LIBERTAS MATEMATICA, vol XXVII (2007)

Functional Monotone VP and Metrical Coercivity

Mihai TURINICI

Dedicated to Academician Radu Miron on his 80th Birthday

Abstract. A functional extension is given for the monotone variational principle in Turinici [An. St. UAIC Iasi, 36 (1990), 329-352]. The obtained facts are then used to establish functional coercivity results via Palais-Smale techniques (involving quasi-order strong slopes) over metric spaces.

Keywords: Metric space, inf-proper lsc function, (monotone) variational principle, normal function, quasi-order, maximal element, partition, coercivity, strong slope, Palais-Smale condition.

Mathematics Subject Classification (2000): Primary 49J53; Secondary 47J30.

1 Introduction

Let (M, d) be a complete metric space; and $\varphi : M \to R \cup \{\infty\}$ some function with

$$\varphi \text{ is inf-proper } (\mathrm{Dom}(\varphi) \neq \emptyset \text{ and } \varphi_* := \inf[\varphi(M)] > -\infty) \tag{1.1}$$

$$\varphi \text{ is } d\text{-lsc } (\liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \to x). \tag{1.2}$$

The following 1974 statement in Ekeland [9] (referred to as Ekeland's variational principle; in short: EVP) is well known.

Theorem E. *Let the precise conditions hold. Then,*
a) for each $u \in \mathrm{Dom}(\varphi)$ *there exists* $v = v(u) \in \mathrm{Dom}(\varphi)$ *with*

$$d(u, v) \leq \varphi(u) - \varphi(v) \ (\textit{hence } \varphi(u) \geq \varphi(v)) \tag{1.3}$$

$$d(v, x) > \varphi(v) - \varphi(x), \quad \textit{for all } x \in M \setminus \{v\}. \tag{1.4}$$

aa) if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ *fulfill* $\varphi(u) - \varphi_* \leq \rho$, *then (1.3) gives*

$$(\varphi(u) \geq \varphi(v) \ \textit{and}) \quad d(u, v) \leq \rho. \tag{1.5}$$

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the 1979 paper by

Ekeland [10] for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of Theorem E were proposed. For example, the dimensional way of extension refers to the support space (R) of Codom(φ) being substituted by a (topological or not) vector space. An account of the results in this area is to be found in the 2003 monograph by Goepfert, Riahi, Tammer and Zălinescu [11, Ch 3]; see also Hamel [12, Ch 4], Isac [13], Khanh [16], Nemeth [20], Rozoveanu [23] and Turinici [30]. Further, the (pseudo) metrical one consists in the conditions imposed to the ambient metric over M being relaxed. The basic result in this direction was obtained in 1992 by Tataru [27], via Ekeland type techniques. Subsequent extensions of it may be found in the 1996 paper by Kada, Suzuki and Takahashi [14]; note that the authors' argument relies on the nonconvex minimization theorem in Takahashi [26]. Finally, we must add to this list the "functional" extension of EVP obtained in 1997 by Zhong [33] (and referred to as Zhong's variational principle; in short: ZVP). Take a function $t \vdash b(t)$ from $R_+ := [0, \infty[$ to itself, with the *normality* properties

$$b \text{ is decreasing and } b(R_+) \subseteq R_+^0 :=]0, \infty[\tag{1.6}$$

$$B(\infty) = \infty, \quad \text{where } B(t) = \int_0^t b(\tau)d\tau,\ t \geq 0. \tag{1.7}$$

Theorem Z. *Let $a \in M$, $\rho > 0$ be given, as well as $u \in \mathrm{Dom}(\varphi)$ with $\varphi(u) - \varphi_* \leq B(r + \rho) - B(r)$, where $r := d(a, u)$. There exists then $v = v(u)$ in $\mathrm{Dom}(\varphi)$ with*

$$d(a, v) \leq r + \rho, \quad \varphi(u) \geq \varphi(v) \tag{1.8}$$

$$b(d(a, v))d(v, x) > \varphi(v) - \varphi(x), \quad \textit{for each } x \in M \setminus \{v\}. \tag{1.9}$$

Clearly, Theorem Z includes (for $b = 1$ and $a = u$) the local version of Theorem E based upon (1.5). The relative form of the same, based upon (1.3) also holds (but indirectly); cf. Bao and Khanh [4]. Summing up, Theorem Z includes Theorem E; further aspects may be found in Suzuki [24].

