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On the Original Dirac Equations with Radiation Term

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Abstract. The well known Lorentz-Dirac equations

$$m\ddot{x}_r = \frac{e}{c}F_{rn}\dot{x}_n + \frac{2}{3}\frac{e^2}{c^3}\left(\dddot{x}_r - \frac{1}{c^2}\ddot{x}_n\ddot{x}_n\dot{x}_r\right) \quad (r = 1, 2, 3, 4)$$

generate difficulties as an appearance of a third derivative, an existence of run-away solutions, a violation of causality over short time intervals. We overcome these difficulties following the original P. A. M. Dirac's physical assumptions. Unlike Dirac we replace his local expansions with nonlocal formulation of the problem. So we obtain instead of the above equations a system of first order neutral functional differential equations of motion with respect to the unknown velocities containing both retarded and advanced arguments. We show also that the fourth equation is a consequence of the first three ones, that is, Dirac system is not overdeterminated. This simplifies calculations in the various applications.

1 Introduction

In a previous paper [1] we have obtained a new form of Lorentz radiation term in one-dimensional case. As a consequence we have shown that Ehrenfest paradox does not exist provided to replace the known Lorentz radiation term [2] by the newly derived one [1]. In the relativistic case the usually accepted radiation term leads to the well known Dirac (or Lorentz-Dirac) equations [3].

$$m\ddot{x}_r = \frac{e}{c}F_{rn}\dot{x}_n + \frac{2}{3}\frac{e^2}{c^3}\left(\dddot{x}_r - \frac{1}{c^2}\ddot{x}_n\ddot{x}_n\dot{x}_r\right) \quad (r = 1, 2, 3, 4) \tag{1}$$

where $x_1(t), x_2(t), x_3(t), x_4(t) = ict$ are the coordinates of the electron, e is its charge, m - its rest mass, c - the speed of light, the dot is a differentiation with respect to arc length, i.e.

$$\dot{x}_n = \frac{dx_n}{ds} = \frac{cdx_n}{\sqrt{c^2 - u^2}dt}$$

and Einstein summation convention is valid. The second term in (1) is the Abraham four - vector of radiation reaction derived also by Dirac [3].

The profound analysis of the above system of equations due to F. Rohrlich [4] put on view the following difficulties: 1) the appearance of a third derivative of positions does not permit a determination of the solutions of these equations in terms of the Newtonian initial conditions of positions and velocities; 2) an existence of runaway solutions; 3) a violation of causality over short time intervals, or an acceleration occurs prior to the applied force.

We would like to point out that here we overcome the above mentioned difficulties following the original physical assumptions due to P.A.M.Dirac [3]. He assumes the delayed and advanced argument to be a small parameter. Unlike Dirac we replace his local expansions with nonlocal formulation of the problem. So we obtain instead of (1) a system of first order neutral functional differential equations of motion with respect to unknown velocity containing both retarded and advanced arguments. (cf. [5]). We propose slightly more different formulation of Dirac's incoming and outgoing fields by comparison with [1] relaxing the conditions on the incoming and outgoing field functions.

In view of the numerous applications of the Dirac equations much work has been devoted to this subject [6]-[29]. Many authors suggest various solutions. Nevertheless V.Ginsburg [24] notices that the problem in question is not clear. In [28] A.Yaghjian considers the Lorentz model of the electron as a charged insulator. He modifies the self electromagnetic force during the short time interval after the external force is applied. The resulting modification of the equations of motion eliminates the noncausal pre-acceleration. In contrast to [28] we consider the electron as a mass-point charge - an assumption due to Dirac. We study also an existence problem for Dirac equations coming from their original form [1], using incoming (retarded) and outgoing (advanced) fields as they has been introduced by Dirac in [1]. The derivation of the new form of the radiation term is based on the previous results [31], [32] using relativistic form of the retarded and advanced Lienard-Wiechert potentials [6]. We stand on the theory of functional differential equations [5], [31], [32] of neutral type with both retarded and advanced arguments caused by the finite propagation of the interaction - the basic assumption of Einstein relativity theory. So Dirac equations become second order neutral equations. As a consequence one can show as in [1] that do not exist runaway solutions. We have also proved that the fourth equation is a consequence of the first three ones. The proof is based on the property of the electromagnetic tensor and the new form of the radiation term. The existence-uniqueness theorem for the equations of motion is proved by means of fixed point theory [34]. Finally we show that Dirac combination of retarded and advanced fields really leads to a "renormalization of mass" (cf. Section IX).

Section II is devoted the strict mathematical formulations of the original Dirac assumptions.

In Section III we show that the fourth equation is a consequence of the first three ones. As far as we know such an assertion is not yet proved anywhere (cf. [6]-[27]). So we consider just 3 equations for 3 unknown functions. This simplifies calculations in the applications.

In Section IV we obtain estimates of the functions $\tau^{ret}(t)$ and $\tau^{adv}(t)$ by the distances between a moving electron and incoming and outgoing fields. In order to apply fixed point theorems from [34] we make in Section V estimates of the right-hand sides of the newly derived Dirac equations.

In the same way in Section VI we obtain Lipschitz estimates of the right-hand sides of the equations of motion.

In Section VII we recall some tools from the fixed point theory [34].

In Section VIII using a suitable fixed point theorem an existence-uniqueness theorem for Dirac equations is proved.

It is shown in Section IX that runaway solutions do not exist.

The theorem proved in Section X implies the finiteness of the energy.

In conclusion remark an estimate of the maximal velocities is made which outlines the frame of applicability of the theory.

2 Derivation of Dirac equations using retarded and advanced potentials

Our goal here is to derive a new form of the radiation term based on the original physical reasoning's due to P.A.Dirac [3]. We follow the approach from [1] but we specify our reasoning's and use the denotations from [30]-[32].

Consider a charge e describing any curve L in space-time. Let $A(x_1(t), x_2(t), x_3(t), ict)$ be any event and let $A^{ret}(\check{x}_1(t), \check{x}_2(t), \check{x}_3(t), ict)$ be the intersection of L with the null-cone drawn into the past from A, and let $A^{adv}(\hat{x}_1(t), \hat{x}_2(t), \hat{x}_3(t), ict)$ be the intersection of L with the null-cone drawn into the future from A where $\check{t} < t$ and $t < \hat{t}$.

Let

$$\check{\lambda} = (\check{\lambda}_1, \check{\lambda}_2, \check{\lambda}_3, \check{\lambda}_4) = \left(\frac{\check{u}_1(\check{t})}{\check{\Delta}}, \frac{\check{u}_2(\check{t})}{\check{\Delta}}, \frac{\check{u}_3(\check{t})}{\check{\Delta}}, \frac{ic}{\check{\Delta}}\right),$$

where $\check{\Delta} = \sqrt{c^2 - \sum_{\gamma=1}^3 \check{u}_\gamma^2(\check{t})}$ be the unit tangent vector to L at A^{ret} and let

$$\xi^{ret} = (\xi_1^{ret}, \xi_2^{ret}, \xi_3^{ret}, \xi_4^{ret}) = (x_1(t) - \check{x}_1(\check{t}), x_2(t) - \check{x}_2(\check{t}), x_3(t) - \check{x}_3(\check{t}), ic(t - \check{t})),\ \check{t} < t$$

be the isotropic vector $A^{ret}A$.

Analogously, let

$$\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4) = \left(\frac{\hat{u}_1(\hat{t})}{\hat{\Delta}}, \frac{\hat{u}_2(\hat{t})}{\hat{\Delta}}, \frac{\hat{u}_3(\hat{t})}{\hat{\Delta}}, \frac{ic}{\hat{\Delta}}\right),$$

where $\hat{\Delta} = \sqrt{c^2 - \sum_{\gamma=1}^3 \hat{u}_\gamma^2(\hat{t})}$ be the unit tangent vector to L at A^{adv} and

$$\xi^{adv} = (\xi_1^{adv}, \xi_2^{adv}, \xi_3^{adv}, \xi_4^{adv}) = (\hat{x}_1(\hat{t}) - x_1(t), \hat{x}_2(\hat{t}) - x_2(t), \hat{x}_3(\hat{t}) - x_3(t), ic(\hat{t} - t)),\ \hat{t} > t$$

be the isotropic vector AA^{adv}.

The unit tangent vector to L at A is

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{u_1(t)}{\Delta}, \frac{u_1(t)}{\Delta}, \frac{u_1(t)}{\Delta}, \frac{ic}{\Delta}\right), \quad \text{where } \Delta = \sqrt{c^2 - \sum_{\gamma=1}^3 u_\gamma^2(t)}.$$

In accordance with Dirac assumptions the radiation term is defined as a half of the difference between both retarded and advanced potentials, that is,

$$F_{kn}^{rad} = \frac{1}{2}\left[\left(\frac{\partial A_n^{ret}}{\partial \check{x}_k} - \frac{\partial A_k^{ret}}{\partial \check{x}_n}\right) - \left(\frac{\partial A_n^{adv}}{\partial \hat{x}_k} - \frac{\partial A_k^{adv}}{\partial \hat{x}_n}\right)\right],$$

where

$$A_n^{ret} = -\frac{e\check{\lambda}_n}{\left\langle \check{\lambda}, \xi^{ret} \right\rangle}, \quad A_n^{adv} = -\frac{e\hat{\lambda}_n}{\left\langle \hat{\lambda}, \xi^{adv} \right\rangle}$$

are Lienard-Wiechert retarded and advanced potentials [6]. So that Dirac physical assumptions lead to the following form of equations (instead of (1)):

$$m\frac{d\lambda_k}{ds} = \frac{e}{c^2}(F_{kn}\lambda_n + F_{kn}^{rad})$$

$$m\frac{d\lambda_k}{ds} = \frac{e}{c^2}\left[F_{kn}\lambda_n + \frac{1}{2}\left(\frac{\partial A_n^{ret}}{\partial \check{x}_k} - \frac{\partial A_k^{ret}}{\partial \check{x}_n}\right)\lambda_n - \frac{1}{2}\left(\frac{\partial A_n^{adv}}{\partial \hat{x}_k} - \frac{\partial A_k^{adv}}{\partial \hat{x}_n}\right)\lambda_n\right] \ (k = 1, 2, 3, 4)$$

or

$$\frac{d\lambda_k}{ds} = \frac{e}{mc^2}F_{kn}\lambda_n + \frac{e}{mc^2}\left[\left(\frac{\partial A_n^{ret}}{\partial \check{x}_k} - \frac{\partial A_k^{ret}}{\partial \check{x}_n}\right)\lambda_n - \left(\frac{\partial A_n^{adv}}{\partial \hat{x}_k} - \frac{\partial A_k^{adv}}{\partial \hat{x}_n}\right)\lambda_n\right] \ (k = 1, 2, 3, 4).$$

Further on we assume (cf. [31], [32]) that

(AR): $t - \check{t} = \tau^{ret}(t), \hat{t} - t = \tau^{adv}(t)$

In fact postulating **(AR)** we extend the relation between the relativistic and absolute time from [33].

Remark 1. Recall Dirac assumptions from [3]:
1) $\tau^{ret}(t) = \tau^{adv}(t) = \tau > 0$;
2) τ is a small parameter.

Dirac derives the radiation term expanding in $\tau > 0$ using Taylor formula. It is known, however, from the theory of functional differential equations with small delays that when $\tau \to 0$ the solution of functional differential equation could not always tend to the solution of corresponding ordinary differential equation.

Since A^{ret} and A^{adv} lie on the trajectory L we obtain

$$\begin{aligned}
\check{x}(t) &= \left(\check{x}_1(\check{t}), \check{x}_2(\check{t}), \check{x}_3(\check{t}), \check{x}_4(\check{t}) = ic\check{t}\right) = \left(x_1(\check{t}), x_2(\check{t}), x_3(\check{t}), ic\check{t}\right) = \\
&= \left(x_1(t - \tau^{ret}(t)), x_2(t - \tau^{ret}(t)), x_3(t - \tau^{ret}(t)), ic(t - \tau^{ret}(t))\right) \\
\hat{x}(t) &= \left(\hat{x}_1(\hat{t}), \hat{x}_2(\hat{t}), \hat{x}_3(\hat{t}), \hat{x}_4(\hat{t}) = ic\hat{t}\right) = \left(x_1(\hat{t}), x_2(\hat{t}), x_3(\hat{t})ic\hat{t}\right) = \\
&= \left(x_1(t + \tau^{adv}(t)), x_2(t + \tau^{adv}(t)), x_3(t + \tau^{adv}(t)), ic(t + \tau^{adv}(t))\right)
\end{aligned}$$

and

$$\check{\lambda} = \left(\frac{u_1(t - \tau^{ret})}{\check{\Delta}}, \frac{u_2(t - \tau^{ret})}{\check{\Delta}}, \frac{u_3(t - \tau^{ret})}{\check{\Delta}}, \frac{ic}{\check{\Delta}}\right), \quad \text{where } \check{\Delta} = \sqrt{c^2 - \sum_{\gamma=1}^{3} u_\gamma^2(t - \tau^{ret})},$$

$$\hat{\lambda} = \left(\frac{u_1(t + \tau^{adv})}{\hat{\Delta}}, \frac{u_2(t + \tau^{adv})}{\hat{\Delta}}, \frac{u_3(t + \tau^{adv})}{\hat{\Delta}}, \frac{ic}{\hat{\Delta}}\right), \quad \text{where } \hat{\Delta} = \sqrt{c^2 - \sum_{\gamma=1}^{3} u_\gamma^2(t + \tau^{adv})}.$$

Therefore

$$\xi^{ret} = \left(x_1(t) - x_1(t-\tau^{ret}(t)), x_2(t) - x_2(t-\tau^{ret}(t)), x_3(t) - x_3(t-\tau^{ret}(t)), ic\tau^{ret}\right)$$

and

$$\xi^{adv} = \left(x_1(t+\tau^{adv}(t)) - x_1(t), x_2(t+\tau^{adv}(t)) - x_2(t), x_3(t+\tau^{adv}(t)) - x_3(t), ic\tau^{adv}\right).$$

The functions $\tau^{ret}(t)$, $\tau^{adv}(t)$ can be obtained as solutions of the functional equations

$$\left\langle \xi^{ret}, \xi^{ret} \right\rangle_4 = 0, \quad \left\langle \xi^{adv}, \xi^{adv} \right\rangle_4 = 0$$

where $\langle \cdot, \cdot \rangle_4$ is the scalar product in the Minkowski space. Let us recall that $\xi^{ret}(t)$, $\xi^{adv}(t)$ are isotropic vectors drawn into the past and into the future respectively.

