LIBERTAS MATHEMATICA, vol XXXI (2011)

Stability Properties and Almost Periodic Solutions of Abstract Functional Differential Equations with Infinite Delay

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Abstract. In order to obtain the existence theorem for almost periodic solutions in an abstract functional differential equation, we consider the relationship between (K, ρ) weakly uniformly asymptotic stabilities and (K, ρ) uniformly asymptotic stabilities.

Keywords: almost periodic solutions, abstract functional equation, weakly uniformly asymmptotically stable.

1 Introduction

For ordinary differential equations and functional differential equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors. One of the most popular method is to assume the certain stability properties [1,2,3,4,14]. Hamaya [5] has shown the existence of almost periodic solutions for abstract functional differential equations on a fading memory space B by assuming the existence of a bounded solution which is (K, ρ)-stability condition. This result is to extend results in Murakami and Yoshizawa [11] to abstract functional differential equations.

In order to obtain the existence theorem for almost periodic solution in ordinary differential equations, Sell [12] introduced a new stability concept which is referred to as the weakly uniformly asymptotically stable. This stability property is weaker than uniformly asymptotically stable (cf.[14]). Recently [13] has studied that the existence of almost periodic solutions of ordinary difference (or differential) equation by using globally quasi-uniformly asymptotically stable.

In this paper, we shall discuss the relationship between weakly uniformly asymptotic stabilities and uniformly asymptotic stabilities in periodic and almost periodic abstract functional differential equations, and show that for a periodic abstract functional differential equation, (K, ρ)-weakly uniformly asymptotic stability and (K, ρ)-uniformly asymptotic stability are equivalent, and moreover, we obtain the existence of almost periodic solutions in abstract functional differential equations by using this (K, ρ)-weakly uniformly asymptotically stable. It seems that the relationship between our weakly uniformly asymptotic stability and globally quasi-uniformly asymptotic stability in [13] is very complicated and sophisticated, however the definition of our stability are clearer and simpler than their in

[13]. Finally, as applications, we show that the existence of almost periodic solutions for nonlinear integrodifferential equations with diffusion.

2 Phase space B and notations

Let X denote Banach space and $|\cdot|_X$ will denote the norm in X. For any interval $I \subset R := (-\infty, \infty)$, we denote by $BC = BC(I : X)$ the set of all bounded continuous functions mapping I into X and set $|\phi|_{BC} = \sup\{|\phi(s)|_X : s \in I\}$. Then, $BC(I : X)$ is a Banach space with the norm $|\phi|_{BC}$.

Now, for any function $x : (-\infty, a) \to X$ and $t < a$, define a function $x_t : R^- = (-\infty, 0] \to X$ by $x_t(s) = x(t+s)$ for $s \in R^-$. Let $B = B(R^- : X)$ be a real linear space of functions mapping R^- into X with a complete seminorm $|\cdot|_B$. We assume the following conditions on the space B.

(A1) There exist positive constants J, L and M with the property that if $x : (-\infty, a) \to X$ is continuous on $[\sigma, a)$ with $x_\sigma \in B$ for some $\sigma < a, \sigma, a \in R$, then for all $t \in [\sigma, a)$,

(*i*) $x_t \in B$,

(*ii*) x_t is continuous in t (with respect to $|\cdot|_B$),

(*iii*) $J|x(t)|_X \le |x_t|_B \le L \sup_{\sigma \le s \le t} |x(s)|_X + M|x_\sigma|_B$,

(A2) If $\{\phi^k\}$ is a sequence in $B \cap BC$ converging to a function ϕ uniformly on any compact interval in R^- and $\sup_k |\phi^k|_{BC} < \infty$, then $\phi \in B$ and $|\phi^k - \phi|_B \to 0$ as $k \to \infty$.

We hold that the space B contains BC and that there is a constant $l > 0$ such that

$$|\phi|_B \le l|\phi|_{BC} \quad \text{for all} \quad \phi \in BC. \tag{2.1}$$

Set $B_0 = \{\phi \in B : \phi(0) = 0\}$, and define an operator $S(t) : B_0 \to B_0$ by

$$[S(t)\phi](s) = \begin{cases} \phi(t+s) & \text{if} \quad s \le -t, \\ 0 & \text{if} \quad -t \le s \le 0 \end{cases}$$

for each $t \ge 0$ and $\phi \in B_0$. By (A1), one can see that the family $\{S(t)\}_{t \ge 0}$ is a strong continuous semigroup of bounded linear operators on B_0. The space B is called a fading memory space for X, if it satisfies the following fading memory condition together with (A1) and (A2).