Now, another way of extending Ekeland's variational principle (Theorem E) is the *quasi-order* one, proposed in the 1990 paper by Turinici [29] (cf. Section 2). So, it is natural asking whether corresponding versions of Theorem Z are available. A positive answer to this will be provided in Section 4. The specific tool of our investigations is the concept of *normal* function we just introduced, and a construction related to the fixed point one in Park and Bae [21]; we refer to Section 3 for details. Finally, some applications of these facts are given to *functional coercivity* criteria over metric/normed structures via Palais-Smale techniques (cf. Sections 5 and 6). The "differential" context for these is based on the concept of *strong slope* of a function, as developed in DeGiorgi, Marino and Tosques [8]. Further aspects will be discussed elsewhere.

2 Monotone VP

(A) Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) ($\leq$) over M; as well as a function $x \vdash \varphi(x)$ from M to R_+. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w).$$

A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [5]:

Proposition 1. *Suppose that*

$$\begin{aligned}&(M,\le) \textit{ is sequentially inductive:}\\ &\textit{each ascending sequence has an upper bound (modulo } (\le))\end{aligned} \tag{2.1}$$

$$\varphi \textit{ is } (\le)\textit{-decreasing } (x \le y \Longrightarrow \varphi(x) \ge \varphi(y)). \tag{2.2}$$

Then, for each $u \in M$ *there exists a* $(\le, \varphi)$*-maximal* $v \in M$ *with* $u \le v$.

Note that $\mathrm{Codom}(\varphi) \subseteq R_+$ is not essential for the conclusion above; cf. Cârjă and Ursescu [6]. Moreover, $(R_+, \ge)$ may be substituted by a separable ordering structure $(P, \le)$ without altering the conclusion above; see Turinici [32] for details.

This principle, including Ekeland's [9,10] (Theorem E) found some useful applications to convex and nonconvex analysis (cf. the above references). For this reason, it was the subject of many extensions; such as the ones in Altman [1], Anisiu [2] and Szaz [25]. The obtained results are interesting from a technical viewpoint. However, we must emphasize that, whenever a maximality principle of this type is to be applied, a substitution of it by the Brezis-Browder's is always possible. This raises the question of to what extent are these enlargements of Proposition 1 effective. As we shall see, the answer is *essentially* negative. This will necessitate some conventions. By a *pseudometric* over M we shall mean any map $d : M \times M \to R_+$. If, in addition, d is *reflexive* $[d(x, x) = 0, \forall x \in M]$, *triangular* $[d(x, z) \le d(x, y) + d(y, z), \forall x, y, z \in M]$ and *symmetric* $[d(x, y) = d(y, x), \forall x, y \in M]$, we say that it is a *semimetric* over M. Suppose that we fixed such an object. Call the point $z \in M$, $(\le, d)$*-maximal*, in case

$$w \in M \quad \text{and } z \le w \text{ imply } d(z, w) = 0.$$

Note that, if (in addition) d is *sufficient* $[d(x, y) = 0$ implies $x = y]$, this property may be written as

$$w \in M, z \le w \Longrightarrow z = w \ \ (z \text{ is (strongly) } (\le)\text{-maximal}).$$

So, existence results involving such points may be viewed as "metrical" versions of the Zorn maximality principle. To get sufficient conditions for these, one may proceed as below. Let (x_n) be an ascending sequence in M. The d-Cauchy property for it is introduced in the usual way: $\forall \varepsilon > 0, \exists n(\varepsilon)$ such that $n(\varepsilon) \le p \le q \Longrightarrow d(x_p, x_q) \le \varepsilon$. Also, call this sequence d*-asymptotic*, when $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Clearly, each (ascending) d-Cauchy sequence is d-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent each other:

$$\text{each ascending sequence is } d\text{-Cauchy} \tag{2.3}$$

$$\text{each ascending sequence is } d\text{-asymptotic.} \tag{2.4}$$

By definition, either of these will be referred to as $(M, \le)$ is *regular* (modulo d). Moreover, this property implies its relaxed version

$$\begin{aligned}&(M,\le) \text{ is weakly regular (modulo } d\text{): } \forall x \in M, \forall \varepsilon > 0,\\ &\exists y = y(x, \varepsilon) \ge x \text{ such that } y \le u \le v \Longrightarrow d(u, v) \le \varepsilon.\end{aligned} \tag{2.5}$$

The following ordering principle is available (cf. Kang and Park [15]):

