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Some Results on Parallel Surfaces in 3 - Dimensional Minkowski Space $\mathbb{R}^3_1$

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Abstract. Parallel surfaces in a 3-dimensional Euclidian space is a classical subject, see [1], [3], [5]. It is much less studied in a 3 - dimensional Minkowski space, see [2], [4].

Let S be a spacelike or timelike orientable surface in 3 - dimensional Minkowski space $\mathbb{R}^3_1$ and let δ be a constant positive real number. The surface $\widetilde{S}$ is parallel to S at distance δ if for each point $P \in S$ we have $\widetilde{P}(u, v) = P(u, v) + \delta \cdot n(u, v)$, where n is the unit normal vector field on S. In this paper we investigate the properties induced, for example, by the conditions as minimality / maximality, developability or umbilicity over these surfaces.

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1 Preliminaries

Let $\mathbb{R}^3$ be a 3 - dimensional real vector space.

Definition 1.1. The pair $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_1)$, denoted $\mathbb{R}^3_1$, where the pseudo - inner product $\langle\cdot,\cdot\rangle_1$ is given by

$$\langle x, y\rangle_1 = -x_1y_1 + x_2y_2 + x_3y_3 \tag{1.1}$$

for every $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, is called 3 - dimensional Minkowski space.

Throughout this paper, S will be an orientable surface from $\mathbb{R}^3_1$ and n its unit normal vector field.

The *Gauss map* is defined to be

$$\gamma : S \longrightarrow S^2(-1) := \{(x, y, z) \in \mathbb{R}^3_1 | \ -x^2 + y^2 + z^2 = -1\} \tag{1.2}$$

if S is spacelike, or

$$\gamma : S \longrightarrow S^2(1) := \{(x, y, z) \in \mathbb{R}^3_1 | \ -x^2 + y^2 + z^2 = 1\} \tag{1.3}$$

if S is timelike.

With the Gauss maps defined above, we still have for Weingarten operator: $A = -\gamma_{*,p}$.

In terms of a local parametrization $P(u,v) = X(u,v)$ of surface S, the *Weingarten equations* are:

$$\begin{cases} n_u = \dfrac{FM-GL}{EG-F^2}X_u + \dfrac{FL-EM}{EG-F^2}X_v \\ n_v = \dfrac{FN-GM}{EG-F^2}X_u + \dfrac{FM-EN}{EG-F^2}X_v \end{cases} \tag{1.4}$$

where $\{E,F,G\}$ and $\{L,M,N\}$ are the coefficients of the first and second fundamental forms of surface S, defined as

$$E = \langle X_u, X_u\rangle_1, F = \langle X_u, X_v\rangle_1, G = \langle X_v, X_v\rangle_1 \tag{1.5}$$

$$L = -\langle n_u, X_u\rangle_1, M = -\langle n_u, X_v\rangle_1 = -\langle n_v, X_u\rangle_1, N = -\langle n_v, X_v\rangle_1 \tag{1.6}$$

The coefficients of X_u and X_v from (1.4) provide the matrix of $-A$, where A is the Weingarten operator.

Let Bn be the vector - valued second fundamental form of S. For the scalar fundamental form B we have $\epsilon B(X_u,X_u) = \langle AX_u, X_u\rangle_1 = L$, $\epsilon B(X_u,X_v) = \langle AX_u, X_v\rangle_1 = M$, $\epsilon B(X_v,X_v) = \langle AX_v, X_v\rangle_1 = N$, with $\epsilon = \langle n,n\rangle_1 = +$ or -1. In Section 3E of [3] the Gaussian curvature is defined as

$$K = \frac{\langle B(U,U)n, B(V,V)n\rangle_1 - \langle B(U,V)n, B(U,V)n\rangle_1}{g(U,U)g(V,V) - g(U,V)g(U,V)}$$

for an arbitrary basis (U,V) in the tangent space to S. Here g is the first fundamental form of S. For $U = X_u$ and $V = X_v$ one gets

$$K = \epsilon\frac{LN-M^2}{EG-F^2}$$

The mean curvature vector field of S is given by

$$\mathbf{H} = Hn$$

with $H = \frac{1}{2}TraceA$. In the natural frame (X_u, X_v) one gets

$$H = \frac{1}{2}\frac{EN-2FM+GL}{EG-F^2}$$

This invariant is defined up to a sign since A depends on the orientation of n.