(A3) $\lim_{t\to\infty} |S(t)\phi|_B = 0, \quad \phi \in B_0$.

It is well known ([7,8]) that the following typical example of fading memory spaces. Let $g : R^- \to [1, \infty)$ be any continuous nonincreasing function such that $g(0) = 1$ and $g(s) \to \infty$ as $s \to -\infty$. We set

$$C_g^0(X) = \{\phi : R^- \to X \text{ is continuous with } \lim_{s\to-\infty} |\phi(s)|_X / g(s) = 0\}.$$

Then the space C_g^0 equipped with the norm

$$|\phi|_g = \sup_{s\leq 0} |\phi(s)|_X / g(s), \qquad \phi \in C_g^0,$$

is a separable Banach space and satisfies (A1), (A2) and (A3).

We introduce an almost periodic function $f(t,x) : R \times B \to X$.

Definition 2.1. $f(t,x)$ is said to be almost periodic in t uniformly for $x \in B$, if for any $\epsilon > 0$ and any compact set K in B, there exists a positive number $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains a τ for which

$$|f(t+\tau, x) - f(t,x)|_X \leq \epsilon \tag{2.2}$$

for all $t \in R$ and all $x \in K$. Such a number τ in (2.2) is called an ϵ-translation number of $f(t,x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above difinition, we discuss the concept of the normality of almost periodic functions. Namely, Let $f(t,x)$ be almost periodic in t uniformly for $x \in B$. Then, for any sequence $\{h_k^{'}\} \subset R$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ and function $g(t,x)$ such that

$$f(t+h_k, x) \to g(t,x) \tag{2.3}$$

uniformly on $R \times K$ as $k \to \infty$, where K is any compact set in B. We shall denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(t,x) = f(t+\tau, x), \qquad \tau \in R \tag{2.4}$$

Let $H(f)$ denote the closure of $T(f)$ in the sense of (2.4). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{t_k\}, t_k \to \infty$ as $k \to \infty$ and $f(t+t_k, x) \to g(t,x)$ uniformly on $R \times S$ for any compact subset S in B. By (2.3), if $f : R \times B \to X$ is almost periodic in t uniformly for $x \in B$, so is a function in $\Omega(f)$. The following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous function (cf.[8,14]).

Definition 2.2. Let $u : [a, \infty) \to X$ be a continuous function. $u(t)$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $p(t)$ and a continuous function $q(t)$ defined on $I^* = [a, \infty) \subset R^+$ which tends to zero as $t \to \infty$, that is,

$$u(t) = p(t) + q(t).$$

$u(t)$ is asymptotically almost periodic if and only if for any sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{t_{k_j}\}$ for which $u(t+t_{k_j})$ converges uniformly on $a \leq t < \infty$, under the additional assumption that the set $\{u(t) : t \geq a\}$ is relatively compact in X.

3 Definitions of (K, ρ)-stabilities

We shall consider the almost periodic solution of an abstract functional differential equation

$$\frac{dx}{dt} = Ax(t) + f(t, x_t), \tag{3.1}$$

where A is the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X and $f : R \times B \to X$. We impose the following assumptions:

(H1) For any $H > 0$, there is an $L_0(H) > 0$ such that $|f(t, \phi)|_X \leq L_0(H)$ for all $t \in R$ and $\phi \in B$ such that $|\phi|_B \leq H$.

(H2) $f(t, \phi) \in C(R \times B : X) :=$ the set of continuous functions defined on $R \times B$ with values in X, and almost periodic in t uniformly for $x \in B$.

(H3) Eq.(3.1) has a bounded solution $u(t)$ defined on R^+ which passes through $(0, u_0)$, that is $|u_t|_B < \infty$ for all $t \in R^+$ and $u_0 \in BC$.

