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Solving Some Classes of Differential Equations Using Maple V

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Abstract. Some types of non homogenous differential equations, having a superior order, claim for their solutions, complicated calculus, even if we use *Laplace transform*. So, in our short didactic work, we tried another way to find, easy and fast, the solutions for a such differential equation.And we hope, that our aim was realized through *MAPLE V computational procedure.*

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1 Preliminaries

All our following considerations will be related to *linear and non-homogenous differential equations*, having *order n and constant coefficients* (see [1],[3],[4]) of the type:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = f(x), \quad (\forall) x \in I, \tag{1}$$

where $f \in C^0(I)$ is a continuous real function on the domain $I \subseteq \mathbf{R}$. A solution for the equation (1) has the general form:

$$y(x) = \widetilde{y}(x) + y_0(x), \quad (\forall) x \in I, \tag{2}$$

where $\widetilde{y}(x)$ is the solution of the *linear and homogenous differential equation*, having *order n and constant coefficients*:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = 0, \quad (\forall) x \in I, \tag{3}$$

and where $y_0(x)$ is *a particular solution* of (1). We know that:

$$\widetilde{y}(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \tag{4}$$

when c_i, $(\forall) i \in \{1, 2, \ldots, n\}$ are real constants, and $y_1(x), \ldots, y_n(x)$ give *a fundamental system of solutions* on the domain I, namely, when the Wronskian:

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix} \neq 0. \tag{5}$$

It is sufficient to solve the characteristic equation obtained from (3), when we impose as solution $y(x) = e^{rx}$. The particular solution $y_0(x)$ can be found, or using *Lagrange method of parameter variation*, or using the method *of the uncertain coefficients*, which gives us the form of $y_o(x)$, looking at $f(x)$ and without to claim quadratures.

The equation (1) is also resolvable through *Laplace transform* $\mathcal{L}$, if there are given *Cauchy initial conditions*:

$$y(0) = y_0,\ y'(0) = y_1, \dots,\ y^{(n-1)}(0) = y_{n-1}. \tag{6}$$

Usually, we note $\mathcal{L}y = Y(p)$, $\mathcal{L}f = F(p)$ and then, applying to the equation (1), the transform of Laplace, we find:

$$\begin{aligned} Y(p)(a_n p^n + a_{n-1}p^{n-1} + \cdots + a_1 p + a_0) &= F(p) + y(0)(a_n p^{n-1} + a_{n-1}p^{n-2} + \cdots + a_1) \\ &+ y'(0)(a_n p^{n-2} + a_{n-1}p^{n-3} + \cdots + a_2) + \cdots + y^{(n-1)}(0)a_n. \end{aligned} \tag{7}$$

From (7), we obtain the form of $Y(p)$, to which we apply *Mellin-Fourier formula*:

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(p)e^{-px}dp, \tag{8}$$

and so, we find the *original*, namely, the *solution* of the equation (1), at *Cauchy conditions* (6).

2 Concrete Analysis for a Given Case

We shall go on, with a *linear and non homogenous differential equation*, having *order 3 and constant coefficients*. To solve it, we firstly apply the *method of the uncertain coefficients*, secondly the method of *Laplace transform* and finally, the procedures from MAPLE V. Our starting point is the equation:

$$y''' + y = \frac{x^2 e^x}{2}, \quad x \geq 0, \tag{9}$$

at Cauchy conditions $y(0) = y'(0) = y''(0) = 0$.

Its homogenous differential equation:

$$y''' + y = 0 \tag{10}$$

has the *characteristic equation* $r^3 + 1 = 0$, with the solutions:

$$r_1 = -1,\ r_{2,3} = \frac{1 \pm i\sqrt{3}}{2}.$$

According to [1], [3] and [4], the solution $\widetilde{y}(x)$ of (10) is:

$$\widetilde{y}(x) = c_1 e^{-x} + c_2 e^{\frac{x}{2}} \cos\frac{\sqrt{3}}{2}x + c_3 e^{\frac{x}{2}} \sin\frac{\sqrt{3}}{2}x \tag{11}$$

and a *particular solution*, for (9), will be chosen as:

$$y_0(x) = e^x(ax^2 + bx + c), \text{ when } a, b, c \in \mathbf{R}. \tag{12}$$

With the derivatives of $y_0(x)$, given by:

$$\begin{aligned} y_0'(x) &= e^x[ax^2 + (2a + b)x + b + c], \\ y_0''(x) &= e^x[ax^2 + (4a + b)x + 2a + 2b + c], \\ y_0'''(x) &= e^x[ax^2 + (6a + b)x + 6a + 3b + c], \end{aligned} \tag{13}$$

in (9), and using the coefficient identification method, we find:

$$a = \frac{1}{4},\ b = -\frac{3}{4},\ c = \frac{3}{8}. \tag{14}$$

Therefore, the general solution, of (9), is:

$$y(x) = c_1e^{-x} + c_2e^{\frac{x}{2}} \cos \frac{\sqrt{3}}{2}x + c_3e^{\frac{x}{2}} \sin \frac{\sqrt{3}}{2}x + \frac{e^x}{4}(x^2 - 3x + \frac{3}{2}). \tag{15}$$