We shall consider the following cases.

Case (i). S is spacelike. Then n is timelike hence $\epsilon = -1$ and we have

$$K = -\frac{LN-M^2}{EG-F^2} \text{ and } H = \frac{1}{2}\frac{EN-2FM+GL}{EG-F^2} \tag{1.7}$$

Case (ii). S is timelike and so n is spacelike, hence $\epsilon = 1$. For the Gaussian and the mean curvature we have the following formulas

$$K = \frac{LN-M^2}{EG-F^2} \text{ and } H = \frac{1}{2}\frac{EN-2FM+GL}{EG-F^2} \tag{1.8}$$

Definition 1.2. A spacelike (timelike) surface with $H = 0$ is called maximal (minimal) surface.

Definition 1.3. Let S be an orientable surface and let n be the unit normal vector field of S. The surface $\widetilde{S}$ is parallel to S at distance δ if the points $\widetilde{P}(u, v) \in \widetilde{S}$ are defined by

$$\widetilde{P}(u, v) = P(u, v) + \delta \cdot n(u, v) \tag{1.9}$$

where δ is a constant positive real number.

Remark 1.1. $\widetilde{n} = \pm\, n$.

Remark 1.2. If the unit normal vector field of a surface S is in the same time the unit normal vector field of another surface $\widetilde{S}$, then the two surfaces S and $\widetilde{S}$ are parallel.

Indeed, if so, the equation of $\widetilde{S}$ may be written in the form $\widetilde{r} = r + \delta \cdot n$ with δ a function of points of S. Differentiating we get $d\widetilde{r} = dr + d\delta \cdot n + \delta \cdot dn$, from where, multiplying scalar with n it follows $d\delta = 0$ and so $\delta = const$.

2 Main results for parallel surfaces

For the beginning, we will compute the coefficients $\widetilde{E}, \widetilde{F}, \widetilde{G}, \widetilde{L}, \widetilde{M}, \widetilde{N}$ of the first and second fundamental forms of surface $\widetilde{S}$ in terms of their analogues coefficients from S.

In order to simplify some calculations, we choose on S the parametrization given by curvatures lines, which means: $(*)$ $F = M = 0$. In this case, the equations (1.4) become:

$$\begin{cases} n_u = -\dfrac{L}{E} X_u \\ n_v = -\dfrac{N}{G} X_v \end{cases}$$

Using the formulas (1.5) and (1.6), written for surface $\widetilde{S}$, we obtain:

$$\widetilde{E} = E - 2\delta \frac{L}{E} E + \delta^2 \frac{L^2}{E^2} E, \quad \widetilde{F} = F + \delta \left(-\frac{N}{G} F - \frac{L}{E} F \right) + \delta^2 \frac{LN}{EG} F$$
$$\widetilde{G} = G - 2\delta \frac{N}{G} G + \delta^2 \frac{N^2}{G^2} G, \quad \widetilde{L} = L - \delta \frac{L^2}{E^2} E, \quad \widetilde{M} = M - \delta \frac{LN}{EG} F, \quad \widetilde{N} = N - \delta \frac{N^2}{G^2} G$$

i.e.

$$\begin{cases} \widetilde{E} = E \left(1 - \delta \dfrac{L}{E}\right)^2, \ \widetilde{F} = 0, \ \widetilde{G} = G \left(1 - \delta \dfrac{N}{G}\right)^2 \\ \widetilde{L} = L \left(1 - \delta \dfrac{L}{E}\right), \ \widetilde{M} = 0, \ \widetilde{N} = N \left(1 - \delta \dfrac{N}{G}\right) \end{cases} \tag{2.1}$$

Remark 2.1. The parametrization (u, v) on $\widetilde{S}$ consists of curvature lines on $\widetilde{S}$.