By (H1) and (H2), it follows that for any $(\tau, \phi) \in R \times B$, there exists a function $y \in C((-\infty, \alpha) : X)$ such that $y_\tau = \phi$ and the following relation holds

$$y(t) = T(t - \tau)\phi(0) + \int_\tau^t T(t - s)f(s, y_s)ds, \qquad \tau \leq t < \alpha,$$

(cf.[8,11]). The function y is called the mild solution of equation (3.1) defined on $[\tau, \alpha)$ through (τ, ϕ) and denoted by $x(\cdot)$. In the above, α, can be taken as $\alpha = \infty$ if $\sup_{t<\alpha} |y(t)|_X < \infty$. We can see that the set of the closure of $\{u(t) : t \in R^+\}$ is compact set in X, $u(t)$ is uniformly continuous on R^+ and the set

$$\Gamma(u) := \text{the closure of } \{u_t : t \in R^+\}$$

is compact in B (cf.[8,11]).

Now we introduce BC-stability properties and ρ-stability properties with respect to the compact set K and the metric ρ.

Definition 3.1. The bounded solution $u(t)$ of Eq.(3.1) is said to be:

(i) (K, ρ)-stable (in short, (K, ρ)-S) if for any $\epsilon > 0$ there exists a $\delta(t_0, \epsilon) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \delta(t_0, \epsilon)$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in \mathrm{K}$ for all $s \leq 0$.

(ii) (K, ρ)-uniformly stable (in short, (K, ρ)-US) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \delta(\epsilon)$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in \mathrm{K}$ for all $s \leq 0$.

(iii) (K, ρ)-equi asymptotically stable (in short, (K, ρ)-EAS) if it is (K, ρ)-S and for any $\epsilon > 0$ there exists a $\delta_0(t_0) > 0$ and a $T(t_0, \epsilon) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \delta_0(t_0)$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0 + T(t_0, \epsilon)$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in \mathrm{K}$ for all $s \leq 0$.

(*iv*) (K, ρ)-weakly uniformly asymptotically stable (in short, (K, ρ)-WUAS) if it is (K, ρ)-US and there exists a $\delta_0 > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \delta_0$, then $\rho(x_t, u_t) \to 0$ as $t \to \infty$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in$ K for all $s \leq 0$.

(*v*) (K, ρ)-uniformly asymptotically stable (in short, (K, ρ)-UAS) if it is (K, ρ)-US and is (K, ρ)-quasi uniformly asymptotically stable, that is, if the δ_0 and the T in above (iii) are independent of t_0: (for any $\epsilon > 0$ there exists a $\delta_0 > 0$ and a $T(\epsilon) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \delta_0$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0 + T(\epsilon)$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in$ K for all $s \leq 0$.)

(*vi*) (K, ρ)-globally equi asymptotically stable (in short, (K, ρ)-GEAS) if it is (K, ρ)-S and for any $\epsilon > 0$, any $\alpha > 0$ there exists a $T(t_0, \epsilon, \alpha) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \alpha$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0 + T(t_0, \epsilon, \alpha)$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in$ K for all $s \leq 0$.

(*vii*) (K, ρ)-globally weakly uniformly asymptotically stable (in short, (K, ρ)-GWUAS) if it is (K, ρ)-US and $\rho(x_t, u_t) \to 0$ as $t \to \infty$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in$ K for all $s \leq 0$.

(*viii*) (K, ρ)-globally uniformly asymptotically stable (in short, (K, ρ)-GUAS) if it is (K, ρ)-US and is (K, ρ)-globally quasi uniformly asymptotically stable, that is, if the T in above (vi) are independent of t_0: (for any $\epsilon > 0$ there exists a $T(\epsilon, \alpha) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \alpha$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0 + T(\epsilon, \alpha)$, where $x(t)$ is a solution of (3.1) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in$ K for all $s \leq 0$.)

(*ix*) (K, ρ)-eventually totally stable (in short, (K, ρ)-ETS) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ and $\alpha(\epsilon)$ such that if $t_0 \geq \alpha(\epsilon)$, $\rho(x_{t_0}, u_{t_0}) < \delta(\epsilon)$ and $h \in BC([t_0, \infty))$ which satisfies $|h|_{[t_0,\infty)} < \delta(\epsilon)$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0$, where $x(t)$ is a solution of

$$\frac{dx}{dt} = Ax(t) + f(t, x_t) + h(t), \tag{3.2}$$

through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in$ K for all $s \leq 0$. If we can choose $\alpha(\epsilon) \equiv 0$, then $u(t)$ is said to be (K, ρ)-totally stable (in short, (K, ρ)-TS). In the case where $h(t) \equiv 0$, this gives the definition of the (K, ρ)-US of $u(t)$.