Proposition 2. *Assume that (2.1) and (2.5) are true. Then, for each $u \in M$ there exists a $(\leq, d)$-maximal $v \in M$ with $u \leq v$.*

As a direct consequence of this, we have (cf. Turinici [28]):

Proposition 3. *Assume that $(M, \leq)$ is sequentially inductive and regular (modulo d). Then, the conclusion of Proposition 2 is retainable.*

Now (see the above reference) Proposition 1 $\Longrightarrow$ Proposition 2. On the other hand, Proposition 2 $\Longrightarrow$ Proposition 3 in a trivial way. Finally, Proposition 3 $\Longrightarrow$ Proposition 1; just take

$$d(x, y) = |\varphi(x) - \varphi(y)|, \quad x, y \in M \qquad \text{(where } \varphi \text{ is the above one).}$$

Summing up, all these variants of the Brezis-Browder ordering principle (Proposition 1) are nothing but logical equivalents of it.

(B) A basic application of these facts is to "monotone" variational principles. Let M be a nonempty set; and $(\leq)$, some quasi-order on it. All notions to be used refer to its *dual* $(\geq)$; but these may be also formulated in terms of $(\leq)$. Precisely, let $d : M \times M \to R_+$ be a *metric* (i.e.: sufficient semimetric) over it. Call the subset Z of M, $(\geq)$-*closed* when the limit of each ascending (modulo $(\geq)$) sequence in Z belongs to Z. Clearly, any closed part of M is $(\geq)$-closed too; but the converse is not in general true. (Just take $M = R$ (endowed with the usual order/metric; and choose $Z = [0, 1[$). Further, call the quasi-order $(\geq)$, *self-closed* provided $M(x, \geq) := \{u \in M; x \geq u\}$ is $(\geq)$-closed, for each $x \in M$; or, equivalently: the limit of each ascending sequence is an upper bound of it (modulo $(\geq)$). For example, this is the case when $(\geq)$ is *semi-closed*: $M(x, \geq)$ is closed, for each $x \in M$. Finally, call the ambient metric d, $(\geq)$-*complete* provided each ascending (modulo $(\geq)$) d-Cauchy sequence converges. As before, if d is complete, then it is $(\geq)$-complete too. The reciprocal is not in general true; take $M = [0, 1[$ endowed with the standard order/metric.

We are now in position to state the announced result. Take a function $\varphi : M \to R \cup \{\infty\}$ fulfilling (1.1) as well as

$$\begin{aligned} &\varphi \text{ is } (\geq)\text{-lsc: } \liminf_n \varphi(x_n) \geq \varphi(x), \\ &\text{whenever } (x_n) \text{ is } (\geq)\text{-ascending and } x_n \to x. \end{aligned} \tag{2.6}$$

Proposition 4. *Let $(\geq)$ be self-closed and d be $(\geq)$-complete. Then*
a) for each $u \in \mathrm{Dom}(\varphi)$ there exists $v \in \mathrm{Dom}(\varphi)$ with

$$u \geq v, d(u, v) \leq \varphi(u) - \varphi(v) \ \textit{(hence } \varphi(u) \geq \varphi(v)\textit{)} \tag{2.7}$$

$$d(v, x) > \varphi(v) - \varphi(x), \textit{ for each } x \in M(v, \geq) \setminus \{v\}. \tag{2.8}$$

aa) if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ fulfill $\varphi(u) - \varphi_* \leq \rho$, then (2.7) gives

$$(\varphi(u) \geq \varphi(v) \textit{ and) } u \geq v, d(u, v) \leq \rho. \tag{2.9}$$

The original argument is that appearing in Turinici [29]. For the sake of completeness, we shall provide it, with some modifications.

Proof of Proposition 4. Denote for simplicity $M[u] = \{x \in M; u \geq x, \varphi(u) \geq \varphi(x)\}$ (where u is the above one). Clearly, $\emptyset \neq M[u] \subseteq \mathrm{Dom}(\varphi)$; moreover, by (2.6) (and the choice of $(\geq)$),

$$M[u] \text{ is } (\geq)\text{-closed; hence } d \text{ is } (\geq)\text{-complete on } M[u]. \tag{2.10}$$

Let $(\preceq)$ stand for the relation (over M)

$$(x, y \in M)\colon x \preceq y \text{ iff } x \geq y,\ d(x, y) + \varphi(y) \leq \varphi(x).$$