A particular solution, of (9), asks firstly, the derivatives for $y(x)$, given from (15). So, it follows:

$$\begin{aligned} y'(x) &= -c_1e^{-x} + \tfrac{1}{2}e^{\frac{x}{2}}[(c_3 - c_2\sqrt{3}) \sin \tfrac{\sqrt{3}}{2}x + (c_2 + \sqrt{3}c_3) \cos \tfrac{\sqrt{3}}{2}x] + \tfrac{e^x}{4}(x^2 - x - \tfrac{3}{2}), \\ y''(x) &= c_1e^{-x} + \tfrac{1}{2}e^{\frac{x}{2}}[(c_3\sqrt{3} - c_2) \cos \tfrac{\sqrt{3}}{2}x - (c_3 + \sqrt{3}c_2) \sin \tfrac{\sqrt{3}}{2}x] + \tfrac{e^x}{4}(x^2 + x - \tfrac{5}{2}). \end{aligned} \tag{16}$$

Going in (9) with (15), (16) and using the coefficient identification method, it occurs:

$$c_1 = -\frac{1}{24},\ c_2 = -\frac{1}{3},\ c_3 = \frac{1}{\sqrt{3}}. \tag{17}$$

Now, cumulating the relations (15) and (17), we conclude that the equation (9) has the finale solution:

$$y(x) = -\frac{1}{24}e^{-x} - \frac{1}{3}e^{\frac{x}{2}} \cos \frac{\sqrt{3}}{2}x + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}x + \frac{e^x}{4}(x^2 - 3x + \frac{3}{2}). \tag{18}$$

In the following, we shall give for (9), a solution based on *Laplace transform.* From the relation (7), it occurs:

$$(p^3 + 1)Y(p) = F(p), \text{ or } Y(p) = \frac{F(p)}{p^3 + 1}, \text{ where } F(p) = \frac{1}{2}\mathcal{L}(e^x x^2). \tag{19}$$

By the substitution $x(p - 1) = u$ and with *Euler* Γ *function*(see [2]), we have:

$$\mathcal{L}(e^x x^2) = \int_0^\infty e^{-x(p-1)}x^2dx = \frac{1}{(p-1)^3}\int_0^\infty e^{-u}u^2du = \frac{\Gamma(3)}{(p-1)^3} = \frac{2}{(p-1)^3}. \tag{20}$$

Therefore, from (19) and (20), we deduce that:

$$Y(p)=\frac{1}{(p-1)^3(p^3+1)}. \tag{21}$$

We rewrite (21), using a decomposition in *simple fractions*, and so, we obtain:

$$\begin{aligned} Y(p) &= \frac{3}{8(p-1)}-\frac{1}{24(p+1)}-\frac{3}{4(p-1)^2}+\frac{1}{2(p-1)^3} \\ &-\frac{p-\frac{1}{2}}{3[(p-\frac{1}{2})^2+\frac{3}{4}]}+\frac{\frac{3}{2}}{3[(p-\frac{1}{2})^2+\frac{3}{4}]}. \end{aligned} \tag{22}$$

Now, we can apply *Mellin-Fourier formula*, namely, with the relations which give us the *original function*, from (22), it occurs:

$$y(x)=\frac{3}{8}e^x-\frac{1}{24}e^{-x}-\frac{3}{4}xe^x+\frac{1}{4}x^2e^x-\frac{1}{3}e^{\frac{x}{2}}\cos\frac{\sqrt{3}}{2}x+\frac{1}{\sqrt{3}}e^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x. \tag{23}$$

Arranging the terms in (23), we shall find the same solution (18), of the equation (9).

3 Maple V approach

We shall transpose the equation (9), in Maple V language as:

```
[> ecdiff:=diff(y(x),x$3)+y(x)=x^2*exp(x)/2;
```

With Maple V statement:

```
[>dsolve({ecdiff,y(0)=0,D(y)(0)=0,D(D(y))(0)=0},y(x));
```

we shall provide the solution of the equation (9):

```
y(x)=1/8*(3-6*x+2*x^2)*exp(x)-1/24*exp(-x)-1/3*exp(1/2*x)*
cos(1/2*3^(1/2)*x)+1/3*3^(1/2)*exp(1/2*x)*sin(1/2*3^(1/2)*x);
```

We shall solve the same equation (9), using *Laplace transforma*, with the following Maple V statement:

```
[> dsolve({eqdiff, y(0) = 0, (D(y))(0) = 0,(D(D(y)))(0) = 0},
y(x), method = laplace);
```

We obtain the same solution:

```
y(x)=-1/24*exp(-x)+1/8*(3-6*x+2*x^2)*exp(x)+1/3*(-cos(1/2*3^(1/2)*x)+
3^(1/2)*sin(1/2*3^(1/2)*x))*exp(1/2*x);
```

and its integral curve (see the Figure 1), found by the following Maple V statement:

```
[>with(plots):
[>plot(-(1/24)*exp(-x)+(1/8*(3-6*x+2*x^2))*exp(x)+(1/3*(-cos((1/2)*
sqrt(3)*x)+sqrt(3)*sin((1/2)*sqrt(3)*x)))*exp((1/2)*x),x=0..infinity);
```

In the Figure 2, we present all our Maple V calculus.