Further, for the parallel surface $\widetilde{S}$ to a spacelike surface S, using the conditions ($*$) in formulas (1.7), written for $\widetilde{S}$, and the coefficients given in (2.1), we obtain:

$$\widetilde{K} = -\frac{\dfrac{LN}{EG}}{1 - \delta\dfrac{EN+GL}{EG} + \delta^2\dfrac{LN}{EG}}$$

from which we get:

$$\widetilde{K} = \frac{K}{1 - 2\delta H - \delta^2 K}$$

and, for the mean curvature:

$$\widetilde{H} = \frac{1}{2}\frac{\dfrac{EN+GL}{EG} - 2\delta\dfrac{LN}{EG}}{1 - \delta\dfrac{EN+GL}{EG} + \delta^2\dfrac{LN}{EG}}$$

and so

$$\widetilde{H} = \frac{H + \delta K}{1 - 2\delta H - \delta^2 K}.$$

In case S is timelike, using formulas (1.8) and making similar computations we obtain

$$\widetilde{K} = \frac{K}{1 - 2\delta H + \delta^2 K}$$

and following the same steps, we get

$$\widetilde{H} = \frac{H - \delta K}{1 - 2\delta H + \delta^2 K}.$$

Thus, we have proved the following results:

Theorem 2.1. *Let S be a spacelike orientable surface with Gaussian curvature K and mean curvature H and let δ be a real positive constant such that $1 - 2\delta H - \delta^2 K \neq 0$. Then, the curvatures $\widetilde{K}$ and $\widetilde{H}$ of the surface $\widetilde{S}$ parallel to S at distance δ are given by:*

$$\widetilde{K} = \frac{K}{1 - 2\delta H - \delta^2 K} \tag{2.2}$$

$$\widetilde{H} = \frac{H + \delta K}{1 - 2\delta H - \delta^2 K} \tag{2.3}$$

Theorem 2.2. *Let S be a timelike orientable surface with Gaussian curvature K and mean curvature H and let δ be a real positive constant such that $1 - 2\delta H + \delta^2 K \neq 0$. Then, the curvatures $\widetilde{K}$ and $\widetilde{H}$ of the surface $\widetilde{S}$ parallel to S at distance δ are given by:*

$$\widetilde{K} = \frac{K}{1 - 2\delta H + \delta^2 K} \tag{2.4}$$

$$\widetilde{H} = \frac{H - \delta K}{1 - 2\delta H + \delta^2 K} \tag{2.5}$$

For a given surface S (spacelike or timelike) we have two posibilies: $K = 0$ (S *is developable*) and $K \neq 0$ (S *is non - developable*). For this two cases we have the following theorems:

Theorem 2.3. *Let S be a spacelike orientable surface without planar points, let K be the Gaussian curvature, H the mean curvature of S and let δ be a real positive constant such that $1 - 2\delta H - \delta^2 K \neq 0$. Let be $\widetilde{S}$ be the parallel surface to S at distance δ. Then:*
a) $\widetilde{S}$ is developable iff S is developable.
b) If S is developable, then $\widetilde{S}$ is maximal iff S is maximal.

Theorem 2.4. *Let S be a timelike orientable surface and let δ be a real positive constant such that $1 - 2\delta H + \delta^2 K \neq 0$. Then, the parallel surface $\widetilde{S}$ to S at distance δ is developable iff S is developable.*

Theorem 2.5. *Let S be a timelike orientable non - developlable non - umbilical surface with constant Gaussian curvature K and constant mean curvature H such that $1-2\delta H+\delta^2 K \neq 0$. Then, any parallel surface $\widetilde{S}$ to S at distance $\delta = \frac{H}{K}$ is minimal.*

The proofs of these theorems are easy and are omitted.

