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An Inequality of Cauchy–Schwarz Type with Application in the Theory of Elastic Rods

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Abstract. In this note we establish a general inequality of Cauchy–Schwarz type. We present this new inequality in both the discrete and the integral forms. The integral version of this inequality appears in the study of mechanical properties of thin elastic rods.

Keywords: Cauchy–Bunyakovsky–Schwarz inequality, bilinear forms, elastic rods.

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1 Introduction

In his monograph published in 1821, A.-L. Cauchy [3] presents an inequality (for finite sums) which will play an important role in mathematics. Later, V.I. Bunyakovsky [2] extends this inequality to its integral version. H.A. Schwarz [5] establishes in 1888 the general form of this inequality, valid in vectorial spaces endowed with a scalar product. The so–called Cauchy–Bunyakovsky–Schwarz inequality (in short CBS inequality) has useful applications in various branches of mathematics such that: real and complex analysis, probability theory and statistics, number theory, etc. The reader can find more details on this subject in [4, 6].

2 Inequality of Cauchy–Schwarz type

In what follows we present a new inequality of CBS type which has emerged from the study of mechanical properties of elastic rods [1].

Proposition 1 *Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be arbitrary elements of $\mathbb{R}^n$. Then, the following inequality holds true*

$$\begin{gathered}\Big[\Big(\sum_{i=1}^{n} p_i\Big)\Big(\sum_{i=1}^{n} p_i\, x_i\, y_i\Big) - \Big(\sum_{i=1}^{n} p_i\, x_i\Big)\Big(\sum_{i=1}^{n} p_i\, y_i\Big)\Big]^2 \leq \\ \Big[\Big(\sum_{i=1}^{n} p_i\Big)\Big(\sum_{i=1}^{n} p_i\, x_i^2\Big) - \Big(\sum_{i=1}^{n} p_i\, x_i\Big)^2\,\Big] \cdot \Big[\Big(\sum_{i=1}^{n} p_i\Big)\Big(\sum_{i=1}^{n} p_i\, y_i^2\Big) - \Big(\sum_{i=1}^{n} p_i\, y_i\Big)^2\,\Big]\,,\end{gathered} \tag{1}$$

for any positive numbers $p_1, p_2, \ldots, p_n$ (weights). The relation (1) becomes an equality if and only if there exists a linear combination of the vectors x and y which is a constant vector (i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha x + \beta y = \text{const}$).

Proof. In the space $\mathbb{R}^n$ we consider the bilinear form $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle x , y \rangle = \Big(\sum_{i=1}^n p_i\Big)\Big(\sum_{i=1}^n p_i\, x_i\, y_i\Big) - \Big(\sum_{i=1}^n p_i\, x_i\Big)\Big(\sum_{i=1}^n p_i\, y_i\Big), \tag{2}$$

and we notice that it possesses the following properties:

1. $\langle x , y \rangle = \langle y , x \rangle , \quad \forall\, x, y \in \mathbb{R}^n$;
2. $\langle x + z , y \rangle = \langle x , y \rangle + \langle z , y \rangle , \quad \forall\, z \in \mathbb{R}^n$;
3. $\langle \lambda\, x , y \rangle = \lambda \langle x , y \rangle , \quad \forall\, \lambda \in \mathbb{R}$;
4. $\langle x , c \rangle = 0, \quad \forall\, c \in \mathbb{R}^n, \;\; c = \text{const}$;
5. $\langle x , x \rangle \geq 0 , \qquad \text{and} \qquad \langle x , x \rangle = 0 \;\Leftrightarrow\; x = \text{const}$.