Solving the last equations with respect to $\tau^{ret}(t)$ and $\tau^{adv}(t)$ we obtain:

$$\tau^{ret}(t) = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - x_\gamma(t-\tau^{ret}(t))\right]^2}, \quad \tau^{adv}(t) = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t+\tau^{adv}(t)) - x_\gamma(t)\right]^2} \tag{2}$$

We would like to point out once again that we follow strictly physical reasoning's due to Dirac [3] and following [31], [32] we replace the usually accepted Dirac equations obtain nonlocal system of neutral functional differential equations with both retarded and advanced arguments.

In what follows we use the calculations from [31], [32] and we obtain

$$\begin{aligned}
\frac{d\lambda_k}{ds} = & \frac{e}{mc^2}F_{kn}\lambda_n + \\
& + \frac{e^2}{2mc^2}\left[\frac{\xi_k^{ret}\left\langle\lambda,\check{\lambda}\right\rangle_4 - \check{\lambda}_k\left\langle\xi^{ret},\lambda\right\rangle_4}{\left\langle\check{\lambda},\xi^{ret}\right\rangle_4^3}\left(1+\left\langle\xi^{ret},\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\right) + \right.\\
& \left. + \frac{\left\langle\xi^{ret},\lambda\right\rangle_4\frac{d\check{\lambda}_k}{ds_{ret}} - \left\langle\lambda,\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\xi_k^{ret}}{\left\langle\check{\lambda},\xi^{ret}\right\rangle_4^2}\right] - \\
& - \frac{e^2}{2mc^2}\left[\frac{\xi_k^{adv}\left\langle\lambda,\hat{\lambda}\right\rangle_4 - \hat{\lambda}_k\left\langle\xi^{adv},\lambda\right\rangle_4}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle_4^3}\left(1+\left\langle\xi^{adv},\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\right) + \right.\\
& \left. + \frac{\left\langle\xi^{adv},\lambda\right\rangle_4\frac{d\hat{\lambda}_k}{ds_{adv}} - \left\langle\lambda,\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\xi_k^{adv}}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle_4^2}\right] \qquad (k=1,2,3,4).
\end{aligned} \tag{3}$$

It is known ([6],[12]-[14],[26],[27]) that the elements of the electromagnetic tensor are:

$$\begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} = \begin{pmatrix} 0 & H_3 & H_2 & iE_1 \\ -H_3 & 0 & H_1 & iE_2 \\ -H_2 & -H_1 & 0 & iE_3 \\ -iE_1 & -iE_2 & -iE_3 & 0 \end{pmatrix},$$

where $\vec{E}(E_1, E_2, E_3)$ is the electric field intensity vector and $\vec{H}(H_1, H_2, H_3)$ is the magnetic field intensity vector. It is easy to see that $F_{kn} = -F_{nk}$.

The above system (3) can be split into "space-like" and "time-like" parts:

$$\begin{aligned}\frac{d\lambda_\alpha}{ds} &= \frac{e}{mc^2}F_{\alpha n}\lambda_n + \\ &+ \frac{e^2}{2mc^2}\left[\frac{\xi^{ret}_\alpha\left\langle\check{\lambda},\lambda\right\rangle_4 - \check{\lambda}_\alpha\left\langle\xi^{ret},\lambda\right\rangle_4}{\left\langle\check{\lambda},\xi^{ret}\right\rangle^3_4}\left(1+\left\langle\xi^{ret},\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\right)+\right. \\ &\qquad\left.+\frac{\left\langle\xi^{ret},\lambda\right\rangle_4\frac{d\check{\lambda}_\alpha}{ds_{ret}} - \left\langle\lambda,\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\xi^{ret}_\alpha}{\left\langle\check{\lambda},\xi^{ret}\right\rangle^2_4}\right] - \\ &- \frac{e^2}{2mc^2}\left[\frac{\xi^{adv}_\alpha\left\langle\hat{\lambda},\lambda\right\rangle_4 - \hat{\lambda}_\alpha\left\langle\xi^{adv},\lambda\right\rangle_4}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle^3_4}\left(1+\left\langle\xi^{adv},\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\right)+\right. \\ &\qquad\left.+\frac{\left\langle\xi^{adv},\lambda\right\rangle_4\frac{d\hat{\lambda}_\alpha}{ds_{adv}} - \left\langle\lambda,\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\xi^{adv}_\alpha}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle^2_4}\right] \qquad (\alpha = 1,2,3).\end{aligned}$$

$$\begin{aligned}\frac{d\lambda_4}{ds} &= \frac{e}{mc^2}F_{4n}\lambda_n + \\ &+ \frac{e^2}{2mc^2}\left[\frac{\xi^{ret}_4\left\langle\check{\lambda},\lambda\right\rangle - \check{\lambda}_4\left\langle\xi^{ret},\lambda\right\rangle}{\left\langle\check{\lambda},\xi^{ret}\right\rangle^3}\left(1+\left\langle\xi^{ret},\frac{d\check{\lambda}}{ds_{ret}}\right\rangle\right)+\right. \\ &\qquad\left.+\frac{\left\langle\xi^{ret},\lambda\right\rangle\frac{d\check{\lambda}_4}{ds_{ret}} - \left\langle\lambda,\frac{d\check{\lambda}}{ds_{ret}}\right\rangle\xi^{ret}_4}{\left\langle\check{\lambda},\xi^{ret}\right\rangle^2}\right] - \\ &- \frac{e^2}{2mc^2}\left[\frac{\xi^{adv}_4\left\langle\hat{\lambda},\lambda\right\rangle - \hat{\lambda}_4\left\langle\xi^{adv},\lambda\right\rangle}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle^3}\left(1+\left\langle\xi^{adv},\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle\right)+\right. \\ &\qquad\left.+\frac{\left\langle\xi^{adv},\lambda\right\rangle\frac{d\hat{\lambda}_4}{ds_{adv}} - \left\langle\lambda,\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle\xi^{adv}_4}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle^2}\right].\end{aligned}$$

We use the denotation $\langle\cdot,\cdot\rangle$ for 3-dimensional scalar product in Euclidean subspace. Following [31], [32] we have to find the relations between the derivatives at past, present and future instants. Indeed, differentiating the relations $t - \check{t} = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - \check{x}_\gamma(\check{t})\right]^2}$

with respect to $\check{t}$ (considering $t = t(\check{t})$) we obtain

$$\frac{dt}{d\check{t}} - 1 = \frac{\sum\limits_{\gamma=1}^{3}\left[x_\gamma(t) - \check{x}_\gamma(\check{t})\right]\left[u_\gamma(t)\dfrac{dt}{d\check{t}} - \check{u}_\gamma(\check{t})\right]}{c\sqrt{\sum\limits_{\gamma=1}^{3}\left[x_\gamma(t) - \check{x}_\gamma(\check{t})\right]^2}}$$

and hence

$$\check{D} \equiv \frac{dt}{d\check{t}} = \frac{c\sqrt{\langle\xi^{ret}, \xi^{ret}\rangle} - \langle\xi^{ret}, \check{u}\rangle}{c\sqrt{\langle\xi^{ret}, \xi^{ret}\rangle} - \langle\xi^{ret}, u\rangle} = \frac{c^2\tau^{ret} - \langle\xi^{ret}, u(t - \tau^{ret})\rangle}{c^2\tau^{ret} - \langle\xi^{ret}, u(t)\rangle}$$

Analogously differentiating $\hat{t} - t = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[\hat{x}_\gamma(\hat{t}) - x_\gamma(t)\right]^2}$ with respect to $\hat{t}$ (considering $t = t(\hat{t})$) we obtain

$$1 - \frac{dt}{d\hat{t}} = \frac{\sum\limits_{\gamma=1}^{3}\left[\hat{x}_\gamma(\hat{t}) - x_\gamma(t)\right]\left[\hat{u}_\gamma(\hat{t}) - u_\gamma(t)\frac{dt}{d\hat{t}}\right]}{c\sqrt{\sum\limits_{\gamma=1}^{3}\left[\hat{x}_\gamma(\hat{t}) - x_\gamma(t)\right]^2}}$$

and hence

$$\hat{D} \equiv \frac{dt}{d\hat{t}} = \frac{c\sqrt{\langle\xi^{adv}, \xi^{adv}\rangle} - \left\langle\xi^{adv}, \hat{u}\right\rangle}{c\sqrt{\langle\xi^{adv}, \xi^{adv}\rangle} - \langle\xi^{adv}, u\rangle} = \frac{c^2\tau^{adv} - \left\langle\xi^{adv}, u(t + \tau^{adv})\right\rangle}{c^2\tau^{adv} - \langle\xi^{adv}, u(t)\rangle}.$$

Further on we have

$$\left\langle\lambda, \check{\lambda}\right\rangle_4 = \frac{\langle u(t), u(t - \tau^{ret}\rangle - c^2}{\Delta\check{\Delta}}; \qquad \left\langle\lambda, \hat{\lambda}\right\rangle_4 = \frac{\left\langle u(t), u(t + \tau^{adv}\right\rangle - c^2}{\Delta\hat{\Delta}};$$

$$\left\langle\xi^{ret}, \lambda\right\rangle_4 = \frac{\langle\xi^{ret}, u(t)\rangle - c^2\tau^{ret}}{\Delta}; \qquad \left\langle\xi^{adv}, \lambda\right\rangle_4 = \frac{\left\langle\xi^{adv}, u(t)\right\rangle - c^2\tau^{adv}}{\Delta};$$

$$\left\langle\check{\lambda}, \xi^{ret}\right\rangle_4 = \frac{\langle\xi^{ret}, u(t - \tau^{ret})\rangle - c^2\tau^{ret}}{\check{\Delta}}; \qquad \left\langle\hat{\lambda}, \xi^{adv}\right\rangle_4 = \frac{\left\langle\xi^{adv}, u(t + \tau^{adv})\right\rangle - c^2\tau^{adv}}{\hat{\Delta}};$$

$$\frac{d\lambda_\alpha}{ds} = \frac{c}{\Delta}\frac{d}{dt}\left(\frac{u_\alpha(t)}{\Delta}\right) = \frac{c}{\Delta^2}\dot{u}_\alpha(t) + \frac{c\,\langle u(t), \dot{u}(t)\rangle}{\Delta^4}u_\alpha(t), \quad (\alpha = 1, 2, 3);$$

$$\frac{d\lambda_4}{ds} = \frac{ic^2}{\Delta}\frac{d}{dt}\left(\frac{1}{\Delta}\right) = \frac{ic^2}{\Delta^4}\langle u(t), \dot{u}(t)\rangle\,; \quad \left(\dot{u}_\alpha(t) \equiv \frac{du_\alpha(t)}{dt}\right);$$

$$\frac{d}{ds_{ret}} = \frac{1}{\check{\Delta}}\frac{d}{d\check{t}} = \frac{1}{\check{\Delta}}\frac{dt}{d\check{t}}\frac{d}{dt} = \frac{1}{\check{\Delta}}\check{D}\frac{1}{dt}, \quad \frac{d}{ds_{adv}} = \frac{1}{\hat{\Delta}}\frac{d}{d\hat{t}} = \frac{1}{\hat{\Delta}}\frac{dt}{d\hat{t}}\frac{d}{dt} = \frac{1}{\hat{\Delta}}\hat{D}\frac{1}{dt}$$

where $\check{D}=\dfrac{dt}{d\check{t}}$, $\quad \hat{D}=\dfrac{dt}{d\hat{t}}$.

$$\frac{d\check{\lambda}_\alpha}{ds_{ret}}=\check{D}\left[\frac{c\dot{u}_\alpha(t-\tau^{ret})}{\check{\Delta}^2}+\frac{cu_\alpha(t-\tau^{ret})\left\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\right\rangle}{\check{\Delta}^4}\right](\alpha=1,2,3),$$

$$\frac{d\check{\lambda}_4}{ds_{ret}}=\frac{ic^2\check{D}\left\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\right\rangle}{\check{\Delta}^4};$$

$$\frac{d\hat{\lambda}_\alpha}{ds_{adv}}=\hat{D}\left[\frac{c\dot{u}_\alpha(t+\tau^{adv})}{\hat{\Delta}^2}+\frac{cu_\alpha(t+\tau^{adv})\left\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^4}\right](\alpha=1,2,3),$$

$$\frac{d\hat{\lambda}_4}{ds_{adv}}=\frac{ic^2\hat{D}\left\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^4};$$

$$\left\langle\xi^{ret},\frac{d\check{\lambda}_\alpha}{ds_{ret}}\right\rangle_4=c\check{D}\left[\frac{\left\langle\xi^{ret},\dot{u}(t-\tau^{ret})\right\rangle}{\check{\Delta}^2}+\frac{\left(\left\langle\xi^{ret},u(t-\tau^{ret})\right\rangle-c^2\tau^{ret}\right)\left\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\right\rangle}{\check{\Delta}^4}\right]$$

$$\left\langle\lambda,\frac{d\check{\lambda}_\alpha}{ds_{ret}}\right\rangle_4=\frac{c\check{D}}{\Delta}\left[\frac{\left\langle u(t),\dot{u}(t-\tau^{ret})\right\rangle}{\check{\Delta}^2}+\frac{\left(\left\langle u(t),u(t-\tau^{ret})\right\rangle-c^2\tau^{ret}\right)\left\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\right\rangle}{\check{\Delta}^4}\right]$$

$$\left\langle\xi^{adv},\frac{d\hat{\lambda}_\alpha}{ds_{adv}}\right\rangle_4=c\hat{D}\left[\frac{\left\langle\xi^{adv},\dot{u}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^2}+\frac{\left(\left\langle\xi^{adv},u(t+\tau^{adv})\right\rangle-c^2\tau^{adv}\right)\left\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^4}\right]$$

$$\left\langle\lambda,\frac{d\hat{\lambda}_\alpha}{ds_{adv}}\right\rangle_4=\frac{c\hat{D}}{\Delta}\left[\frac{\left\langle u(t),\dot{u}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^2}+\frac{\left(\left\langle u(t),u(t+\tau^{adv})\right\rangle-c^2\tau^{adv}\right)\left\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\right\rangle}{\hat{\Delta}^4}\right]$$

