; i.e., (b04) holds. Summing up, Theorem 2.2 applies to these data; and we are done. □

(B) A different proof of the above statement is available from the following statement in Maia [4, Theorem 2]:

Proposition 2.2. *Suppose that $T : X \to X$ fulfills, for some $\lambda > 1$,*

(b06) $g(x, y) := \sum_{n \geq 0} \lambda^n d(T^n x, T^n y) < \infty$, *for all $x, y \in X$.*

Then **j)** *g is a metric over X,* **jj)** *T is (g, μ)-contractive for $\mu \geq 1/\lambda$ (hence, T is g-contractive),* **jjj)** *$d \leq g$ [i.e.: d is subordinated to g].*

Proof. (**Sketch**) The first and third part are clear. For the second one, it will suffice noting that, by the very definition above,

$$g(x, y) = d(x, y) + \lambda g(Tx, Ty) \geq \lambda g(Tx, Ty), \ \forall x, y \in X.$$

Hence the conclusion. □

Having this established, we may now pass to an alternate

Proof. (**Theorem 2.1**) Let $\alpha \in]0,1[$ be the number appearing in (a02). Given $x, y \in X$, there exists, from (b03), a $<>$-chain $\{z_1, ..., z_k\}$ (where $k \geq 2$) connecting them. By monotonicity, $T^n z_i <> T^n z_{i+1}$, $\forall i \in \{1, ..., k-1\}$, $\forall n$; hence, combining with (a02), $d(T^n z_i, T^n z_{i+1}) \leq \alpha^n d(z_i, z_{i+1})$, for the same ranks (i, n). But then

$$d(T^n x, T^n y) \leq \sum_{i=1}^{k-1} d(T^n z_i, T^n z_{i+1}) \leq M\alpha^n, \quad \forall n;$$

where $M := \sum_{i=1}^{k-1} d(z_i, z_{i+1})$; so that, for each $\lambda \in]1, 1/\alpha[$,

$$\sum_{n\geq 0} \lambda^n d(T^n x, T^n y) \leq M \sum_{n \geq 0} (\lambda\alpha)^n < \infty;$$

i.e., (b06) holds for our data. Taking Proposition 2.2 into account, the result follows, via Theorem 2.2. □

3 Extensions of Maia's result

From these developments, it follows that Maia's result [4, Theorem 1] is an outstanding tool in the area; so, the question of enlarging it is of interest. A positive answer to this – in a pseudometric setting – will be described below.

(A) Let X be a nonempty set. By a *pseudometric* over X we shall mean any map $d : X \times X \to R_+$. If d is reflexive, we call it a *r-pseudometric*; and if d is reflexive triangular, we shall talk about a *rt-pseudometric* (on X). Note that this last object is not a *semimetric* over X; because the symmetry is not accepted here. Given the pseudometric d, let us say that the sequence (x_n) is *d-Cauchy* when [$\forall \varepsilon > 0$, $\exists n(\varepsilon)$: $n(\varepsilon) \leq n \leq m \Longrightarrow d(x_n, x_m) \leq \varepsilon$]. Further, define a d-convergence structure on X by the convention: $x_n \xrightarrow{d} x$ iff $d(x_n, x) \to 0$ as $n \to \infty$; referred to as: x is a d-limit of (x_n). The set of all these will be denoted $\lim_n x_n$; when it is nonempty, we call (x_n), *d-convergent*. Note that, by the lack of symmetry, a relationship like [(for each sequence) d-convergent $\Longrightarrow$ d-Cauchy] is not in general true. However, we say that d is *complete* when each d-Cauchy sequence in X is d-convergent. Call d, *Hausdorff admissible* when, for each sequence (x_n), $\lim_n x_n$ is a singleton whenever it is nonempty. Finally, we say that $T : X \to X$ is *d-continuous* provided $x_n \xrightarrow{d} x$ implies $Tx_n \xrightarrow{d} Tx$.

(B) Now, take a relation ∇ over X; supposed to be *reflexive* [$x\nabla x$, $\forall x \in X$]. Denote, for each $x \in X$: $X(x, \nabla) = \{y \in X; x\nabla y\}$ (the *x-section* of ∇ in X). Let us associate to (∇) another relation $(\preceq)$ over X according to

(c01) $x \preceq y$ iff there exists a ∇-chain $\{z_1, ..., z_k\}$ (for $k \geq 2$) between x and y.