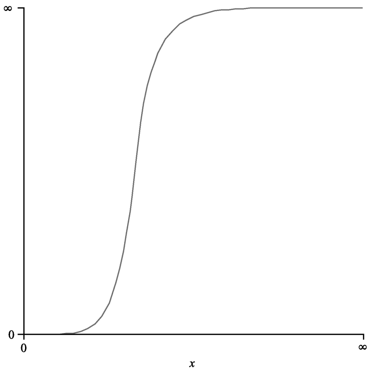
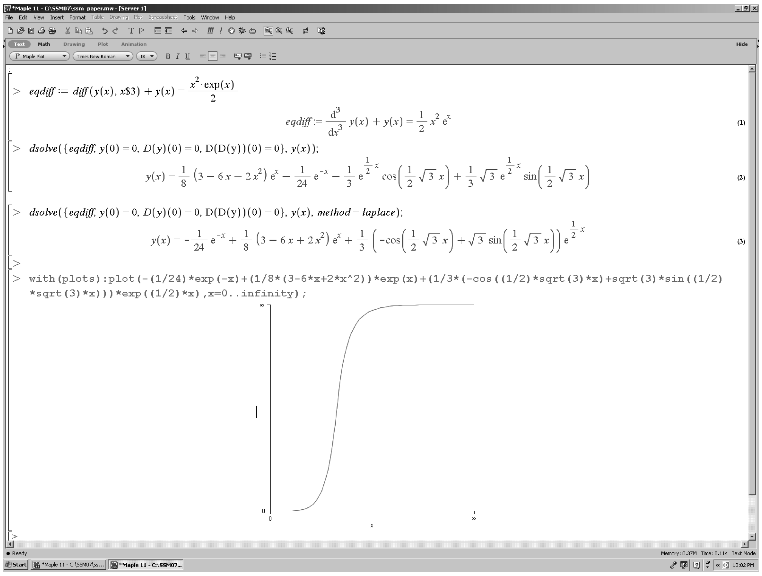


Figure 1. The integral curve of the equation (9)



> $eqdiff := diff(y(x), x\$3) + y(x) = \frac{x^2 \cdot \exp(x)}{2}$

$$eqdiff := \frac{d^3}{dx^3}\, y(x) + y(x) = \frac{1}{2}\, x^2\, e^x \qquad (1)$$

> $dsolve(\{eqdiff, y(0) = 0, D(y)(0) = 0, D(D(y))(0) = 0\}, y(x));$

$$y(x) = \frac{1}{8}\left(3 - 6\,x + 2\,x^2\right) e^x - \frac{1}{24}\, e^{-x} - \frac{1}{3}\, e^{\frac{1}{2}x} \cos\left(\frac{1}{2}\sqrt{3}\, x\right) + \frac{1}{3}\sqrt{3}\, e^{\frac{1}{2}x} \sin\left(\frac{1}{2}\sqrt{3}\, x\right) \qquad (2)$$

> $dsolve(\{eqdiff, y(0) = 0, D(y)(0) = 0, D(D(y))(0) = 0\}, y(x), method = laplace);$

$$y(x) = -\frac{1}{24}\, e^{-x} + \frac{1}{8}\left(3 - 6\,x + 2\,x^2\right) e^x + \frac{1}{3}\left(-\cos\left(\frac{1}{2}\sqrt{3}\, x\right) + \sqrt{3}\,\sin\left(\frac{1}{2}\sqrt{3}\, x\right)\right) e^{\frac{1}{2}x} \qquad (3)$$

```
>
> with(plots):plot(-(1/24)*exp(-x)+(1/8*(3-6*x+2*x^2))*exp(x)+(1/3*(-cos((1/2)*sqrt(3)*x)+sqrt(3)*sin((1/2)
  *sqrt(3)*x)))*exp((1/2)*x),x=0..infinity);
```

Figure 2. Maple V calculus proof

4 Conclusions

When we solved the equation (9) with Maple V, we used the *symbolic calculus* and not at all the *numerical approach.* Although the time, to find the solution (18), was not longer that 10 seconds, we emphasize that Maple V can not be used by a person who has not Mathematics skills. Therefore, the ideas of some unspecialists that the utilitary software, dedicated to Applied Mathematics, could be used by anyone, is one absolutely *false*! We surely remain the permanent sustainers of the strong idea that in all the technical and scientific faculties, the education of young people who will use software packages, must be harmoniously realized by a curriculum which must compulsory contain lectures in Mathematics and Informatics.

References

[1] Corduneanu C., *Differential and Integral Equations* (In Romanian), Iaşi University Press, 1971.

[2] Modan L., *Euler's Gamma Function Analyzing the Open Question OQ 1352*, OCTOGON, Math. Mag., v.12, nr.2B, 2004 Braşov, pg.1056-1057.

[3] Rogai E., *Differential and Integral Equation by Exercises and Problems* (in Romanian), Technical Printing House, 1965, Bucharest.

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