Therefore the above system becomes

$$\frac{c}{\Delta^2}\frac{du_\alpha(t)}{dt}+\frac{c}{\Delta^4}\left\langle u(t),\frac{du(t)}{dt}\right\rangle u_\alpha(t)=\frac{e^2}{mc^2}\frac{1}{\Delta}\left[\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta(t)-E_\alpha\right]+$$

$$+\frac{e^2}{2mc^2}\left[\frac{\xi_\alpha^{ret}\left\langle\check{\lambda},\lambda\right\rangle_4-(\check{u}_\alpha/\check{\Delta})\left\langle\xi^{ret},\lambda\right\rangle_4}{\left\langle\check{\lambda},\xi^{ret}\right\rangle_4^3}\left(1+\left\langle\xi^{ret},\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\right)+\right.$$

$$\left.+\frac{\dfrac{\check{D}}{\check{\Delta}^2}\left(\dfrac{du_\alpha(t-\tau^{ret})}{dt}+\dfrac{u_\alpha(t-\tau^{ret})}{\check{\Delta}^2}\left\langle u(t-\tau^{ret}),\dfrac{du(t-\tau^{ret})}{dt}\right\rangle\right)\left\langle\xi^{ret},\lambda\right\rangle_4-\left\langle\lambda,\dfrac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\xi_\alpha^{ret}}{\left\langle\check{\lambda},\xi^{ret}\right\rangle_4^2}\right]-$$

$$-\frac{e^2}{2mc^2}\left[\frac{\xi_\alpha^{adv}\left\langle\hat{\lambda},\lambda\right\rangle_4-(\hat{u}_\alpha/\hat{\Delta})\left\langle\xi^{adv},\lambda\right\rangle_4}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle_4^3}\left(1+\left\langle\xi^{adv},\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\right)+\right.$$

$$\left.+\frac{\dfrac{\hat{D}}{\hat{\Delta}^2}\left(\dfrac{du_\alpha(t+\tau^{adv})}{dt}+\dfrac{u_\alpha(t+\tau^{adv})}{\hat{\Delta}^2}\left\langle u(t+\tau^{ret}),\dfrac{du(t+\tau^{adv})}{dt}\right\rangle\right)\left\langle\xi^{adv},\lambda\right\rangle_4-\left\langle\lambda,\dfrac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\xi_\alpha^{adv}}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle_4^2}\right],$$

$$(\alpha = 1, 2, 3).$$

$$\frac{ic^2}{\Delta^4}\left\langle u(t), \frac{du(t)}{dt}\right\rangle = \frac{e^2}{mc^2}\left(-ic\frac{\langle E, u(t)\rangle}{\Delta}\right)+$$

$$+\frac{e^2}{2mc^2}\left[\frac{ic\tau^{ret}\left\langle\check{\lambda},\lambda\right\rangle_4 - (ic/\check{\Delta})\left\langle\xi^{ret},\lambda\right\rangle_4}{\left\langle\check{\lambda},\xi^{ret}\right\rangle_4^3}\left(1+\left\langle\xi^{ret},\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4\right)+\right.$$

$$\left.+\frac{ic\frac{\check{D}}{\check{\Delta}^4}\left\langle u(t-\tau^{ret}), \frac{du(t-\tau^{ret})}{dt}\right\rangle\left\langle\xi^{ret},\lambda\right\rangle_4 - ic\tau^{ret}\left\langle\lambda,\frac{d\check{\lambda}}{ds_{ret}}\right\rangle_4}{\left\langle\check{\lambda},\xi^{ret}\right\rangle_4^2}\right]-$$

$$-\frac{e^2}{2mc^2}\left[\frac{ic\tau^{adv}\left\langle\hat{\lambda},\lambda\right\rangle_4 - (ic/\hat{\Delta})\left\langle\xi^{adv},\lambda\right\rangle_4}{\left\langle\hat{\lambda},\xi^{adv}\right\rangle_4^3}\left(1+\left\langle\xi^{adv},\frac{d\hat{\lambda}}{ds_{adv}}\right\rangle_4\right)+\right.$$

$$\left.+\frac{ic\frac{\hat{D}}{\hat{\Delta}^4}\left\langle u(t+\tau^{adv}), \frac{du(t+\tau^{adv})}{dt}\right\rangle\left\langle\xi^{adv},\lambda\right\rangle_4 - ic\tau^{adv}\left\langle\lambda,\frac{d\check{\lambda}}{ds_{adv}}\right\rangle_4}{\left\langle\check{\lambda},\xi^{adv}\right\rangle_4^2}\right].$$

The last equation should be divided by ic.

3 The fourth equation is a consequence of the first three ones

After obvious transformations we obtain the following systems of four equations

$$\dot{u}_\alpha(t) + \frac{\langle u(t), \dot{u}(t)\rangle}{\Delta^2}u_\alpha = \frac{e^2\Delta}{mc^3}\left\{\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta(t) - E_\alpha+\right.$$

$$+\frac{\check{H}}{2}\cdot\frac{\xi_\alpha^{ret}\left(\langle u(t), u(t-\tau^{ret}\rangle - c^2\right) - u_\alpha(t-\tau^{ret})\left(\langle\xi^{ret}, u(t)\rangle - c^2\tau^{ret}\right)}{\left[\langle\xi^{ret}, u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^3}+$$

$$+\frac{\check{D}}{2}\left[\frac{\left(\langle\xi^{ret}, u(t)\rangle - c^2\tau^{ret}\right)\left(\dot{u}_\alpha(t-\tau^{ret}) + u_\alpha(t-\tau^{ret})\frac{\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2}\right)}{\left[\langle\xi^{ret}, u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^2}-\right.$$

$$\left.-\frac{\xi_\alpha^{ret}\frac{\langle u(t), \dot{u}(t-\tau^{ret})\rangle\check{\Delta}^2 + \left(\langle u(t), u(t-\tau^{ret})\rangle - c^2\right)\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2}}{\left[\langle\xi^{ret}, u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^2}\right]-$$

$$-\frac{\hat{H}}{2}\cdot\frac{\xi_\alpha^{adv}\left(\langle u(t), u(t-\tau^{adv}\rangle - c^2\right) - u_\alpha(t-\tau^{adv})\left(\langle\xi^{adv}, u(t)\rangle - c^2\tau^{adv}\right)}{\left[\langle\xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^3}-$$

$$-\frac{\hat{D}}{2}\left[\frac{\left(\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}\right)\left(\dot{u}_\alpha(t+\tau^{adv})+u_\alpha(t+\tau^{adv})\frac{\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2}\right)}{\left[\langle\xi^{adv},u(t-\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}-\right.$$

$$\left.\left.-\frac{\xi_\alpha^{adv}\frac{\langle u(t),\dot{u}(t+\tau^{adv})\rangle\hat{\Delta}^2+\left(\langle u(t),u(t+\tau^{adv})\rangle-c^2\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2}}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}\right]\right\} \quad (4.\alpha)$$

$$\frac{\langle u(t),\dot{u}(t)\rangle}{\Delta^2}=-\frac{e^2\Delta}{mc^3}\langle E,u(t)\rangle+\frac{e^2\Delta}{2mc^3}\left\{\check{H}\frac{\tau^{ret}\langle u(t),u(t-\tau^{ret})\rangle-\langle\xi^{ret},u(t)\rangle}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^3}+\right.$$

$$+\check{D}\left[\frac{\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle\left(\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}\right)}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}-\right.$$

$$\left.-\tau^{ret}\frac{\check{\Delta}^2\langle u(t),\dot{u}(t-\tau^{ret})\rangle+\left(\langle u(t),u(t-\tau^{ret})\rangle-c^2\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}\right]-$$

$$-\hat{H}\frac{\tau^{adv}\langle u(t),u(t+\tau^{adv})\rangle-\langle\xi^{adv},u(t)\rangle}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^3}-\hat{D}\left[\frac{\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle\left(\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}\right)}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}-\right.$$

$$\left.\left.-\tau^{adv}\frac{\hat{\Delta}^2\langle u(t),\dot{u}(t+\tau^{adv})\rangle+\left(\langle u(t),u(t+\tau^{adv})\rangle-c^2\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\left[\langle\xi^{adv},u(t+\tau^{ret})\rangle-c^2\tau^{adv}\right]^2}\right]\right\}, \quad (4.4)$$

where

$$\check{H}=\check{\Delta}^2+\check{D}\frac{\check{\Delta}^2\langle\xi^{ret},\dot{u}(t-\tau^{ret})\rangle+\left(\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2},$$

$$\hat{H}=\hat{\Delta}^2+\hat{D}\frac{\hat{\Delta}^2\langle\xi^{adv},\dot{u}(t+\tau^{adv})\rangle+\left(\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right)\langle u(t+\tau^{ret}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2}.$$

Then multiplying $(4.\alpha)$ by $u_\alpha(t)$, summing up in α and dividing into c^2 we obtain (4.4). Indeed, in details, for the left-hand side of $(4.\alpha)$ we have

$$\langle u(t),\dot{u}(t)\rangle+\frac{1}{\Delta^2}\langle u(t),u(t)\rangle\langle u(t),\dot{u}(t)\rangle=\langle u(t),\dot{u}(t)\rangle\left[1+\frac{\langle u(t),\dot{u}(t)\rangle}{\Delta^2}\right]=$$

$$=\langle u(t),\dot{u}(t)\rangle\frac{c^2-\langle u(t),u(t)\rangle+\langle u(t),u(t)\rangle}{\Delta^2}=\frac{c^2\langle u(t),\dot{u}(t)\rangle}{\Delta^2}.$$

For the right-hand side we put $F=\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}F_{\alpha\beta}u_\alpha u_\beta=\sum_{\alpha,\beta=1}^{3}F_{\alpha\beta}u_\alpha u_\beta$. Then in view of $F_{\alpha\beta}=-F_{\beta\alpha}$ it follows $F_{\alpha\alpha}=0$. Therefore

$$F=\sum_{\alpha<\beta}^{3}F_{\alpha\beta}u_\alpha u_\beta+\sum_{\alpha>\beta}^{3}F_{\alpha\beta}u_\alpha u_\beta=$$

$$= \sum_{\alpha<\beta}^{3} F_{\alpha\beta} u_\alpha u_\beta + \sum_{\alpha<\beta}^{3} F_{\beta\alpha} u_\alpha u_\beta = \sum_{\alpha<\beta}^{3} F_{\alpha\beta} u_\alpha u_\beta - \sum_{\alpha<\beta}^{3} F_{\alpha\beta} u_\alpha u_\beta = 0.$$

Consequently

$$\sum_{\alpha}\sum_{\beta} F_{\alpha\beta} u_\alpha u_\beta - c^2 \sum_{\gamma=1}^{3} E_\gamma u_\gamma = -c^2 \sum_{\gamma=1}^{3} E_\gamma u_\gamma = -c^2 \langle E, u(t)\rangle.$$

For the second term in the right-hand side of (4.α) we obtain

$$\langle \xi^{ret}, u(t)\rangle \left(\langle u(t), u(t-\tau^{ret}\rangle - c^2\right) - \langle u(t), u(t-\tau^{ret}\rangle \left(\langle \xi^{ret}, u(t) - c^2\tau^{ret}\rangle\right) =$$
$$= c^2 \left(\tau^{ret}\langle u(t), u(t-\tau^{ret}\rangle - \langle \xi^{ret}, u(t)\rangle\right).$$

For the corresponding term from (4.4) we have

$$\tau^{ret}\langle u(t), u(t-\tau^{ret}\rangle - \langle \xi^{ret}, u(t)\rangle.$$

For the third term in the right-hand side of (4.α) we obtain

$$\left(\langle \xi^{ret}, u(t)\rangle - c^2\tau^{ret}\right)\left(\check{\Delta}^2\langle u(t), \dot{u}(t-\tau^{ret})\rangle + \langle u(t), u(t-\tau^{ret})\rangle\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle\right) -$$
$$-\langle \xi^{ret}, u(t)\rangle \left[\langle u(t), \dot{u}(t-\tau^{ret})\rangle\check{\Delta}^2 + \left(\langle u(t), u(t-\tau^{ret})\rangle - c^2\right)\langle u(t), \dot{u}(t-\tau^{ret})\rangle\right] =$$
$$= c^2\left[-\check{\Delta}^2\tau^{ret}\langle u(t), \dot{u}(t-\tau^{ret})\rangle - \tau^{ret}\langle u(t), u(t-\tau^{ret})\rangle\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle + \right.$$
$$\left. + \langle \xi^{ret}, u(t)\rangle\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle\right].$$

For the corresponding term from (4.4) we have

$$\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle\left(\langle \xi^{ret}, u(t)\rangle - c^2\tau^{ret}\right) -$$
$$-\tau^{ret}\left[\check{\Delta}^2\langle u(t), \dot{u}(t-\tau^{ret})\rangle + \left(\langle u(t), u(t-\tau^{ret})\rangle - c^2\right)\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle\right] =$$
$$= \langle \xi^{ret}, u(t)\rangle\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle -$$
$$-\check{\Delta}^2\tau^{ret}\langle u(t), \dot{u}(t-\tau^{ret})\rangle - \tau^{ret}\langle u(t), u(t-\tau^{ret})\rangle\langle u(t-\tau^{ret}), \dot{u}(t-\tau^{ret})\rangle.$$

For the remaining terms corresponding to the advanced potential the verification can be accomplished in the same way.