[Here, the last convention means: $z_1 = x$, $z_k = y$, and $z_i \nabla z_{i+1}$, $i \in \{1, ..., k-1\}$]. Clearly, $(\preceq)$ is reflexive and transitive (hence, a quasi-order) on X. Let $\mathcal{F}(R_+)$ stand for the class of all functions $\varphi : R_+ \to R_+$; and $\mathcal{F}_1(R_+)$, the subclass of all $\varphi \in \mathcal{F}(R_+)$ with $\varphi(0) = 0$ and $\varphi(t) < t$, $\forall t > 0$. We say that $\varphi \in \mathcal{F}_1(R_+)$ is a strong comparison function if φ=increasing and $\sum_n \varphi^n(t) < \infty$, for all $t > 0$. (Note that $\varphi \in \mathcal{F}_1(R_+)$ follows from these; cf. Matkowski [5]; but, this is not essential for us). Further, let $e(.,.)$ be a pseudometric over X; and $T : X \to X$ be a selfmap of X. Given $\varphi \in \mathcal{F}_1(R_+)$, we say that T is $(e, \nabla; \varphi)$-*contractive* if

(c02) $e(Tx, Ty) \le \varphi(e(x, y))$, $\forall x, y \in X$, $x\nabla y$;

if this holds for at least one strong comparison function φ, then we say that T is *extended strong* (e, ∇)-*contractive*. Finally, given the pseudometric d on X, let us say that it is ∇-*subordinated* to e, provided $d(x, y) \le e(x, y)$, when $x\nabla y$.

Theorem 3.1. *Let the reflexive relation ∇ be such that*

(c03) $X(T, \nabla) := \{x \in X; x\nabla Tx\}$ *is nonempty*

(c04) T *is ∇-increasing* ($x\nabla y \Longrightarrow Tx\nabla Ty$).

In addition, let the rt-pseudometric d and the r-pseudometric e be such that d is complete and Hausdorff admissible, T is d-continuous and extended strong (e, ∇)-contractive and d is ∇-subordinated to e. Then, for each $x_0 \in X(T, \nabla)$, there exists $x^* \in \mathrm{Fix}(T)$ with $T^n x \xrightarrow{d} x^*$ as $n \to \infty$, for each $x \in X(x_0, \succeq)$.

Proof. Put $x_n = T^n x_0$, $n \in N$; and let $\varphi \in \mathcal{F}_1(R_+)$ be the strong comparison function appearing in the extended strong (e, ∇)-contractive property of T. By (c04) and (c02), $e(x_{n+1}, x_{n+2}) \le \varphi(e(x_n, x_{n+1}))$, for all n. This yields (by the ∇-subordination property of d with respect to e)

$$d(x_n, x_{n+1}) \le e(x_n, x_{n+1}) \le \varphi^n(e(x_0, x_1)), \quad \forall n;$$

wherefrom (by the triangular property of d), (x_n) is a d-Cauchy sequence. As d is complete, $x_n \xrightarrow{d} x^*$ as $n \to \infty$, for some $x^* \in X$. This, along with the d-continuity of T, yields $x_{n+1} = Tx_n \xrightarrow{d} Tx^*$; wherefrom, $x^* \in \mathrm{Fix}(T)$ (as d is Hausdorff admissible). Finally, take some $x \in X(x_0, \succeq)$. By definition, there exists a ∇-chain $\{z_1, ..., z_k\}$ (for some $k \ge 2$) with $z_1 = x$, $z_k = x_0$, $z_i\nabla z_{i+1}$, $i \in \{1, ..., k-1\}$. From (c04) and (c02), this yields, for all such i, and all $n \ge 0$

$$T^n z_i \nabla T^n z_{i+1}, d(T^n z_i, T^n z_{i+1}) \le e(T^n z_i, T^n z_{i+1}) \le \varphi^n(e(z_i, z_{i+1}));$$

wherefrom (again by the triangular property of d) $\lim_n T^n z_1 = ... = \lim_n T^n z_k = \{x^*\}$; and this gives the conclusion we want. □

In particular, when $\nabla = X \times X$ (=the trivial quasi-order on X) and the comparison function φ is linear ($\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in]0, 1[$), the obtained result includes Maia's (Theorem 2.2). In fact, Theorem 3.1 also covers the statement in Nieto and Rodriguez-Lopez [6]; we do not give details. Some related facts may be found in Rus [10, Ch 3, Sect 3.3].

4 Further aspects

As already mentioned, another extension of Theorem 1.1 was provided by O'Regan and Petruşel [7, Theorem 3.3]. To state it, we need some preliminaries.

Let $(X, \leq; d)$ be an ordered metric space; and $T : X \to X$, be a self-map of X. The question to be solved is the one in Section 1. Denote

(d01) $X_{(\leq)} = (\leq) \cup (\geq)$ (i.e.: $(x, y) \in X_{(\leq)}$ iff either $x \leq y$ or $x \geq y$).

Note that $X_{(\leq)}$ is just the graph of the relation $<>$ over X introduced as in (a01); so, it may be identified with this relation. As a consequence, $X_{(\leq)}$ is *reflexive* $[(x, x) \in X_{(\leq)}$, for each $x \in X]$ and *symmetric* $[(x, y) \in X_{(\leq)}$ iff $(y, x) \in X_{(\leq)}]$. However, $X_{(\leq)}$ is not in general transitive, as simple examples show. We say that $\varphi \in \mathcal{F}_1(R_+)$ is a comparison function if φ=increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$. (As before, $\varphi \in \mathcal{F}_1(R_+)$ follows from these; we do not give details). Given $\varphi \in \mathcal{F}_1(R_+)$, let us say that T is $(d, <>; \varphi)$-*contractive* provided