(*x*) (K, ρ)-attracting in $\Omega(f, F)$ (in short, (K, ρ)-A in $\Omega(f, F)$) if there exists a $\delta_0 > 0$ such that if $t_0 \geq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(x_{t_0}, v_{t_0}) < \delta_0$, then $\rho(x_t, v_t) \to 0$ as $t \to \infty$, where $x(t)$ is a solution of

$$\frac{dx}{dt} = Ax(t) + g(t, x_t), \tag{3.3}$$

through (t_0, ψ) such that $x_{t_0}(s) = \psi(s) \in$ K for all $s \leq 0$.

(*xi*) (K, ρ)-weakly uniformly asymptotically stable in $\Omega(f, F)$ (in short, (K, ρ)-WUAS in $\Omega(f, F)$) if it is (K, ρ)-US in $\Omega(f, F)$, that is if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $t_0 \geq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(x_{t_0}, v_{t_0}) < \delta(\epsilon)$, then $\rho(x_t, v_t) < \epsilon$ for all $t \geq t_0$, where $x(t)$ is a solution of (3.3) through (t_0, ψ) such that $x_{t_0}(s) = \psi(s) \in$ K for all $s \leq 0$, and (K, ρ)-A in $\Omega(f, F)$.

For (*iv*) and (*v*) in the above Definition 3.1, actually, the (K, ρ)-WUAS is weaker than the (K, ρ)-UAS as [13, Example 3.1] shows.

4 Stability of Bounded Solutions in Periodic and Almost Periodic Systems

As there are complete proofs of these results in [cf.6], we omit the proofs of theses theorems for simplicity.

Theorem 4.1. *Under the assumptions (H3) and (H4), if the bounded solution $u(t)$ of Eq.(3.1) is (K, ρ)-WUAS, then it is (K, ρ)-EAS.*

For the periodic system, we have the following theorem.

Theorem 4.2. *Under the assumptions (H1),(H3) and (H4), if the bounded solution $u(t)$ of Eq.(3.1) is (K, ρ)-WUAS, then it is (K, ρ)-UAS.*

For the almost periodic system (3.1), we have the following theorem.

Theorem 4.3. *Under the above assumptions (H2), (H3) and (H4), if the zero solution $u(t) \equiv 0$ of Eq.(3.1) is (K, ρ)-WUAS then it is (K, ρ)-UAS.*

The following Theorem can be proved by the same argument as in the proof of Theorem 4.1.

Theorem 4.4. *Under the assumptionss (H3) and (H4), if the bounded solution $u(t)$ of Eq.(3.1) is (K, ρ)-GWUAS, then it is (K, ρ)-GEAS.*

Theorem 4.5. *Assume conditions (H1), (H3) and (H4). Then the solution $u(t)$ of Eq.(3.1) is (K, ρ)-GWUAS implies the solution $u(t)$ of Eq.(3.1) is (K, ρ)-GUAS.*

For the ordinary differential equation, it is well known that a example in [14, pp 81] is of a scalar almost periodic equation such that the zero solution is GWUAS but is not GUAS.

Theorem 4.6. *Under the assumption (H2), (H3) and (H4), if the solution $u(t)$ of Eq.(3.1) is (K, ρ)-WUAS in $\Omega(f, F)$, then the solution $u(t)$ of Eq.(3.1) is (K, ρ)-ETS. Moreover, if $u(t)$ is the unique solution of Eq.(3.1) through (0, ϕ^0), then $u(t)$ is (K, ρ)-TS.*

We have the following existence theorem of an almost periodic solution for Eq.(3.1).

Theorem 4.7. *Under the assumption (H2), (H3) and (H4), if the solution $u(t)$ of Eq.(3.1) is (K, ρ)-WUAS in $\Omega(f, F)$ and $u(t)$ is the unique solution of Eq.(3.1) through (0, ϕ^0), then the Eq.(3.1) has an almost periodic solution.*