Theorem 2.6. *The parallel surface $\widetilde{S}$ at distance $\delta = -\dfrac{1}{H}$ of a constant negative mean curvature spacelike non - maximal umbilical surface S is maximal.*

Proof. By the mean inequality we have $K \leq H^2$ with equality if S is umbilical. Replacing $K = H^2$ and $\delta = -\dfrac{1}{H}$ in (12) we get $\widetilde{H} = 0$, so $\widetilde{S}$ is maximal. □

Theorem 2.7. *The parallel surface at distance $\delta = \dfrac{1}{H}$ of a constant mean curvature timelike non - minimal non - umbilical surface is a constant mean curvature surface.*

Proof. For non-umbilical surfaces we have $H^2 - K > 0$ and making $\delta = \dfrac{1}{H}$ in (2.5) we obtain $\widetilde{H} = -H$ and the proof is complete. □

Theorem 2.8. *A timelike non – developable constant Gaussian curvature surface with constant mean curvature S and its parallel surface $\widetilde{S}$ at distance $\delta = \dfrac{2H}{K}$ are locally isometric.*

Proof. Making $\delta = \dfrac{2H}{K}$ in (2.4) we get $\widetilde{K} = K$ and according to Minding's theorem, S and $\widetilde{S}$ are isometric. □

Similarly, we can prove

Theorem 2.9. *The parallel surface $\widetilde{S}$ at distance $\delta = \dfrac{2H}{K}$ of a timelike non – developable constant Gaussian curvature and constant mean curvature surface S is a constant mean curvature surface.*

Theorem 2.10. *Let S be an orientable spacelike non – developable surface. If S has constant mean curvature $H > 0$, then there exist two surfaces parallel to S such that one has constant negative gaussian curvature $\widetilde{K} = -4H^2$ and the other has negative constant mean curvature $\widetilde{H} = -H$.*

Proof. From (2.2) and $\delta = \dfrac{1}{2H}$, using the conditions from the hypothesis, we obtain $\widetilde{K} = -4H^2$. Putting $\delta = \dfrac{1}{H}$ in (2.3) we get $\widetilde{H} = -H$. □

Theorem 2.11. *Let S be an orientable timelike non – developable surface. If S has constant mean curvature $H > 0$, then there exist two surfaces parallel to S such that one has constant positive Gaussian curvature $\widetilde{K} = 4H^2$ and the other one has negative constant mean curvature $\widetilde{H} = -H$.*

Proof. For $\delta = \dfrac{1}{2H}$, (2.4) becomes $\widetilde{K} = 4H^2$ and for $\delta = \dfrac{1}{H}$ in (2.5) we get $\widetilde{H} = -H$, which end the proof. □

Definition 2.1. A surface on which the Gaussian curvature is everywhere positive (negative) is called synclastic (anticlastic).

Theorem 2.12. *The parallel surface $\widetilde{S}$ to a timelike (spacelike) non – developable surface surface S at distance $\delta = \dfrac{1}{2H}$, H a positive constant, is synclastic (respectively, anticlastic).*

Theorem 2.13. *Let S be an orientable surface, n - the unit normal vector field on S and $\epsilon = \langle n, n\rangle_1$. Let $\delta > 0$ be a real constant and let $\widetilde{S}$ be the parallel surface to S at distance δ. We have*

(*i*) *If $H \equiv 0$ then $\widetilde{H} = -\epsilon\delta\widetilde{K}$.*
(*ii*) *If $H = \epsilon\delta K$ then $\widetilde{H} \equiv 0$.*
(*iii*) *The point P is umbilic on S iff it is umbilic on $\widetilde{S}$.*
(*iv*) *The point P is planar on S iff it is planar on $\widetilde{S}$.*

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