(d02) $d(Tx, Ty) \leq \varphi(d(x, y))$, $\forall x, y \in X$, $x <> y$;

if this holds for at least one comparison function φ, then T is referred to as *extended* $(d, <>)$-*contractive*. Finally, let us say that T is **h)** *orbitally continuous* if: $T^{n(i)}x \to z$ as $i \to \infty$ imply $T^{n(i)+1}x \to Tz$ as $i \to \infty$; and **hh)** *orbitally* $X_{(\leq)}$-*continuous* provided: $T^{n(i)}x \to z$ as $i \to \infty$ and $(T^{n(i)}x, z) \in X_{(\leq)}$, for all $i \geq 0$, imply $T^{n(i)+1}x \to Tz$ as $i \to \infty$; cf. Petruşel and Rus [8].

We are now in position to give the announced result.

Theorem 4.1. *Assume that d is complete, T is extended* (d,<>)*-contractive and* (a03)+(b02) *hold, as well as*

(d03) $X_{(\leq)}$ *is transitive:* $(x, y), (y, z) \in X_{(\leq)} \Rightarrow (x, z) \in X_{(\leq)}$

(d04) $(x, y) \notin X_{(\leq)} \Rightarrow \exists\ c = c(x, y) \in X$*:* $(x, c), (y, c) \in X_{(\leq)}$.

In addition, assume that one of the conditions below is fulfilled:

(d05) *T is orbitally continuous*

(d06) *T is orbitally $X_{(\leq)}$-continuous and there exists $x_0 \in X(T, <>)$ and a subsequence $(T^{n(i)}x_0; i \geq 0)$ of $(T^n x_0; n \geq 0)$ in such a way that $(T^{n(i)}x_0, x^*) \in X_{(\leq)}$, for each i.*

Then, T is a Picard map (modulo d).

This result is claimed to include the 2004 contribution in the area due to Ran and Reurings [9] (i.e., Theorem 1.1 in our notations), obtained under a linear comparison function: $\varphi(t) = \alpha t$, $t \in R_+$, for some $\alpha \in]0, 1[$. However, this is not true; because, (d03)+(d04) give (by the symmetry of $X_{(\leq)}$)

$$X \times X = X_{(\leq)} \text{ (i.e.: the ambient order } (\leq) \text{ is total).}$$

In fact, let $x, y \in X$ be arbitrary fixed. If $(x, y) \in X_{(\leq)}$, we are done; so, assume $(x, y) \notin X_{(\leq)}$. By (d04), there exists $c = c(x, y) \in X$ such that $(x, c) \in X_{(\leq)}$, $(y, c) \in X_{(\leq)}$. This, along with the symmetry of $X_{(\leq)}$, gives $(c, y) \in X_{(\leq)}$; hence, by (d03), $(x, y) \in X_{(\leq)}$; and the claim follows. As a consequence, Theorem 4.1 cannot extend the Ran-Reurings result (based on non-total orderings). In addition, the sentence (d06) is invalid; because x^* is not appearing as a variable for the basic existential ($\exists$) or universal ($\forall$) operators. Perhaps its corrected form is the one appearing in [7, Theorem 3.10]; namely

(d07) T is orbitally $X_{(\leq)}$-continuous and there exists $x_0 \in X(T, <>)$
and a rank sequence $(n(i); i \geq 0)$ in such a way that
$(\forall x^*)$: $[T^{n(i)}x_0 \to x^*] \Longrightarrow (T^{n(i)}x_0, x^*) \in X_{(\leq)}$, for all $i \geq 0$;

we shall discuss these facts elsewhere. Further aspects may be found in Agarwal, El-Gebeily and O'Regan [1].

References

[1] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces,* Appl. Anal., 87 (2008), 109-116.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales.* Fund. Math., 3 (1922), pp. 133-181.

[3] J. Chandra and B. A. Fleishman, *On a generalization of the Gronwall-Bellman lemma in partially ordered Banach spaces,* J. Math. Anal. Appl., 31 (1970), 668-681.

[4] M. G. Maia, *Un'osservazione sulle contrazioni metriche,* Rend. Sem. Mat. Univ. Padova, 40 (1968), 139-143.

[5] J. Matkowski, *Fixed point theorems for mappings with a contractive iterate at a point,* Proc. Amer. Math. Soc., 62 (1977), 344-348.

[6] J.J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,* Order, 22 (2005), 223-239.

[7] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces,* J. Math. Anal. Appl., 341 (2008), 1241-1252.

[8] A. Petruşel and I. A. Rus, *Fixed point theorems in ordered L-spaces,* Proc. Amer. Math. Soc., 134 (2006), 411-418.

[9] A. C. M. Ran and M. C. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proc. Amer. Math. Soc., 132 (2004), 1435-1443.

[10] I. A. Rus, *Generalized Contractions and Applications* Cluj University Press, Cluj-Napoca, 2001.

[11] M. Turinici, *Fixed points for monotone iteratively local contractions,* Dem. Math., 19 (1986), 171-180.