The properties 1–4 can be verified directly, taking into account the definition (2). Property 5 is proved using the classical CBS inequality:

$$\begin{aligned}\langle x , x \rangle &= \Big(\sum_{i=1}^n p_i\Big)\Big(\sum_{i=1}^n p_i\, x_i^2\Big) - \Big(\sum_{i=1}^n p_i\, x_i\Big)^2 = \\ &= \Big(\sum_{i=1}^n p_i\Big)\Big(\sum_{i=1}^n p_i\, x_i^2\Big) - \Big(\sum_{i=1}^n \sqrt{p_i}\cdot\sqrt{p_i}\, x_i\Big)^2 \geq 0 ,\end{aligned}$$

where the inequality becomes equality if and only if the vectors $(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_n})$ and $(\sqrt{p_1}\, x_1, \sqrt{p_2}\, x_2, \ldots, \sqrt{p_n}\, x_n)$ are linearly dependent, i.e. $x = (x_1, x_2, \ldots, x_n)$ is a constant vector. With these notations, the inequality (1) can be written in the form

$$\langle x , y \rangle^2 \leq \langle x , x \rangle \cdot \langle y , y \rangle , \quad \forall\, x, y \in \mathbb{R}^n. \tag{3}$$

To prove this inequality we follow the same ideas as in the classical situation. Let us distinguish two cases:

I. $x = \text{const}$ or $y = \text{const}$.

In view of property 4, in this case the relation (3) reduces to the equality $0 = 0$.

II. $x \neq \text{const}$ and $y \neq \text{const}$.

By virtue of property 5, we have

$$\langle \lambda x + \mu y , \lambda x + \mu y \rangle \geq 0 , \quad \forall\, \lambda, \mu \in \mathbb{R}, \tag{4}$$

which can be written as

$$\lambda^2 \langle x , x \rangle + 2\lambda\, \mu \langle x , y \rangle + \mu^2 \langle y , y \rangle \geq 0 , \quad \forall\, \lambda, \mu \in \mathbb{R}. \tag{5}$$

In our case the coefficients $\langle x , x \rangle$ and $\langle y , y \rangle$ are positive, so that the inequality (5) is satisfied (for every λ, μ) if and only if the inequality (3) holds true.

Clearly, the inequality (3) reduces to an equality if and only if the quadratic form (5) in the variables λ, μ has a root of multiplicity 2, i.e. there exist $\alpha, \beta \in \mathbb{R}$ $(\alpha, \beta \neq 0)$ such that $\langle \alpha x + \beta y , \alpha x + \beta y \rangle = 0$. Due to the property 5, this is equivalent to $\alpha x + \beta y = \text{const}$.

In both cases we obtain that the inequality (3) is valid for any $x, y \in \mathbb{R}^n$, and that the equality in (3) holds if and only if $\alpha x + \beta y = \text{const}$, for some $\alpha, \beta \in \mathbb{R}$. The proof is complete. ∎

Remarks. 1. In the particular case when $p_1 = p_2 = \cdots = p_n = 1$, from (1) we obtain the inequality

$$\Big[n\sum_{i=1}^{n} x_i\, y_i - \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i\Big]^2 \le \Big[n\sum_{i=1}^{n} x_i^2 - \Big(\sum_{i=1}^{n} x_i\Big)^2\Big] \cdot \Big[n\sum_{i=1}^{n} y_i^2 - \Big(\sum_{i=1}^{n} y_i\Big)^2\Big], \tag{6}$$

which holds true for any $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$.

2. If we consider the space l_2, then we obtain in the same way the inequality

$$\begin{gathered}\Big[\Big(\sum_{i=1}^{\infty} p_i\Big)\Big(\sum_{i=1}^{\infty} p_i\, x_i\, y_i\Big) - \Big(\sum_{i=1}^{\infty} p_i\, x_i\Big)\Big(\sum_{i=1}^{\infty} p_i\, y_i\Big)\Big]^2 \le \\ \Big[\Big(\sum_{i=1}^{\infty} p_i\Big)\Big(\sum_{i=1}^{\infty} p_i\, x_i^2\Big) - \Big(\sum_{i=1}^{\infty} p_i\, x_i\Big)^2\Big] \cdot \Big[\Big(\sum_{i=1}^{\infty} p_i\Big)\Big(\sum_{i=1}^{\infty} p_i\, y_i^2\Big) - \Big(\sum_{i=1}^{\infty} p_i\, y_i\Big)^2\Big],\end{gathered} \tag{7}$$

which is valid for any $x = \big(x_i\big)_i$ and $y = \big(y_i\big)_i$ in l_2, provided the weights $p = \big(p_i\big)_i$ satisfy the conditions : (i) $p_i > 0,\ \ i \in \mathbb{N}$; and (ii) $\sum\limits_{i=1}^{\infty} p_i < \infty$.
We mention that condition (ii) insures the convergence of all series appearing in (7).