Consequently we should consider the following system of 3 equations for the 3 unknown functions obtained after obvious transformations from (4.α):

$$\dot{u}_\alpha(t) + \frac{\langle u(t), \dot{u}(t)\rangle}{\Delta^2} = \frac{e^2\Delta}{2mc^3}\left\{2\left(\sum_{\beta=1}^{3} F_{\alpha\beta}u_\beta(t) - E_\alpha\right) + \right.$$
$$+\xi_\alpha^{ret}\left[\check{H}\frac{\langle u(t), u(t-\tau^{ret}\rangle - c^2}{\left[\langle \xi^{ret}, u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^3} + \right.$$

$$+\check{D}\frac{\langle u(t),\dot{u}(t-\tau^{ret})\rangle\check{\Delta}^2+\left(\langle u(t),u(t-\tau^{ret})\rangle-c^2\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}\Bigg]+$$

$$+u_\alpha(t-\tau^{ret})\left[-\check{H}\frac{\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^3}+\right.$$

$$\left.+\check{D}\frac{\left(\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}\right]+$$

$$+\dot{u}_\alpha(t-\tau^{ret})\check{D}\frac{\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}-$$

$$-\xi_\alpha^{adv}\left[\hat{H}\frac{\langle u(t),u(t+\tau^{adv}\rangle-c^2}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^3}+\right.$$

$$\left.+\hat{D}\frac{\langle u(t),\dot{u}(t+\tau^{adv})\rangle\hat{\Delta}^2+\left(\langle u(t),u(t+\tau^{adv})\rangle-c^2\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}\right]-$$

$$-u_\alpha(t+\tau^{adv})\left[-\hat{H}\frac{\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^3}+\right.$$

$$\left.+\hat{D}\frac{\left(\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}\right]-$$

$$-\dot{u}_\alpha(t+\tau^{adv})\hat{D}\frac{\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}}{\hat{\Delta}^2\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}\equiv G_\alpha. \qquad (5.\alpha)$$

Denoting by $G_\alpha(\alpha=1,2,3)$ the right-hand sides of $(5.\alpha)$ we have to solve the following system with respect to $\dot{u}_1(t),\dot{u}_2(t),\dot{u}_3(t)$:

$$\left|\begin{array}{l}\dot{u}_1(t)+\dfrac{u_1(t)}{\Delta^2}\left(u_1(t)\dot{u}_1(t)+u_2(t)\dot{u}_2(t)+u_3(t)\dot{u}_3(t)\right)=G_1\\ \dot{u}_2(t)+\dfrac{u_2(t)}{\Delta^2}\left(u_1(t)\dot{u}_1(t)+u_2(t)\dot{u}_2(t)+u_3(t)\dot{u}_3(t)\right)=G_2\\ \dot{u}_3(t)+\dfrac{u_3(t)}{\Delta^2}\left(u_1(t)\dot{u}_1(t)+u_2(t)\dot{u}_2(t)+u_3(t)\dot{u}_3(t)\right)=G_3\end{array}\right.$$

or

$$\left|\begin{array}{l}\left(1+\dfrac{u_1^2(t)}{\Delta^2}\right)\dot{u}_1(t)+\dfrac{u_1(t)u_2(t)}{\Delta^2}\dot{u}_2(t)+\dfrac{u_1(t)u_3(t)}{\Delta^2}\dot{u}_3(t)=G_1\\ \dfrac{u_1(t)u_2(t)}{\Delta^2}\dot{u}_1(t)+\left(1+\dfrac{u_2^2(t)}{\Delta^2}\right)\dot{u}_2(t)+\dfrac{u_2(t)u_3(t)}{\Delta^2}\dot{u}_3(t)=G_2\\ \dfrac{u_1(t)u_3(t)}{\Delta^2}\dot{u}_1(t)+\dfrac{u_2(t)u_3(t)}{\Delta^2}\dot{u}_2(t)+\left(1+\dfrac{u_3^2(t)}{\Delta^2}\right)\dot{u}_3(t)=G_3\end{array}\right. \qquad (5.4)$$

We make the following:

Assumption (C): Velocities satisfy the inequality $|u(t)|=\sqrt{u_1^2(t)+u_2^2(t)+u_3^2(t)}\le\bar{c}<c$ where the constant $\bar{c}$ will be discussed later.

Therefore $c^2 - |u(t)|^2 \geq c^2 - \bar{c}^2$ and the determinant of the above system is $\delta = \frac{c^2}{\Delta^2} \Longrightarrow 1 \leq \delta \leq \dfrac{c^2}{c^2 - \bar{c}^2}$.

Consequently we obtain from (5.4):

$$\begin{aligned}
\dot{u}_1 &= \frac{1}{c^2}\left[\left(\Delta^2 + u_2^2 + u_3^2\right) G_1 - u_1 u_2 G_2 - u_1 u_3 G_3\right] \\
\dot{u}_2 &= \frac{1}{c^2}\left[-u_1 u_2 G_1 + \left(\Delta^2 + u_1^2 + u_3^2\right) G_2 - u_2 u_3 G_3\right] \\
\dot{u}_3 &= \frac{1}{c^2}\left[-u_1 u_3 G_1 - u_2 u_3 G_2 + \left(\Delta^2 + u_1^2 + u_3^2\right) G_3\right]
\end{aligned}$$

or

$$\begin{aligned}
\dot{u}_1 &= \frac{1}{c^2}\left[\left(c^2 - u_1^2\right) G_1 - u_1 u_2 G_2 - u_1 u_3 G_3\right] \\
\dot{u}_2 &= \frac{1}{c^2}\left[-u_1 u_2 G_1 + \left(c^2 - u_2^2\right) G_2 - u_2 u_3 G_3\right] \\
\dot{u}_3 &= \frac{1}{c^2}\left[-u_1 u_3 G_1 - u_2 u_3 G_2 + \left(c^2 - u_3^2\right) G_3\right].
\end{aligned} \tag{6}$$

4 Estimations of the functions τ^{ret} and τ^{adv}

Introduce denotations:

$$\check{r}(t) = \sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - \check{x}_\gamma(t)\right]^2}, \quad \hat{r}(t) = \sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - \hat{x}_\gamma(t)\right]^2}$$

$$\bar{r}(t) = \sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_{\gamma 0}(t) - x_{\gamma 0}(t) + \beta t\right]^2}, \quad \bar{\bar{r}}(t) = \sqrt{\sum_{\gamma=1}^{3}\left[\hat{x}_{\gamma 0}(t) - x_{\gamma 0}(t) + \beta t\right]^2}$$

$$X_0 = \min\left\{\sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_{\gamma 0}(t) - x_{\gamma 0}(t)\right]^2}, \quad \sqrt{\sum_{\gamma=1}^{3}\left[\hat{x}_{\gamma 0}(t) - x_{\gamma 0}(t)\right]^2}\right\},$$

where $\beta = const. > 0$ and $t \geq 0$.

Obviously

$$\bar{r}(t) = \sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_{\gamma 0}(t) - x_{\gamma 0}(t) + \beta t\right]^2} \geq \sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_{\gamma 0}(t) - x_{\gamma 0}(t)\right]^2} \geq X_0,$$

$$\bar{\bar{r}}(t) = \sqrt{\sum_{\gamma=1}^{3}\left[\hat{x}_{\gamma 0}(t) - x_{\gamma 0}(t) + \beta t\right]^2} \geq \sqrt{\sum_{\gamma=1}^{3}\left[\hat{x}_{\gamma 0}(t) - x_{\gamma 0}(t)\right]^2} \geq X_0.$$

The following inequalities are valid (cf. [30], [31]):

$$\tau^{ret}(t) = \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[x_\gamma(t) - \check{x}_\gamma(t - \tau^{ret}(t))\right]^2} \geq$$

$$\geq \frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t)]^2}-\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}[\check{x}_\gamma(t)-\check{x}_\gamma(t-\tau^{ret}(t))]^2}\geq\frac{\check{r}(t)}{c}-\frac{c\tau^{ret}(t)}{c}\implies$$

$$\tau^{ret}(t)\geq\frac{\check{r}(t)}{2c}\iff\frac{1}{\tau^{ret}(t)}\leq\frac{2c}{\check{r}(t)}. \qquad (7-ret)$$

In the same way we obtain

$$\tau^{adv}(t)\geq\frac{\hat{r}(t)}{2c}\iff\frac{1}{\tau^{adv}(t)}\leq\frac{2c}{\hat{r}(t)}. \qquad (7-adv)$$

Let us recall the standard denotations:

$$x_\gamma=x_{\gamma0}+u_{\gamma0}t+\int_0^t\int_0^s w_\gamma(\theta)d\theta ds, u_\gamma=u_{\gamma0}t+\int_0^t w_\gamma(s)ds,$$
$$\check{x}_\gamma=\check{x}_{\gamma0}+\check{u}_{\gamma0}t+\int_0^t\int_0^s \check{w}_\gamma(\theta)d\theta ds, \check{u}_\gamma=\check{u}_{\gamma0}t+\int_0^t \check{w}_\gamma(s)ds,$$
$$\hat{x}_\gamma=\hat{x}_{\gamma0}+\hat{u}_{\gamma0}t+\int_0^t\int_0^s \hat{w}_\gamma(\theta)d\theta ds, \hat{u}_\gamma=\hat{u}_{\gamma0}+\int_0^t \hat{w}_\gamma(s)ds,$$

Lemma 1. *Let the following conditions (W) be satisfied:*

(W1) *the coordinate functions of the acceleration satisfy the inequalities $w_\gamma(t)\leq w_0(t)$, where $w_0(\cdot)\in L_1^\infty(R^1)\cap L_\infty^\infty(R^1)$ and $w_0(t)\geq 0$*

(W2) *the initial dates are such that $\check{u}_{\gamma0}-u_{\gamma0}-\beta>0, \hat{u}_{\gamma0}-u_{\gamma0}-\beta>0$, $\check{x}_{\gamma0}-x_{\gamma0}>0, \hat{x}_{\gamma0}-x_{\gamma0}>0\,(\gamma=1,2,3)$;*

(W3) *the following inequalities hold true*

$$\int_{-\infty}^{\infty}w_\gamma(s)ds\leq\frac{\check{u}_{\gamma0}-u_{\gamma0}-\beta}{2},\int_{-\infty}^{\infty}w_\gamma(s)ds\leq\frac{\hat{u}_{\gamma0}-u_{\gamma0}-\beta}{2}.$$

Then the following estimates are valid

$$\tau^{ret}(t)\geq\frac{1}{2c}\sqrt{\sum_{\gamma=1}^{3}[\check{x}_{\gamma0}-x_{\gamma0}+\beta t]}\geq\frac{\overline{r}(t)}{2c}\text{ and }\tau^{adv}(t)\geq\frac{1}{2c}\sqrt{\sum_{\gamma=1}^{3}[\hat{x}_{\gamma0}-x_{\gamma0}+\beta t]}\geq\frac{\overline{\overline{r}}(t)}{2c} \qquad (8)$$

Proof.

$$\int_0^t w_\gamma(s)ds-\int_0^t\check{w}_\gamma(s)ds\leq\int_0^t|w_\gamma(s)-\check{w}_\gamma(s)|ds\leq 2\int_0^t w_0(s)ds\leq\check{u}_{\gamma0}-u_{\gamma0}-\beta\implies$$

$$\implies u_{\gamma0}+\int_0^t w_\gamma(s)ds-\check{u}_{\gamma0}-\int_0^t\check{w}_\gamma(s)ds\geq\beta\iff\check{u}_\gamma(t)-u_\gamma(t)\geq\beta\implies$$

$$\implies\int_0^t u_\gamma(s)ds-\int_0^t\check{u}_\gamma(s)ds\geq\beta.$$

Therefore

$$\check{r}(t)=\sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_\gamma(t)-x_\gamma(t)\right]^2}=\sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_{\gamma 0}-x_{\gamma 0}+\int_0^t \check{u}_\gamma(s)ds-\int_0^t u_\gamma(s)ds\right]^2}\geq$$

$$\geq\sqrt{\sum_{\gamma=1}^{3}\left[\check{x}_{\gamma 0}-x_{\gamma 0}+\beta t\right]^2}=\bar{r}(t).$$

It follows from (7) $\tau^{ret}(t)\geq\frac{\bar{r}(t)}{2c}$ and analogously $\tau^{adv}(t)\geq\frac{\bar{\bar{r}}(t)}{2c}$. The Lemma 1 is thus proved.

5 Estimations of the right-hand sides of the newly derived Dirac equations

First we introduce denotations $v=\dfrac{\bar{c}}{c}$,

$$A_\alpha=2\left(\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta(t)-E_\alpha\right),$$

$$B^{ret}=\check{H}\frac{\langle u(t),u(t-\tau^{ret}\rangle-c^2}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^3}+$$
$$+\check{D}\frac{\langle u(t),\dot{u}(t-\tau^{ret})\rangle\check{\Delta}^2+\left(\langle u(t),u(t-\tau^{ret})\rangle-c^2\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2},$$

$$C^{ret}=-\check{H}\frac{\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^3}+$$
$$+\check{D}\frac{\left(\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2},$$

$$D^{ret}=\check{D}\frac{\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2};$$

$$B^{adv}=\hat{H}\frac{\langle u(t),u(t+\tau^{adv}\rangle-c^2}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^3}+$$
$$+\hat{D}\frac{\langle u(t),\dot{u}(t+\tau^{adv})\rangle\hat{\Delta}^2+\left(\langle u(t),u(t+\tau^{adv})\rangle-c^2\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2},$$

$$C^{adv}=-\hat{H}\frac{\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}}{\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^3}+$$
$$+\hat{D}\frac{\left(\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2},$$

$$D^{adv}=\hat{D}\frac{\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}}{\hat{\Delta}^2\left[\langle\xi^{adv},u(t+\tau^{adv})\rangle-c^2\tau^{adv}\right]^2}.$$

Then (5.α) become

$$\dot{u}_\alpha(t)+\frac{\langle u(t),\dot{u}(t)\rangle}{\Delta^2}=\frac{e^2\Delta}{2mc^3}\left[A_\alpha+B^{ret}\xi^{ret}_\alpha+C^{ret}u_\alpha(t-\tau^{ret})+D^{ret}\dot{u}_\alpha(t-\tau^{ret})-\right.$$
$$\left.-B^{adv}\xi^{adv}_\alpha-C^{adv}u_\alpha(t+\tau^{adv})-D^{adv}\dot{u}_\alpha(t+\tau^{adv})\right]\,(\alpha=1,2,3). \qquad (9.\alpha)$$