5 Applications in population model

We shall consider the existence of a strictly positive componentwise almost periodic solution of a system of Volterra differential equations

$$\begin{aligned} \frac{\partial u_1}{\partial t}(t,x) &= \Delta u_1(t,x) + u_1(t,x)\Big\{b_1(t) - a_1(t)u_1(t,x) - c_2(t)\int_{-\infty}^{t} K_2(t-s)u_2(s,x)ds\Big\} \\ \frac{\partial u_2}{\partial t}(t,x) &= \Delta u_2(t,x) + u_2(t,x)\Big\{-b_2(t) - a_2(t)u_2(t,x) + c_1(t)\int_{-\infty}^{t} K_1(t-s)u_1(s,x)ds\Big\} \\ \frac{\partial u_i}{\partial n}(t,x) &= 0 \quad t>0,\ \ x\in\partial\Omega, \end{aligned} \tag{5.1}$$

which describes a model of the dynamics of a prey-predator system with diffusion in mathematical ecology. Function $u_1, u_2 \in C(R^+ \times \bar{\Omega}, R)$ is called a (classical) solution of (5.1) if $\partial u_i/\partial t$, $\partial u_i/\partial x$ and $\partial^2 u_i/\partial x^2$ for $i = 1,2$ belong to the space $C(R^+ \times \Omega)$, $\partial u_i/\partial n$ exist on $R^+ \times \partial\Omega$ and (5.1) is identically satisfied. From [8], we can show that the existence of solution is guranteed for (5.1) whenever the initial function

$$u_i(\theta,x) = \Phi_i(\theta,x) \ge 0, x\in\bar{\Omega}, \quad \text{belong to the } (\theta,x)\in(-\infty,0]\times C^1(\bar{\Omega}) \ \ (i=1,2).$$

We can regard Eq.(3.1) as the following functional differential equation with axiomatic phase space B and (K, ρ) topology (cf. [4,7]):

$$\frac{dx}{dt} = Ax(t) + h(t,x_t), \quad t\in R^+, \tag{5.2}$$

where $h : R^+ \times B \to R^2$. Then, we also hold Theorem (4.1, 4.2, 4.3, 4.4, 4.5, 4.6 and) 4.7 for (5.2), and we can treat the system (5.1) as an application of Eq.(5.2). For system (5.1), it is sufficient for the proofs of our results that we tret the change of diffusion terms $\dfrac{\partial u_i}{\partial t}(t,x)$ to $\dfrac{du_i(t)}{dt}$, $(i = 1,2)$. Then, system (5.1) can be written as

$$\begin{aligned} \frac{du_1(t)}{dt} &= u_1(t)\Big\{b_1(t) - a_1(t)u_1(t) - c_2(t)\int_{-\infty}^{t} K_2(t-s)u_2(s)ds\Big\} \\ \frac{du_2(t)}{dt} &= u_2(t)\Big\{-b_2(t) - a_2(t)u_2(t) + c_1(t)\int_{-\infty}^{t} K_1(t-s)u_1(s)ds\Big\}. \end{aligned} \tag{5.3}$$

In (5.3), setting $a_i(t)$ and $b_i(t)$ are R-valued bounded almost periodic function in R, $a_i = \inf_{t\in R} a_i(t)$, $A_i = \sup_{t\in R} a_i(t)$, $b_i = \inf_{t\in R} b_i(t)$, $B_i = \sup_{t\in R} b_i(t)$, $c_i = \inf_{t\in R} c_i(t)$ and $C_i = \sup_{t\in R} c_i(t)$ $(i = 1,2)$, and $K_i : R^+ \to R^+$ $(i = 1,2)$ denote delay kernels such that

$$K_i(s) \ge 0, \quad \int_0^\infty K_i(s)ds = 1 \quad \text{and} \quad \int_0^\infty sK_i(s)ds < \infty \ \ (i=1,2).$$

We set

$$\alpha_1 = B_1/a_1,\ \ \alpha_2 = B_1C_1/a_1a_2,\ \ \beta_1 = b_1/A_1 - B_1C_1C_2/a_1a_2A_1,$$

and

$$\beta_2 = b_1c_1/A_1A_2 - B_2/A_2 - B_1c_1C_1C_2/a_1a_2A_1A_2$$

(cf. [4.1 Application 4, 7]). We now make the following assumptions:

(i) $a_i > 0$, $b_i > 0$ $(i = 1, 2)$ and $c_1 > 0$, $c_2 \geq 0$,

(ii) $b_1 > B_1C_1C_2/a_1a_2$, and $B_2 < b_1c_1/A_1 - B_1c_1C_1C_2/a_1a_2A_1$,

(iii) there exists a positive constant m such that

$$a_i > C_i + m \quad (i = 1, 2).$$

Then, we have $0 < \beta_i < \alpha_i$ for each $i = 1, 2$. If $u(t) = (u_1(t), u_2(t))$ is a solution of (5.3) through $(0, \phi)$ such that $\beta_i \leq \phi(s) \leq \alpha_i (i = 1, 2)$ for all $s \leq 0$, then we have $\beta_i \leq u_i(t) \leq \alpha_i (i = 1, 2)$ for all $t \geq 0$. Let K be the closed bounded set in R^2 such that

$$K = \{(u_1, u_2) \in R^2; \beta_i \leq u_i \leq \alpha_i \text{ for each } i = 1, 2\}.$$