3. The completeness of the spaces in which we are working does not play any role for this problem.

Let us present now the integral version of the CBS type inequality (1).

Proposition 2 *Let the functions $f, g, p : D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous on the compact domain D, and let p be positive. The following inequality holds true*

$$\begin{aligned}&\Big(\int_D p(x)\,\mathrm{d}x \cdot \int_D p(x)f(x)g(x)\,\mathrm{d}x - \int_D p(x)f(x)\,\mathrm{d}x \cdot \int_D p(x)g(x)\,\mathrm{d}x\Big)^2 \le \\ &\qquad \le \Big[\int_D p(x)\,\mathrm{d}x \cdot \int_D p(x)f^2(x)\,\mathrm{d}x - \Big(\int_D p(x)f(x)\,\mathrm{d}x\Big)^2\Big] \cdot \\ &\qquad\qquad \cdot \Big[\int_D p(x)\,\mathrm{d}x \cdot \int_D p(x)g^2(x)\,\mathrm{d}x - \Big(\int_D p(x)g(x)\,\mathrm{d}x\Big)^2\Big].\end{aligned} \tag{8}$$

The relation (8) reduces to equality if and only if there exist the scalars $\alpha, \beta \in \mathbb{R}$ such that $\alpha f + \beta g = \mathrm{const}$.

Proof. The inequality (8) can be put in the form

$$\langle\, f\,, g\,\rangle^2 \le \langle\, f\,, f\,\rangle \cdot \langle\, g\,, g\,\rangle\,,$$

where we denote by

$$\langle\, f\,, g\,\rangle = \int_D p(x)\,\mathrm{d}x \cdot \int_D p(x)f(x)g(x)\,\mathrm{d}x - \int_D p(x)f(x)\,\mathrm{d}x \cdot \int_D p(x)g(x)\,\mathrm{d}x\,.$$

In an analogous manner as in the proof of Proposition 1 we verify that the properties 1–5 are satisfied and we prove the desired results. ■

Remarks. **1.** If we take $D = [a,b] \subset \mathbb{R}$ and $p(x) = 1$, $\forall\, x \in [a,b]$, then from (8) we get the inequality

$$\Big[(b-a)\int_a^b f(x)g(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(x)\,\mathrm{d}x\Big]^2 \leq$$
$$\Big[(b-a)\int_a^b f^2(x)\,\mathrm{d}x - \Big(\int_a^b f(x)\,\mathrm{d}x\Big)^2\Big] \cdot \Big[(b-a)\int_a^b g^2(x)\,\mathrm{d}x - \Big(\int_a^b g(x)\,\mathrm{d}x\Big)^2\Big],$$

for any continuous functions $f,\, g : [a,b] \to \mathbb{R}$.

2. We observe that the inequality (8) remains valid also for functions $f,\, g,\, p$ in the space $L_2(D)$.

3 Final remark

Let us mention the context of the mechanical problem which provides an instance for the application of the above inequality of CBS type.

Within the theory of deformable continuum media, the inequality (8) for the case $n = 2$ appears in the study of mechanical properties of thin elastic rods. In this context, the two–dimensional domain D represents the cross–section of the rod, the function p is the mass density of the material, and $x = (x_1, x_2)$ stand for the cross–section coordinates. Then the inequality (8) justifies the (physical) property that the kinetic energy of the rod is a positive definite function of its arguments (see [1], Section 3).

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