In what follows we derive some inequalities (cf. [31], [32]):

$$|\check{D}|\le\frac{c^2\tau^{ret}+c\tau^{ret}\bar{c}}{c^2\tau^{ret}-c\tau^{ret}\bar{c}}=\frac{c+\bar{c}}{c-\bar{c}}=\frac{1+v}{1-v},\ |\hat{D}|\le\frac{1+v}{1-v}.$$

$$A_\alpha=2\left(\sum_{\beta=1}^{3}F_{\alpha\beta}u_\beta(t)-E_\alpha\right)\le2\sqrt{\sum_{\beta=1}^{3}F^2_{\alpha\beta}}\sqrt{\sum_{\beta=1}^{3}u^2_\alpha}+2\sqrt{\sum_{\beta=1}^{3}E^2_\alpha}\le2\bar{c}\sqrt{\sum_{\beta=1}^{3}H^2_\alpha}+2\sqrt{\sum_{\beta=1}^{3}E^2_\alpha}.$$

$$|\check{H}|=\left|\check{\Delta}^2+\check{D}\frac{\check{\Delta}^2\langle\xi^{ret},\dot{u}(t-\tau^{ret})\rangle+\left(\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2}\right|\le$$
$$\le c^2+\frac{1+v}{1-v}\frac{c^2c\tau^{ret}w_0(t)+\left(c\tau^{ret}\bar{c}+c^2\tau^{ret}\right)\bar{c}w_0(t)}{c^2(1-v^2)}\le c^2+\frac{3\sqrt{3}c\tau^{ret}w_0(t)}{(1-v)^2}.$$

$$\left|B^{ret}\right|=\left|\check{H}\frac{\langle u(t),u(t-\tau^{ret}\rangle-c^2}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^3}+\right.$$
$$\left.+\check{D}\frac{\langle u(t),\dot{u}(t-\tau^{ret})\rangle\check{\Delta}^2+\left(\langle u(t),u(t-\tau^{ret})\rangle-c^2\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}\right|\le$$
$$\le\left[c^2+\frac{3\sqrt{3}c\tau^{ret}w_0(t)}{(1-v)^2}\right]\frac{\bar{c}^2+c^2}{(c^2\tau^{ret}-c\tau^{ret}\bar{c})^3}+\frac{1+v}{1-v}\frac{(\bar{c}c^2+\bar{c}^3+c^2\bar{c})w_0(t)\sqrt{3}}{c^2(1-v^2)(c^2\tau^{ret}-c\tau^{ret}\bar{c})^2}\le$$
$$\le\left[c^2+\frac{3\sqrt{3}c\tau^{ret}w_0(t)}{(1-v)^2}\right]\frac{v^2+1}{c^4(\tau^{ret})^3(1-v)^3}+\frac{3\sqrt{3}w_0(t)}{c^3(1-v)^4(\tau^{ret})^2}\le$$
$$\le\frac{2}{c^2(\tau^{ret})^3(1-v)^3}+\frac{6\sqrt{3}w_0(t)}{c^3(1-v)^5(\tau^{ret})^2}+\frac{3\sqrt{3}w_0(t)}{c^3(1-v)^4(\tau^{ret})^2}\le$$
$$\le\frac{2}{c^2(\tau^{ret})^3(1-v)^3}+\frac{9\sqrt{3}w_0(t)}{c^3(1-v)^5(\tau^{ret})^2};$$

$$\left|C^{ret}\right|=\left|-\check{H}\frac{\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}}{\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^3}+\right.$$
$$\left.+\check{D}\frac{\left(\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle-c^2\tau^{ret}\right]^2}\right|\le$$
$$\le\left[c^2+\frac{3\sqrt{3}c\tau^{ret}w_0(t)}{(1-v)^2}\right]\frac{c\tau^{ret}\bar{c}+c^2\tau^{ret}}{(c^2\tau^{ret}-c\tau^{ret}\bar{c})^3}+\frac{1+v}{1-v}\frac{(c\tau^{ret}\bar{c}+c^2\tau^{ret})\bar{c}\sqrt{3}w_0(t)}{c^2(1-v^2)(c^2\tau^{ret}-c\tau^{ret}\bar{c})^2}\le$$

$$\leq \left[c^2 + \frac{3\sqrt{3}c\tau^{ret}w_0(t)}{(1-v)^2}\right] \frac{(v+1)c^2}{c^6(1-v)^3(\tau^{ret})^2} + \frac{(v^2+v)w_0(t)\sqrt{3}}{c^3(1-v)^4\tau^{ret}} \leq$$

$$\leq \frac{2}{c^2(1-v)^3(\tau^{ret})^2} + \frac{6\sqrt{3}w_0(t)}{c^3(1-v)^5\tau^{ret}} + \frac{2\sqrt{3}w_0(t)}{c^3(1-v)^5\tau^{ret}} \leq$$

$$\leq \frac{2}{c^2(1-v)^3(\tau^{ret})^2} + \frac{8\sqrt{3}w_0(t)}{c^3(1-v)^5\tau^{ret}};$$

$$\left|D^{ret}\right| = \left|\check{D}\frac{\langle\xi^{ret}, u(t)\rangle - c^2\tau^{ret}}{\check{\Delta}^2\left[\langle\xi^{ret}, u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^2}\right| \leq \frac{1+v}{1-v}\frac{c\tau^{ret}\bar{c} + c^2\tau^{ret}}{(c^2\tau^{ret} - c\tau^{ret}\bar{c})^2} \leq \frac{4}{c^2(1-v)^3\tau^{ret}};$$

$$\left|\hat{H}\right| \leq c^2 + \frac{3\sqrt{3}c\tau^{adv}w_0(t)}{(1-v)^2};$$

$$\left|B^{adv}\right| \leq \frac{2}{c^2(\tau^{adv})^3(1-v)^3} + \frac{9\sqrt{3}w_0(t)}{c^3(1-v)^5(\tau^{adv})^2};$$

$$\left|C^{adv}\right| \leq \frac{2}{c^2(1-v)^3(\tau^{adv})^2} + \frac{8\sqrt{3}w_0(t)}{c^3(1-v)^5\tau^{adv}}; \qquad \left|D^{adv}\right| \leq \frac{4}{c^2(1-v)^3\tau^{adv}}.$$

We have to estimate the right-hand sides of the system:

$$\begin{aligned}
\dot{u}_1 &= \frac{1}{c^2}\left[\left(c^2 - u_1^2\right)G_1 - u_1u_2G_2 - u_1u_3G_3\right] \\
\dot{u}_2 &= \frac{1}{c^2}\left[-u_1u_2G_1 + \left(c^2 - u_2^2\right)G_2 - u_2u_3G_3\right] \\
\dot{u}_3 &= \frac{1}{c^2}\left[-u_1u_3G_1 - u_2u_3G_2 + \left(c^2 - u_3^2\right)G_3\right].
\end{aligned} \tag{10}$$

Indeed, the inequalities

$$\frac{e^2\Delta}{2mc^3}\left[|A_\alpha| + \left|B^{ret}\right|\left|\xi_\alpha^{ret}\right| + \left|C^{ret}\right|\left|u_\alpha(t-\tau^{ret})\right| + \left|D^{ret}\right|\left|\dot{u}_\alpha(t-\tau^{ret})\right|\right] \leq$$

$$\leq \frac{e^2\Delta}{2mc^3}\left[2\bar{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2} + 2\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2} + c\tau^{ret}\left(\frac{2}{c^2(\tau^{ret})^3(1-v)^3} + \frac{9\sqrt{3}w_0(t)}{c^3(1-v)^5(\tau^{ret})^2}\right) + \right.$$

$$\left. +\bar{c}\left(\frac{6\sqrt{3}w_0(t)}{c^3(1-v)^5\tau^{ret}} + \frac{2\sqrt{3}w_0(t)}{c^3(1-v)^5\tau^{ret}}\right) + \frac{4w_0(t)}{c^2(1-v)^3\tau^{ret}}\right] \leq$$

$$\leq \frac{e^2}{2m}\left[\frac{2\bar{c}}{c^2}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2} + \frac{2}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2} + \frac{2}{c^3(\tau^{ret})^2(1-v)^3} + \frac{9\sqrt{3}w_0(t)}{c^4(1-v)^5\tau^{ret}} + \right.$$

$$\left. +\frac{2}{c^3(1-v)^3(\tau^{ret})^2} + \frac{8\sqrt{3}w_0(t)}{c^4(1-v)^5\tau^{ret}} + \frac{4w_0(t)}{c^4(1-v)^3\tau^{ret}}\right]$$

imply

$$|G_\alpha| \leq \frac{e^2\Delta}{2mc^3}\Big[\,|A_\alpha| + \left|B^{ret}\right|\left|\xi_\alpha^{ret}\right| + \left|C^{ret}\right|\left|u_\alpha(t-\tau^{ret})\right| + \left|D^{ret}\right|\left|\dot{u}_\alpha(t-\tau^{ret})\right| +$$

$$+\left|B^{adv}\right|\left|\xi_\alpha^{adv}\right|+\left|C^{adv}\right|\left|u_\alpha(t+\tau^{adv})\right|+\left|D^{adv}\right|\left|\dot{u}_\alpha(t+\tau^{adv})\right|\Big]\le$$

$$\le\frac{e^2}{m}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{2}{c^3(\tau^{ret})^2(1-v)^3}+\frac{(17\sqrt{3}+4)w_0(t)}{c^4(1-v)^5\tau^{ret}}+\right.$$

$$\left.+\frac{2}{c^3(\tau^{adv})^2(1-v)^3}+\frac{(17\sqrt{3}+4)w_0(t)}{c^4(1-v)^5\tau^{adv}}\right].$$

Consequently denoting the right-hand sides of (10) by P_α $(\alpha=1,2,3)$ and we obtain

$$|P_1|\le\frac{1}{c^2}\Big[|c^2-u_1^2|\,|G_1|+|u_1u_2|\,|G_2|+|u_1u_3|\,|G_3|\Big]\le$$

$$\le\frac{3e^2}{m}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{8c^2}{c^3\overline{r}^2(t)(1-v)^3}+\frac{(34\sqrt{3}+8)w_0(t)c}{c^4(1-v)^5\overline{r}(t)}+\right.$$

$$\left.+\frac{8c^2}{c^3\overline{\overline{r}}^2(t)(1-v)^3}+\frac{(34\sqrt{3}+8)w_0(t)c}{c^4(1-v)^5\overline{\overline{r}}(t)}\right]\le$$

$$\le\frac{3e^2}{m}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{8}{c(1-v)^3}\left(\frac{1}{\overline{r}^2(t)}+\frac{1}{\overline{\overline{r}}^2(t)}\right)+\frac{34\sqrt{3}+8}{c^3(1-v)^5X_0}w_0(t)\right].\qquad(11-1)$$

Analogously

$$|P_2|\le\frac{1}{c^2}\Big[|u_1u_2|\,|G_1|+|c^2-u_2^2|\,|G_2|+|u_2u_3|\,|G_3|\Big]\le$$

$$\le\frac{3e^2}{m}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{8}{c(1-v)^3}\left(\frac{1}{\overline{r}^2(t)}+\frac{1}{\overline{\overline{r}}^2(t)}\right)+\frac{34\sqrt{3}+8}{c^3(1-v)^5X_0}w_0(t)\right]\qquad(11-2)$$

and

$$|P_3|\le\frac{1}{c^2}\Big[|u_1u_3|\,|G_1|+|u_2u_3|\,|G_2|+|c^2-u_3^2|\,|G_3|\Big]\le$$

$$\le\frac{3e^2}{m}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{8}{c(1-v)^3}\left(\frac{1}{\overline{r}^2(t)}+\frac{1}{\overline{\overline{r}}^2(t)}\right)+\frac{34\sqrt{3}+8}{c^3(1-v)^5X_0}w_0(t)\right].\qquad(11-3)$$

6 Lipschitz estimates of the right-hand sides

In what follows we have to obtain Lipschitz estimates of the right-hand sides G_α $(\alpha=1,2,3)$ with respect to the arguments $w_\alpha(t-\tau^{ret}),w_\alpha(t+\tau^{adv})$:

$$\frac{\partial G_\alpha}{\partial w_\alpha(t-\tau^{ret})}=\frac{e^2\Delta}{2mc^3}\left\{\frac{\xi_\alpha^{ret}\check{D}u_\alpha(t)}{\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}+\right.$$

$$+\frac{\xi_\alpha^{ret}\check{D}\left(\langle u(t),u(t-\tau^{ret})\rangle-c^2\right)u_\alpha(t)}{\check{\Delta}^2\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}+$$
$$+u_\alpha(t-\tau^{ret})\frac{\check{D}(\langle\xi^{ret},u(t)\rangle-c^2)u_\alpha(t-\tau^{ret})}{\check{\Delta}^2\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}+$$
$$\left.+\check{D}\frac{\langle\xi^{ret},u(t)\rangle-c^2\tau^{ret}}{\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}\right\},$$

$$\frac{\partial G_\alpha}{\partial w_\alpha(t+\tau^{adv})}=\frac{e^2\Delta}{2mc^3}\left\{\frac{\xi_\alpha^{adv}\hat{D}u_\alpha(t)}{\left(c^2\tau^{adv}-\langle\xi^{adv},u(t+\tau^{adv})\rangle\right)^2}+\right.$$
$$+\frac{\xi_\alpha^{adv}\hat{D}\left(\langle u(t),u(t+\tau^{adv})\rangle-c^2\right)u_\alpha(t)}{\hat{\Delta}^2\left(c^2\tau^{adv}-\langle\xi^{adv},u(t+\tau^{adv})\rangle\right)^2}+$$
$$+u_\alpha(t+\tau^{adv})\frac{\hat{D}(\langle\xi^{adv},u(t)\rangle-c^2)u_\alpha(t+\tau^{adv})}{\hat{\Delta}^2\left(c^2\tau^{adv}+\langle\xi^{ret},u(t+\tau^{adv})\rangle\right)^2}+$$
$$\left.+\hat{D}\frac{\langle\xi^{adv},u(t)\rangle-c^2\tau^{adv}}{\left(c^2\tau^{adv}-\langle\xi^{adv},u(t+\tau^{adv})\rangle\right)^2}\right\},$$

and for $\beta\neq\alpha$

$$\frac{\partial G_\alpha}{\partial w_\beta(t-\tau^{ret})}=\frac{e^2\Delta}{2mc^3}\left\{\frac{\xi_\alpha^{ret}\check{D}u_\beta(t)}{\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}+\right.$$
$$+\frac{\xi_\alpha^{ret}\check{D}\left(\langle u(t),u(t-\tau^{ret})\rangle-c^2\right)u_\beta(t)}{\check{\Delta}^2\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}+$$
$$\left.+u_\alpha(t-\tau^{ret})\frac{\check{D}(\langle\xi^{ret},u(t)\rangle-c^2)u_\beta(t-\tau^{ret})}{\check{\Delta}^2\left(c^2\tau^{ret}-\langle\xi^{ret},u(t-\tau^{ret})\rangle\right)^2}\right\},$$

$$\frac{\partial G_\alpha}{\partial w_\beta(t+\tau^{adv})}=\frac{e^2\Delta}{2mc^3}\left\{\frac{\xi_\alpha^{adv}\hat{D}u_\beta(t)}{\left(c^2\tau^{adv}-\langle\xi^{adv},u(t+\tau^{adv})\rangle\right)^2}+\right.$$
$$+\frac{\xi_\alpha^{adv}\hat{D}\left(\langle u(t),u(t+\tau^{adv})\rangle-c^2\right)u_\beta(t)}{\hat{\Delta}^2\left(c^2\tau^{adv}-\langle\xi^{adv},u(t+\tau^{adv})\rangle\right)^2}+$$
$$\left.+u_\alpha(t+\tau^{adv})\frac{\hat{D}(\langle\xi^{adv},u(t)\rangle-c^2)u_\beta(t+\tau^{adv})}{\hat{\Delta}^2\left(c^2\tau^{adv}+\langle\xi^{ret},u(t+\tau^{adv})\rangle\right)^2}\right\}.$$