Then K is invariant for system (5.3), that is we can see that for any $t_0 \in R$ and any φ such that $\varphi(s) \in K$, $s \leq 0$, every solution of (5.3) through (t_0, φ) remains in K for all $t \geq t_0$. And hence K is invariant for its limiting equations. Now we shall see that the existence of a strictly positive almost periodic solution of (5.3) can be obtained under conditions (i), (ii) and (iii). We now consider Liapunov functional

$$V(u(t), x(t)) = \sum_{i=1}^{2} \{|\log u_i(t) - x_i(t)| + \int_0^\infty K_i(s) \int_{t-s}^{t} c_i(s + l)|u_i(l) - x_i(l)| dl ds\},$$

where u is a solution of (5.3) which remains in K. Calculating the derivative, we have

$$\dot{V}(u(t), x(t)) \leq -m \sum_{i=1}^{2} |u_i(t) - x_i(t)|.$$

Thus $\sum_{i=1}^{2} |u_i(t) - x_i(t)| \to 0$ as $t \to \infty$, and hence $\rho(u_t, x_t) \to 0$ as $t \to \infty$. Moreover, by using this Liapunov functional, we can show that $u(t)$ is (K, ρ)-WUAS in Ω of (5.3). Thus, from Theorem 4.6, $u(t)$ is (K, ρ)-ETS, because K is invariant. Therefore, it follows from Theorem 4.7 that system (5.3) has an almost periodic solution $p(t)$ such that $\beta_i \leq p_i(t) \leq \alpha_i$, $(i = 1, 2)$, for all $t \in R$.

References

[1] Corduneanu C., Almost Periodic Functions, *Chelsea Publishing Company*, New York, 1989.

[2] Fink A. M., Almost Periodic Differential Equations, *Lecture Notes in Mathematics 377*, *Springer-Verlag*, New York, 1974.

[3] Hamaya Y., Periodic solutions of nonlinear integrodifferential equations, *Tohoku Math. J.*, 41(1989), 105-116.

[4] Hamaya Y., Total stability property in limiting equations of integrodifferential equations, *Funkcialaj Ekvacioj*, 33(1990), 345-362.

[5] Hamaya Y., Relationship between BC-s.d.$\Omega(f)$ and (K, ρ)-s.d.$\Omega(f)$ in an abstract functional differential equation with infinite delay, *International J. Differential Equations and Applications*, 4(2002), 303-321.

[6] Hamaya Y. and Itokazu T., Almost periodic solutions of nonlinear Volterra difference equations with unbounded delay, *J. Difference Equations and Applications*, submitting.

[7] Hino Y., Murakami S. and Naito T., Functional Differential Equations with Infinite Delay, Lecture Notes in Mathematics 1473, *Springer-Verlag, New York*, 1991.

[8] Hino Y., Naito T., Minh N. V. and Shin J. S., Almost Periodic Solutions of Differential Equations in Banach Spaces, *Taylor and Francis*, 2002.

[9] Itokazu T. and Hamaya Y., Almost periodic solutions of prey-predator discrete models with delay, *Advances in Difference Equations*, in press, 2009.

[10] Kato J., Martynyuk A. A. and Shestakov A. A., Stability of Motion of Nonautonomous Systems (Method of Limiting Equations), *Gordon and Breach Publishers*, 1996.

[11] Murakami S. and Yoshizawa T., Relationships between BC-stabilities and ρ-stabilities in functional differential equations with infinite delay, *Tohoku Math. J.*, 44(1992), 45-57.

[12] Sell G. R., Periodic solutions and asymptotic stability, *Journal of Differential Equations*, 2(1996), 143-157.

[13] Xia Y. and Cheng S. S., Quasi-uniformly asymptotic stability and existence of almost periodic solutions of difference equations with applications in population dynamic systems, *J. Difference Equations and Applications*, 14(2008), 59-81.

[14] Yoshizawa T., Stability Theory and Existence of Periodic Solutions and Almost Periodic Solutions, Applied Mathematical Sciences 14, *Springer-Verlag, New York*, 1975.