Therefore we have to estimate $\left|\frac{\partial G_\alpha}{\partial w_\alpha(t-\tau^{ret})}\right|$, $\left|\frac{\partial G_\alpha}{\partial w_\alpha(t+\tau^{adv})}\right|$ because for $\beta\neq\alpha$ the last term vanishes. Indeed,

$$\left|\frac{\partial G_\alpha}{\partial w_\alpha(t-\tau^{ret})}\right|\le\frac{e^2}{2mc^2}\frac{1+v}{1-v}\left\{\frac{\bar{c}c\tau^{ret}}{c^4(1-v)^2(\tau^{ret})^2}+\frac{c\tau^{ret}(c^2+\bar{c}^2)\bar{c}}{(c^2-\bar{c}^2)c^4(1-v)^2(\tau^{ret})^2}+\right.$$
$$\left.+\frac{\bar{c}^2(c^2\tau^{ret}+c\tau^{ret}\bar{c})}{(c^2-\bar{c}^2)c^4(1-v)^2(\tau^{ret})^2}+\frac{c^2\tau^{ret}+c\tau^{ret}\bar{c}}{c^4(1-v)^2(\tau^{ret})^2}\right\}\le$$

$$\leq \frac{e^2}{2mc^2}\left\{\frac{2}{c^2(1-v)^3\tau^{ret}}+\frac{2}{c^2(1-v)^4\tau^{ret}}+\frac{2}{c^2(1-v)^4\tau^{ret}}+\frac{4}{c^2(1-v)^3\tau^{ret}}\right\}\leq$$
$$\leq \frac{e^2}{2mc^4\tau^{ret}}\left\{\frac{6}{(1-v)^3}+\frac{4}{(1-v)^4}\right\},$$

$$\left|\frac{\partial G_\alpha}{\partial w_\alpha(t+\tau^{adv})}\right|\leq \frac{e^2}{2mc^2}\frac{1+v}{1-v}\left\{\frac{\bar{c}c\tau^{adv}}{c^4(1-v)^2(\tau^{adv})^2}+\frac{c\tau^{adv}(c^2+\bar{c}^2)\bar{c}}{(c^2-\bar{c}^2)c^4(1-v)^2(\tau^{adv})^2}+\right.$$
$$\left.+\frac{\bar{c}^2(c^2\tau^{adv}+c\tau^{adv}\bar{c})}{(c^2-\bar{c}^2)c^4(1-v)^2(\tau^{adv})^2}+\frac{c^2\tau^{adv}+c\tau^{adv}\bar{c}}{c^4(1-v)^2(\tau^{adv})^2}\right\}\leq$$
$$\leq \frac{e^2}{2mc^2}\left\{\frac{2}{c^2(1-v)^3\tau^{adv}}+\frac{2}{c^2(1-v)^4\tau^{adv}}+\frac{2}{c^2(1-v)^4\tau^{adv}}+\frac{4}{c^2(1-v)^3\tau^{adv}}\right\}\leq$$
$$\leq \frac{e^2}{2mc^4\tau^{adv}}\left\{\frac{6}{(1-v)^3}+\frac{4}{(1-v)^4}\right\}.$$

Therefore

$$\Big|G_\alpha\left(w_1(t-\tau^{ret}),w_2(t-\tau^{ret}),w_3(t-\tau^{ret}),w_1(t+\tau^{adv}),w_2(t+\tau^{adv}),w_3(t+\tau^{adv})\right)-$$
$$-G_\alpha\left(\bar{w}_1(t-\tau^{ret}),\bar{w}_2(t-\tau^{ret}),\bar{w}_3(t-\tau^{ret}),\bar{w}_1(t+\tau^{adv}),\bar{w}_2(t+\tau^{adv}),\bar{w}_3(t+\tau^{adv})\right)\Big|\leq$$
$$\leq\sum_{\beta=1}^{3}\left|\frac{\partial G_\alpha}{\partial w_\beta(t-\tau^{ret})}\right|\left|w_\beta(t-\tau^{ret})-\bar{w}_\beta(t-\tau^{ret})\right|+$$
$$+\sum_{\beta=1}^{3}\left|\frac{\partial G_\alpha}{\partial w_\beta(t+\tau^{adv})}\right|\left|w_\beta(t+\tau^{adv})-\bar{w}_\beta(t+\tau^{adv})\right|\leq$$
$$\leq\frac{e^2}{2mc^4\tau^{ret}}\left\{\frac{6}{(1-v)^3}+\frac{4}{(1-v)^4}\right\}\sum_{\beta=1}^{3}\left|w_\beta(t-\tau^{ret})-\bar{w}_\beta(t-\tau^{ret})\right|+$$
$$+\frac{e^2}{2mc^4\tau^{adv}}\left\{\frac{6}{(1-v)^3}+\frac{4}{(1-v)^4}\right\}\sum_{\beta=1}^{3}\left|w_\beta(t+\tau^{adv})-\bar{w}_\beta(t+\tau^{adv})\right|\leq$$
$$\leq\frac{e^2}{mc^4\tau^{ret}}\frac{5}{(1-v)^4}\sum_{\beta=1}^{3}\left|w_\beta(t-\tau^{ret})-\bar{w}_\beta(t-\tau^{ret})\right|+$$
$$+\frac{e^2}{2mc^4\tau^{adv}}\frac{5}{(1-v)^4}\sum_{\beta=1}^{3}\left|w_\beta(t+\tau^{adv})-\bar{w}_\beta(t+\tau^{adv})\right|\leq$$
$$\leq\frac{10}{mc^3(1-v)^4\bar{r}(t)}\sum_{\beta=1}^{3}\left|w_\beta(t-\tau^{ret})-\bar{w}_\beta(t-\tau^{ret})\right|+$$
$$+\frac{10}{mc^3(1-v)^4\bar{\bar{r}}(t)}\sum_{\beta=1}^{3}\left|w_\beta(t+\tau^{adv})-\bar{w}_\beta(t+\tau^{adv})\right|.$$

7 Tools from fixed point theory

We briefly recall some notations and definitions from [34].

By $(X, \mathcal{A})$ we mean a sequentially complete $T2$-separated uniform space with a uniformity generated by a saturated family of pseudometrics $\mathcal{A} = \{\rho_\alpha(x, y) : \alpha \in A\}$, A being an index set. Let $j : A \to A$ be a mapping of the index set into itself whose iterates are defined as follows $j^k(\alpha) = j(j^{k-1})(\alpha)\,(k = 1, 2, 3)$.

Introduce a family of functions (called contractive functions) $(\Phi) = \{\Phi_\alpha(t) : \alpha \in A\}$, $\Phi_\alpha(t) : \Re^1_+ \to \Re^1_+ \equiv \big[0, \infty\big)$ with the following properties:

(**Φ1**) $\Phi_\alpha(t)$ is non-decreasing, continuous from the right on $\Re^1_+$ and $\Phi_\alpha(t) > 0$ for $t > 0$;

(**Φ2**) for every $\alpha \in A$ there is a function $\bar{\Phi}_\alpha(t)$ which possesses property (**Φ2**), $\sup\{\Phi_{j^k(\alpha)(t):k=0,1,2,\ldots}\} \le \bar{\Phi}_\alpha(t)$ and $\bar{\Phi}_\alpha(t)/t$ is non-decreasing.

The operator $T : (X, \mathcal{A}) \to (X, \mathcal{A})$ is said to be Φ-contractive if $\rho_\alpha(T(x), T(y)) \le \Phi_\alpha(\rho_{j(\alpha)}(x, y))$ for every $x, y \in X$ and $\alpha \in A$. The space $(X, \mathcal{A})$ is called j-bounded if for every $x, y \in X$ and $\alpha \in A$ there is a constant $Q = Q(\alpha, x, y)$ such that $\rho_{j(\alpha)}(x, y) \le Q$, $k = 0, 1, 2, \ldots$.

Theorem 1. *If $T : (X, \mathcal{A}) \to (X, \mathcal{A})$ is Φ-contractive and there is $x_0 \in X$ for which there exists $Q = Q(\alpha, x_0, T(x_0))$ so that $\rho_{j^k(\alpha)}(x_0, T(x_0)) \le Q\,(k = 0, 1, 2, \ldots)$ then T has at least one fixed point.*

Theorem 2. *If, in addition, $(X, \mathcal{A})$ is j-bounded, then the fixed point is unique.*

In what follows we extend above theorems to a product of uniform spaces.

Let $\{(X_i, \mathcal{A}_i)\}_{i=1}^n$ be a finite family of uniform spaces where $\mathcal{A}_i = \{\rho^{(i)}_\alpha(x, y) : \alpha \in A\}$. The index sets $A_1 = A_2 = \ldots = A_n = A$ coincide. The elements of the Cartesian product $X = \prod_{i=1}^n X_i$ we denote by $x = (x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$.

The product X can be endowed with a family of pseudometrics $\mathcal{A} = \{\rho_\alpha(x, y) : \alpha \in A\}$ where $\rho_\alpha(x, y) = \max\{\rho^{(i)}_\alpha(x, y) : i = 1, 2, ..., n\}$. It is not difficult to check that

$$\rho_\alpha(x^{(0)}, x^{(k)}) \xrightarrow[k\to\infty]{} 0 \iff \left\{ \begin{array}{l} \rho_\alpha\big(x^{(0)}_1, x^{(k)}_1\big) \xrightarrow[k\to\infty]{} 0, \\ \rho_\alpha\big(x^{(0)}_2, x^{(k)}_2\big) \xrightarrow[k\to\infty]{} 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \rho_\alpha\big(x^{(0)}_n, x^{(k)}_n\big) \xrightarrow[k\to\infty]{} 0, \end{array} \right.$$

So the product space $(X, \mathcal{A})$ is a sequentially complete uniform space if every $(X_i, \mathcal{A}_i)$ is a sequentially complete uniform space.

Theorem 3. *Let $T_i : (X_i, \mathcal{A}_i) \to (X_i, \mathcal{A}_i)\,(i = 1, 2, ..., n)$ be Φ-contractive mappings and $T : (X, \mathcal{A}) \to (X, \mathcal{A})$ be a mapping defined as follows:*

$$T(x_1, x_2, \ldots, x_n) = (T_1(x_1), T_2(x_2), \ldots, T_n(x_n)) \text{ for every } (x_1, x_2, \ldots, x_n) \in X.$$

If there are elements $x^{(0)}_i \in X_i$ $(i = 1, 2, ..., n)$ such that $\rho^{(i)}_{j^k(\alpha)}\Big(x^{(0)}_i, T\big(x^{(0)}_i\big)\Big) \le Q^{(i)}(\alpha)$ $(k = 0, 1, 2, \ldots)$ then T is Φ-contractive and has a fixed point in $(X, \mathcal{A})$.

Theorem 4. *If in addition, we suppose all spaces $(X_i, \mathcal{A}_i)$ to be j-bounded it follows that $(X, \mathcal{A})$ is j-bounded and consequently T has a unique fixed point in $(X, \mathcal{A})$.*

8 Existence-uniqueness theorem for Dirac equations

Let us put $w_\alpha(t) = \dot{u}_\alpha(t), (\alpha = 1,2,3)$. Then we can formulate the main initial value problem: to find a solution $\left(\tau^{ret}(t), \tau^{adv}(t), w_1(t), w_2(t), w_3(t)\right)$ of the system

$$\tau^{ret}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-\tau^{ret}(t))]^2},\ \tau^{adv}(t)=\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}\left[\hat{x}_\gamma(t+\tau^{adv}(t))-x_\gamma(t)\right]^2}, t\in\Re^1,$$

$$\begin{aligned}
w_1(t) &= \frac{1}{c^2}\left[\left(c^2-u_1^2(t)\right)G_1-u_1(t)u_2(t)G_2-u_1(t)u_3(t)G_3\right]\\
w_2(t) &= \frac{1}{c^2}\left[-u_1(t)u_2(t)G_1+\left(c^2-u_2^2(t)\right)G_2-u_2(t)u_3(t)G_3\right],\ t>0 \qquad (12)\\
w_3(t) &= \frac{1}{c^2}\left[-u_1(t)u_3(t)G_1-u_2(t)u_3(t)G_2+\left(c^2-u_3^2(t)\right)G_3\right]\\
w_1(t) &= \overline{w}_1(t), t\le 0,\ w_2(t)=\overline{w}_2(t), t\le 0,\ w_3(t)=\overline{w}_3(t), t\le 0,
\end{aligned}$$

where $\overline{w}_\gamma(t)\,(\gamma=1,2,3)$ are prescribed functions on $\left(-\infty,0\right]$ (and $x_\gamma(t)=\overline{x}_\gamma(t), t\le 0$, $x_\gamma(0)=\overline{x}_{\gamma0}$, $u_\gamma(t)=\overline{u}_\gamma(t), t\le 0$, $u_\gamma(0)=\overline{u}_{\gamma0}$).

In what follows we prove an existence–uniqueness theorem for the above system.

Theorem 5. *The function $w_0\in L^\infty(\Re^1)\cap L^\infty(\Re^1)$ $(w_0(t)\ge 0)$ satisfies the inequalities*

5.1) $$\frac{3e^2}{m(1-A)}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{8}{c(1-v)^3}\left(\frac{1}{\overline{r}^2(t)}+\frac{1}{\overline{\overline{r}}^2(t)}\right)\right]\le w_0(t), \qquad (13)$$

where $A=\dfrac{3e^2(34\sqrt{3}+8)}{mc^3(1-v)^5X_0}<1$;

5.2) $$\frac{3e^2}{m(1-A)}\int_0^\infty\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}H_\alpha^2}+\frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3}E_\alpha^2}+\frac{8}{c(1-v)^3}\left(\frac{1}{\overline{r}^2(t)}+\frac{1}{\overline{\overline{r}}^2(t)}\right)\right]dt\le$$
$$\le\int\limits_{-\infty}^{\infty}w_0(t)dt\le\min\left\{\frac{\bar{c}}{\sqrt{3}}-u_0,\frac{\check{u}_{\gamma0}-u_{\gamma0}-\beta}{2},\frac{\hat{u}_{\gamma0}-u_{\gamma0}-\beta}{2}\,(\gamma=1,2,3)\right\} \qquad (14)$$
$(\check{u}_{\gamma0}-u_{\gamma0}-\beta>0, \hat{u}_{\gamma0}-u_{\gamma0}-\beta>0)$.

If the initial trajectories are such that $\check{r}(t)\ge r_0>0$ and $\hat{r}(t)\ge r_0>0$ for $t\le 0$, $|\overline{w}_\alpha(t)|\le w_0(t)$ for $t\le 0$ and $x_{\gamma0}^{(2)}>x_{\gamma0}^{(1)}$ then there is a unique solution $\left(\tau^{ret},\tau^{adv},w_1,w_2,w_3\right)$ of (12) such that $|w_\alpha(t)|\le w_0(t)\,(\alpha=1,2,3)$.

Proof of Theorem 5. Let $(X_1,\mathcal{A})$ be a uniform space consisting of all continuous functions $\tau(t):\Re^1\to\Re^1$ with saturated family of pseudometrics $\rho_I(\tau,\overline{\tau})=\sum\{|\tau(t)-\overline{\tau}(t)|: t\in I\}$ where $I\in A$ runs over all compact subsets $I\subset\Re^1$. Define the operator

$$\left(T^r\tau^{ret}\right)(t)\coloneqq\frac{1}{c}\sqrt{\sum_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-\tau^{ret}(t))]^2}$$

Under the assumption $\sqrt{\sum_{\gamma=1}^{3}\left(u_\gamma^{(p)}(t)\right)^2}\le\bar{c}<c\,(p=1,2)$ the operator T has a unique

fixed point. Indeed, if we denote by $P(t,y)=\dfrac{1}{c}\sqrt{\sum\limits_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-\tau^{ret}(t))]^2}$ then

$$\left|\frac{\partial P}{\partial y}\right|=\left|\frac{1}{c}\frac{\sum\limits_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-y)]\,\check{u}_\gamma(t-y)}{\sqrt{\sum\limits_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-y)]^2}}\right|\le\frac{\bar{c}}{c}\frac{\sqrt{\sum\limits_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-y)]^2}}{\sqrt{\sum\limits_{\gamma=1}^{3}[x_\gamma(t)-\check{x}_\gamma(t-y)]^2}}=\frac{\bar{c}}{c}.$$

Therefore $|(T^r\tau^{ret}-T^r\bar{\tau}^{ret})(t)|\le\dfrac{\bar{c}}{c}\rho_I(\tau^{ret},\bar{\tau}^{ret})$ and then $\rho_I(T^r\tau^{ret},T^r\bar{\tau}^{ret})\le\dfrac{\bar{c}}{c}\rho_I(\tau^{ret},\bar{\tau}^{ret})$.

Then Theorem 2 implies the existence of unique solutions for each of the equations $\tau^{ret}(t)=\dfrac{1}{c}\sqrt{\langle\xi^{ret},\xi^{ret}\rangle}$ and $\tau^{adv}(t)=\dfrac{1}{c}\sqrt{\langle\xi^{adv},\xi^{adv}\rangle}$ belonging to $X_1=X_2=C(\Re^1)$. (In some inequalities above we have replaced $\sqrt{\langle\xi^{ret},\xi^{ret}\rangle}$ by $c\tau^{ret}$ in order to obtain $\check{D}=\dfrac{c^2\tau^{ret}-\langle\check{u}^{ret},\xi^{ret}\rangle}{c^2\tau^{ret}-\langle u^{ret},\xi^{ret}\rangle}$).

Let $X_3=L^\infty_{loc}(\Re^1)\times L^\infty_{loc}(\Re^1)\times L^\infty_{loc}(\Re^1)$ be the uniform space with a saturated family of pseudometrics $\mathcal{A}=\{\rho_I(\cdot,\cdot):I\in A\}$ where the index set A consists of all compact subsets $I\subset\Re^1$ and

$$\rho_I\left(\{f_1,f_2,f_3\},\{\bar{f}_1,\bar{f}_2,\bar{f}_3\}\right)=\max\left\{\rho_I^1\left(f_1,\bar{f}_1\right),\rho_I^2\left(f_2,\bar{f}_2\right),\rho_I^3\left(f_3,\bar{f}_3\right)\right\},$$

$$\rho_I^k\left(f_k,\bar{f}_k\right)=\operatorname{ess\,sup}\left\{\left|f_k(t)-\bar{f}_k(t)\right|:t\in I\right\}\ (k=1,2,3).$$

Define the operator $T:X_3\to X_3$ by the formulas

$$(Tw)_\alpha(t)=\begin{cases}P_\alpha(w)(t),t>0\\ \\ \overline{w}_\alpha(t),t\le 0\end{cases}\quad(\alpha=1,2,3)$$

where the right hand side of (11) is denoted by $P_\alpha(w)$.

Introduce the set $M\subset X_3$ in the following way:

$$M=\{(f_1,f_2f_3)\in X_3:|f_k(t)|\le w_0(t),t\in\Re^1\ (k=1,2,3)\}$$

and show that T maps M into itself. We use the above estimates $\tau^{ret}(t)\ge\dfrac{\check{r}(t)}{2c}$. We need two more inequalities, namely $\check{r}(t)\ge\bar{r}(t)$ and $\hat{r}(t)\ge\bar{\bar{r}}(t)$. But Lemma 1 guarantees them. It is easy to verify that

$$\max\left\{\frac{(102\sqrt{3}+24)e^2}{mc^3(1-v)^5X_0},\frac{10e^2}{mc^3(1-v)^4X_0},\frac{10e^2}{mc^3(1-v)^4X_0}\right\}=\frac{(102\sqrt{3}+24)e^2}{mc^3(1-v)^5X_0}=A<1.$$

The inequalities from Section 5 and conditions of the Theorem 1 imply

$$|(Tw)_\alpha(t)| = |P_\alpha| \le \frac{3e^2}{m}\left[\frac{1}{c}\sqrt{\sum_{\alpha=1}^{3} H_\alpha^2} + \frac{1}{c^2}\sqrt{\sum_{\alpha=1}^{3} E_\alpha^2} + \frac{8c^2}{c^3\bar{r}^2(t)(1-v)^3} + \right.$$
$$\left. + \frac{(34\sqrt{3}+8)w_0(t)c}{c^4(1-v)^5\bar{r}(t)} + \frac{8c^2}{c^3\bar{\bar{r}}^2(t)(1-v)^3} + \frac{(34\sqrt{3}+8)w_0(t)c}{c^4(1-v)^5\bar{\bar{r}}(t)}\right] \le$$
$$\le \frac{3}{c}\frac{e^2}{m}\sqrt{\sum_{\alpha=1}^{3} H_\alpha^2} + \frac{3}{c^2}\frac{e^2}{m}\sqrt{\sum_{\alpha=1}^{3} E_\alpha^2} +$$
$$+ \frac{24e^2}{mc(1-v)^3}\left(\frac{1}{\bar{r}^2(t)} + \frac{1}{\bar{\bar{r}}^2(t)}\right)\frac{(102\sqrt{3}+24)e^2}{mc^3(1-v)^5X_0}w_0(t) \le w_0(t)$$

i.e. $Tw \in M$.

In what follows we show that T is k-contractive operator using the inequalities from Section 6 and (14) from the conditions of Theorem 5:

$$|(Tw)_\alpha(t) - (T\overline{w})_\alpha(t)| \le k\rho_{j(I)}(w,\overline{w}) \tag{15}$$

where $j_\alpha(I_{\alpha+2}) = \{t - \tau^{ret} : t \in I_{\alpha+2}\} \cup \{t - \tau^{ret} : t \in I_{\alpha+2}\}$ $(\alpha = 1, 2, 3)$.

Now we are able to define the Cartesian product $X = X_1 \times X_1 \times M$ with a saturated family of pseudometrics

$$\rho_I\left(\{\tau^{ret}, \tau^{adv}, f_1, f_2, f_3\}, \{\bar{\tau}^{ret}, \bar{\tau}^{adv}, \bar{f}_1, \bar{f}_2, \bar{f}_3\}\right) =$$
$$= \max\left\{\rho_I\left(\tau^{ret}, \bar{\tau}^{ret}\right), \rho_I\left(\tau^{adv}, \bar{\tau}^{adv}\right), \rho_I^1\left(f_1, \bar{f}_1\right), \rho_I^2\left(f_2, \bar{f}_2\right), \rho_I^3\left(f_3, \bar{f}_3\right)\right\},$$

where the elements of X are ordered 5-ples $\left(\tau^{ret}, \tau^{adv}, w_1, w_2, w_3\right)$.

Define the operator T by the formula

$$T\left(\tau^{ret}, \tau^{adv}, w_1, w_2, w_3\right) = \left(T^r\left(\tau^{ret}\right), T^r\left(\tau^{adv}\right), P_1(w), P_2(w), P_3(w)\right).$$

The index set for the saturated family of pseudometrics consists of all ordered 5-tuples i.e. $A = \{I_1, I_2, I_3, I_4, I_5\}$ where every I_k runs over all compact subsets of $\Re^1$. Define the map $J : A \to A$ in the following way $J\big(I_1, I_2, I_3, I_4, I_5\big) = \big(I_1, I_2, j_1\big(I_3\big), j_2\big(I_4\big), j_3\big(I_5\big)\big)$ where $j_\alpha\big(I_{\alpha+2}\big)$ are defined above. Further on $J^2\big(I_1, I_2, I_3, I_4, I_5\big) = \big(I_1, I_2, j_1^2\big(I_3\big), j_2^2\big(I_4\big), j_3^2\big(I_5\big)\big)$ and so on. Since the first components of J are identities it remains to show that M is j-bounded. Indeed, for $k = 1, 2, 3$ we have

$$\rho^k_{j_k^n(I_{k+2})}\big(f_k, \bar{f}_k\big) = \operatorname{esssup}\left\{|w_k(t) - \overline{w}_k(t)| : t \in j_k^n(I_{k+2})\right\} \le \operatorname{esssup}\left\{|2w_0(t)| : t \in \Re^1\right\} < \infty$$

$(n = 1, 2, \ldots)$.

Therefore the space $X = X_1 \times X_1 \times M$ is j- bounded.

The above estimates imply that T is a l-contractive operator with $l = \max\{k, v\} < 1$. Then Theorem 4 guarantees an existence of a unique solution for the initial value problem (12).

Theorem 5 is thus proved.

9 Non-existence of runaway solutions

We proceed as in [1] considering the one-dimensional case. We take the first equation without the external force.

Put $\dot{u}_1 = \dot{u}$, $w_1 = w$, $G_1 = G(G_2 = 0, G_3 = 0)$ and obtain

$$\dot{u} = \frac{1}{c^2}\left[\left(c^2 - u^2\right) G\right] = P_1 \implies w = \dot{u} = \frac{\Delta^2}{c^2} G.$$

We take $G = G_1$ from $(9.\alpha), \alpha = 1$ and obtain:

$$\dot{u} = \frac{e^2\Delta}{2mc^3}[B^{ret}\xi_1^{ret} + C^{ret}u(t-\tau^{ret}) + D^{ret}\dot{u}(t-\tau^{ret}) - \\ - B^{adv}\xi_1^{adv} - C^{adv}u(t+\tau^{adv}) - D^{adv}\dot{u}(t+\tau^{adv})]$$

Then we rewrite (with $\xi_1 = \xi$) the last equation as

$$w(t) = \frac{e^2\Delta}{2mc^3}[B^{ret}\xi^{ret} + C^{ret}u(t-\tau^{ret}) + D^{ret}w(t-\tau^{ret}) - \\ - B^{adv}\xi^{adv} - C^{adv}u(t+\tau^{adv}) - D^{adv}w(t+\tau^{adv})] \qquad (16)$$

Assume that $\lim_{t\to\infty} w(t) = \infty$. Therefore $\lim_{t\to\infty} w_0(t) = \infty$. It follows from Section 4 that $\lim_{t\to\infty} \tau^{adv}(t) = \infty$, $\tau^{adv}(t) \geq \beta t$.

Divide (16) into $w_0(t)$ and let $t \to \infty$. Then in the left hand-side of (16) we obtain a constant, different from zero, while in the right hand-side we have:

$$\frac{1}{w_0(t)}|B^{adv}\xi_1^{adv}| \leq \frac{2\bar{c}\tau^{adv}(t)}{c^2 w_0(t)\big(\tau^{adv}(t)\big)^3(1-v)^3} + \frac{9\sqrt{3}\bar{c}\tau^{adv}(t)}{c^3(1-v)^5 w_0(t)\big(\tau^{adv}(t)\big)^2} \to 0;$$

$$\frac{1}{w_0(t)}|C^{adv}u(t+\tau^{adv})| \leq \frac{2\bar{c}\tau^{adv}(t)}{c^2(1-v)^3 w_0(t)\big(\tau^{adv}(t)\big)^2} + \frac{8\sqrt{3}\bar{c}w_0(t)}{c^3(1-v)^5 w_0(t)\tau^{adv}(t)} \to 0;$$

$$\frac{1}{w_0(t)}|D^{adv}w(t+\tau^{adv})| \leq \frac{4w_0(t)}{c^2(1-v)^3 w_0(t)\tau^{adv}(t)} \to 0$$

and analogously for $|B^{ret}\xi_1^{adv}|$, ..., $|D^{ret}w(t+\tau^{adv})|$ i.e. (16) is violated for sufficiently large t. Consequently a runaway solution $\lim_{t\to\infty} w(t) = \infty$ does not exist.

10 The finiteness of the electron energy obtained as a solution of a neutral equation

For the kinetic energy of the electron we have

$$E_{kin}(t) = \frac{mc^2}{\sqrt{1 - \dfrac{u^2(t)}{c^2}}} = \frac{mc^3}{\sqrt{c^2 - u^2(t)}} = \frac{mc^3}{\Delta}$$

$$\frac{dE_{kin}(t)}{dt} = -\frac{1}{2}\frac{mc^3(-2)\langle u(t),\dot{u}(t)\rangle}{\Delta^3} = \frac{mc^3\langle u(t),\dot{u}(t)\rangle}{\Delta^3}$$

Consequently

$$\frac{dE_{kin}(t-\tau^{ret})}{dt} = \frac{mc^3\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\Delta^3},$$
$$\frac{dE_{kin}(t-\tau^{adv})}{dt} = \frac{mc^3\langle u(t-\tau^{adv}),\dot{u}(t-\tau^{adv})\rangle}{\Delta^3}.$$

Then we can transform the fourth equation (4.4). Indeed we have

$$\begin{aligned}\frac{\langle u(t),\dot{u}(t)\rangle}{\Delta^2} &= -\frac{e^2\Delta}{mc^3}\langle E,u(t)\rangle + \frac{e^2\Delta}{2mc^3}\Bigg\{\check{H}\frac{\tau^{ret}\langle u(t),u(t-\tau^{ret})\rangle - \langle\xi^{ret},u(t)\rangle}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^3} + \\ &+ \check{D}\Bigg[\frac{\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle\left(\langle\xi^{ret},u(t)\rangle - c^2\tau^{ret}\right)}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^2} - \\ &- \tau^{ret}\frac{\check{\Delta}^2\langle u(t),\dot{u}(t-\tau^{ret})\rangle + \left(\langle u(t),u(t-\tau^{ret})\rangle - c^2\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^2}\Bigg] - \\ &- \hat{H}\frac{\tau^{adv}\langle u(t),u(t+\tau^{adv})\rangle - \langle\xi^{adv},u(t)\rangle}{[\langle\xi^{adv},u(t+\tau^{adv})\rangle - c^2\tau^{adv}]^3} - \hat{D}\Bigg[\frac{\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle\left(\langle\xi^{adv},u(t)\rangle - c^2\tau^{adv}\right)}{[\langle\xi^{adv},u(t+\tau^{adv})\rangle - c^2\tau^{adv}]^2} - \\ &- \tau^{adv}\frac{\hat{\Delta}^2\langle u(t),\dot{u}(t+\tau^{adv})\rangle + \left(\langle u(t),u(t+\tau^{adv})\rangle - c^2\right)\langle u(t+\tau^{adv}),\dot{u}(t+\tau^{adv})\rangle}{[\langle\xi^{adv},u(t+\tau^{ret})\rangle - c^2\tau^{adv}]^2}\Bigg]\Bigg\},\end{aligned} \tag{4.4}$$

where

$$\check{H} = \check{\Delta}^2 + \check{D}\frac{\check{\Delta}^2\langle\xi^{ret},\dot{u}(t-\tau^{ret})\rangle + \left(\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right)\langle u(t-\tau^{ret}),\dot{u}(t-\tau^{ret})\rangle}{\check{\Delta}^2},$$
$$\hat{H} = \hat{\Delta}^2 + \hat{D}\frac{\hat{\Delta}^2\langle\xi^{adv},\dot{u}(t+\tau^{adv})\rangle + \left(\langle\xi^{adv},u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right)\langle u(t+\tau^{ret}),\dot{u}(t+\tau^{adv})\rangle}{\hat{\Delta}^2}.$$

After obvious transformations we obtain

$$\begin{aligned}\frac{dE_{kin}(t)}{dt} &= -e^2\langle E,u(t)\rangle + \frac{e^2}{2}\Bigg\{\check{\Delta}^2\frac{\tau^{ret}\langle u(t),u(t-\tau^{ret})\rangle - \langle\xi^{ret},u(t)\rangle}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^3} + \\ &+ \frac{\tau^{ret}\langle u(t),u(t-\tau^{ret})\rangle - \langle\xi^{ret},u(t)\rangle}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^3}\check{D}\langle\xi^{ret},\dot{u}(t-\tau^{ret})\rangle - \\ &- \frac{\check{D}\tau^{ret}\check{\Delta}^2\langle u(t),\dot{u}(t-\tau^{ret})\rangle}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^2} + \\ &+ \Bigg[\frac{\tau^{ret}\langle u(t),u(t-\tau^{ret})\rangle - \langle\xi^{ret},u(t)\rangle}{[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}]^2}\frac{\check{D}\check{\Delta}}{mc^3} + \frac{\check{D}\check{\Delta}^3\left(\langle\xi^{ret},u(t)\rangle - c^2\tau^{ret}\right)}{mc^3\left[\langle\xi^{ret},u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^2} -\end{aligned}$$

$$- \frac{\check{D}\tau^{ret}\check{\Delta}^3\left(\langle u(t), u(t-\tau^{ret})\rangle - c^2\right)}{mc^3\left[\langle \xi^{ret}, u(t-\tau^{ret})\rangle - c^2\tau^{ret}\right]^2}\Bigg] \frac{dE_{kin}(t-\tau^{ret})}{dt} -$$

$$- \Bigg[\hat{\Delta}^2 \frac{\tau^{adv}\langle u(t), u(t+\tau^{adv})\rangle - \langle \xi^{adv}, u(t)\rangle}{\left[\langle \xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^3} +$$

$$+ \frac{\tau^{adv}\langle u(t), u(t+\tau^{adv})\rangle - \langle \xi^{adv}, u(t)\rangle}{\left[\langle \xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^3}\hat{D}\langle \xi^{adv}, \dot{u}(t-\tau^{adv})\rangle -$$

$$- \frac{\hat{D}\tau^{adv}\hat{\Delta}^2\langle u(t), \dot{u}(t+\tau^{adv})\rangle}{\left[\langle \xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^2}\Bigg] -$$

$$- \Bigg[\frac{\tau^{adv}\langle u(t), u(t+\tau^{adv})\rangle - \langle \xi^{adv}, u(t)\rangle}{\left[\langle \xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^2}\frac{\hat{D}\hat{\Delta}}{mc^3} + \frac{\hat{D}\hat{\Delta}^3\left(\langle \xi^{adv}, u(t)\rangle - c^2\tau^{adv}\right)}{mc^3\left[\langle \xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^2} -$$

$$- \frac{\hat{D}\tau^{adv}\hat{\Delta}^3\left(\langle u(t), u(t+\tau^{adv})\rangle - c^2\right)}{mc^3\left[\langle \xi^{adv}, u(t+\tau^{adv})\rangle - c^2\tau^{adv}\right]^2}\Bigg] \frac{dE_{kin}(t+\tau^{adv})}{dt}\Bigg\}.$$

Therefore the above equation is a conservation law for the energy. After obvious denotations we obtain:

$$\frac{dE_{kin}(t)}{dt} = -e^2\langle E, u(t)\rangle + \frac{e^2}{2}\left[A^{ret} + B^{ret}\frac{dE_{kin}(t-\tau^{ret})}{dt} - A^{adv} - B^{adv}\frac{dE_{kin}(t+\tau^{adv})}{dt}\right] \tag{17}$$

The first term in the right-hand side of (17) is the work done by internal forces, while the second one is the energy loss of the accelerated electron. From the mathematical point of view the last equation is a neutral functional differential equation with both retarded and advanced arguments. One can formulate the following initial value problem: to find a solution of (17) for $t > 0$ and $E_{kin}(t) = E^0_{kin}(t)$, $\frac{dE_{kin}(t)}{dt} = \frac{dE^0_{kin}(t)}{dt}$ for $t \le 0$ where $E^0_{kin}(t)$ is a prescribed function. The last problem can be reformulated as the one for a functional equation, namely putting

$$K(t) = \frac{dE_{kin}}{dt}, t > 0 \text{ and } K_0(t) = \frac{dE^0_{kin}}{dt}, t \le 0$$

$$K(t) = -e^2\langle E, u(t)\rangle + \frac{e^2}{2}\left[A^{ret} + B^{ret}K(t-\tau^{ret}) - A^{adv} - B^{adv}E(t+\tau^{adv})\right] \equiv \Pi(K) \tag{18}$$

$$K(t) = K_0(t), t \le 0$$

We show that there exists a bounded solution of the above equation, i.e. $|K(t)| \le \breve{E}$ which implies the finiteness of the energy. As above we have for the right-hand side of (18):

$$|\Pi(K)(t)| \le e^2|E|\bar{c} + \frac{e^2}{(1-v)^4}\left[\frac{4}{\left(\bar{r}(t)\right)^2} + \frac{2w_0(t)}{\bar{r}(t)}\left(\frac{4}{c^2}+1\right)\right] + \frac{4e^2}{\bar{r}(t)m(1-v)^3}\left(\frac{1}{c^5}+\frac{2}{c^3}\right)\breve{E} +$$

$$+ \frac{e^2}{(1-v)^4}\left[\frac{4}{\left(\bar{\bar{r}}(t)\right)^2} + \frac{2w_0(t)}{\bar{\bar{r}}}\left(\frac{4}{c^2}+1\right)\right] + \frac{4e^2}{\bar{\bar{r}}(t)m(1-v)^3}\left(\frac{1}{c^5}+\frac{2}{c^3}\right)\breve{E} \le$$

$$\leq e^2|E|\bar{c} + \frac{2e^2}{(1-v)^4}\left[\frac{4}{X_0^2} + \frac{3w_0(t)}{X_0}\right] + \frac{6e^2}{X_0 m(1-v)^3 c^3}\breve{E}\,. \qquad (19)$$

Theorem 6. *If the initial function satisfies $|K_0(t)| \leq \breve{E}$, $t \leq 0$ and the following inequalities are valid*

$$\mathbf{6.1)}\ k = \frac{6e^2}{X_0 m(1-v)^3 c^3} < 1; \quad \mathbf{6.2)}\ e^2|E|\bar{c} + \frac{2e^2}{(1-v)^4}\left[\frac{4}{X_0^2} + \frac{3w_0(t)}{X_0}\right] \leq (1-k)\,\breve{E},$$

then the initial value problem (18) *has a unique bounded solution.*

Proof. Introduce the set $M \subset L^\infty(\Re^1)$ in the following way:

$$M = \left\{K(\cdot) \in L^\infty(\Re^1) : K_0(t)| \leq \breve{E}, t > 0 \text{ and } K(t) = K_0(t), t \leq 0\right\}$$

It turns out into a uniform space by the saturated family of pseudomerics $\mathcal{A} = \left\{\rho_I\big(K, \overline{K}\big) : t \in I\right\}$, where $\rho_I\big(K, \overline{K}\big) = \left\{|K(t) - \overline{K}(t)| : t \in I\right\}$ and A consists of all compact subsets of $\Re^1$. Define the mapping $j : A \to A$ by the formula

$$j(I) = \left\{t - \tau^{ret}(t) : t \in I\right\} \cap \left\{t + \tau^{adv}(t) : t \in I\right\}.$$

It is easy to see that M is j-bounded. Indeed, $\rho_{j^n(I)}\big(K, \overline{K}\big) \leq 2\breve{E} < \infty\,(n = 0, 1, 2, \ldots)$.

Define the operator $T : M \to M$ by the formula $(TK)(t) := \begin{cases} \Pi(K)(t), t > 0 \\ K_0(t), t \leq 0 \end{cases}$.

The inequalities (19) and condition **6.2)** imply that $|(TK)(t)| \leq \breve{E}$, that is T maps M into itself. On the other hand **6.1)** shows that $\rho_I\big(TK, T\hat{K}\big) \leq k\rho_{j(I)}(K, \hat{K})$. Therefore the unique fixed point of T is a solution of (18) which proves the Theorem 6.

11 Conclusion remarks

The obtained Dirac equations do not possess unphysical solutions which are already proved in the one-dimensional case [1]. For the applications we present some estimates of the upper bound of the velocities, namely $v = \frac{\bar{c}}{c} < 1$. In order to justify the introducing of $\bar{c}$ we recall that in Newton theory the maximum of the velocities is ∞ but every body moves with finite velocity $|u| < \infty$. In the same way in Einstein relativity theory we can assume that $|u| < c$, i.e. c could not be reached such like ∞.

The estimate one can obtained from (13) is

$$\frac{(102\sqrt{3} + 24)e^2}{mc^3(1-v)^5 X_0} = A < 1 \implies \sqrt[5]{\frac{(102\sqrt{3} + 24)e^2}{mc^3(1-v)^5 X_0}} < 1 - v \implies v < 1 - \sqrt[5]{\frac{(102\sqrt{3} + 24)e^2}{mc^3(1-v)^5 X_0}}$$

and consequently

$$\sqrt[5]{\frac{(102\sqrt{3} + 24)e^2}{mc^3(1-v)^5 X_0}} \approx \sqrt[5]{\frac{(102.(1,73) + 24)(1,6.10^{-19})^2}{9.10^{-31}.27.10^{24} X_0}} \approx \sqrt[5]{\frac{(2,11).10^{-31})}{X_0}}.$$

We would like to point out: to speak about electron radius (and hence for a radiation time) is incorrect because bodies with finite size could not be considered in the special relativity. It is well-known that Coulomb potentials become infinity at the point where the electron is. The contradiction generates the Coulomb law because his formulation assumes that the interaction propagates with infinite velocity in analogy with gravitational Newton law. Consequently we can speak about radiation distance at the initial instant, denoted here by X_0. Obviously our method does not insist some restrictions on X_0. If we choose $X_0 \approx 1$. then $\bar{c} \leq \left(1 - 10^{-6}\right)c$ which is sufficient for the applications (cf. [12]-[14], [26]-[29]). Obviously, if $X_0 \to \infty$ then $\bar{c} \to c$.

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