It is not hard to see that $(\preceq)$ acts as an *order* (antisymmetric quasi-order) on $\mathrm{Dom}(\varphi)$; so, it remains as such on $M[u]$. We claim that conditions of Proposition 3 are fulfilled on $(M[u], d; \preceq)$. In fact, let (x_n) be an ascending (modulo $(\preceq)$) sequence in $M[u]$:

$$x_n \geq x_m \text{ and } d(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \text{ if } n \leq m. \tag{2.11}$$

The sequence $(\varphi(x_n))$ is descending and (by (1.1)) bounded from below; hence a Cauchy one. This, along with the preceding relation shows that (x_n) is an ascending (modulo $(\preceq)$) d-Cauchy sequence; wherefrom $(M[u], \preceq)$ is regular (modulo d). Moreover, the obtained properties give us (by (2.10)) some $y \in M[u]$ with $x_n \to y$. Combining with (2.11) one derives (via (2.6) and the choice of $(\leq)$)

$$x_n \geq y, d(x_n, y) \leq \varphi(x_n) - \varphi(y), \text{ (i.e.: } x_n \preceq y), \text{ for all } n.$$

In other words, $y \in M[u]$ is an upper bound (modulo $(\preceq)$) of (x_n); and this shows that $(M[u], \preceq)$ is sequentially inductive. By Proposition 3 it then follows that, for the starting $u \in M[u]$ there exists $v \in M[u]$ with

$$u \preceq v \text{ and } v \text{ is } (\preceq, d)\text{-maximal in } M[u].$$

The former of these is just (2.7). And the latter one gives at once (2.8); because it reads: $x \in M[u]$ and $v \preceq x$ imply $v = x$. The last part is evident; so, the conclusion follows. ■

A basic particular case of our developments corresponds to the choice $(\leq) = (\geq) = M \times M$ (=the *trivial* quasi-order on M). The regularity condition (2.6) may then be written as in (1.2); and Proposition 4 is nothing but Ekeland's variational principle (Theorem E). On the other hand, the same requirement holds under

$$\varphi \text{ is } (\leq)\text{-increasing } (x \leq y \Longrightarrow \varphi(x) \leq \varphi(y)) \tag{2.12}$$

and the self-closeness of $(\geq)$. For this reason, Proposition 4 will be called the *monotone* version of Theorem E. Note that, by the remarks above, it may be also derived from Proposition 1 as well. Finally, an interesting question is that of Proposition 4 being deductible from Theorem E; we conjecture that the answer is positive.

3 Normal functions

(A) Let $b : R_+ \to R_+$ stand for a *normal* function (in the sense of (1.6)+(1.7)). In particular, it is Riemann integrable on each compact interval of R_+ and

$$\int_p^q b(\xi)d\xi = (q - p)\int_0^1 b(p + \tau(q - p))d\tau, \quad 0 \leq p < q < \infty. \tag{3.1}$$

Some basic facts involving the couple (b, B) (where $B : R_+ \to R_+$ is that of (1.7)) are precise in

Lemma 1. *The following are valid*

(i) B is topord (continuous order isomorphism of R_+); hence, so is B^{-1} (its functional inverse)

(ii) $b(s) \le (B(s) - B(t))/(s - t) \le b(t), \forall t, s \in R_+, t < s$

(iii) B is almost concave: $t \vdash [B(t + s) - B(t)]$ is decreasing on R_+, $\forall s \in R_+$

(iv) B is concave: $B(t + \lambda(s - t)) \ge (1 - \lambda)B(t) + \lambda B(s)$, for all $t, s \in R_+$ with $t < s$ and all $\lambda \in [0, 1]$

(v) B is sub-additive (hence B^{-1} is super-additive).

The proof is immediate, by (3.1) above; so, we do not give details. Note that the properties in (iii) and (iv) are equivalent to each other, under (i). This follows at once from the (non-differential) mean value theorem in Bantaş and Turinici [3].

(B) Now, let (M, d) be a (complete or not) metric space; and $\Gamma : M \to R_+$, some function with

$$|\Gamma(x) - \Gamma(y)| \le d(x, y),\ \forall x, y \in M \text{ (non-expansiveness).} \tag{3.2}$$

Given the inf-proper function $\varphi : M \to R \cup \{\infty\}$ let us attach it the function $\psi = \psi(B, \Gamma; \varphi)$ from M to $R_+ \cup \{\infty\}$ as

$$\psi(x) = B^{-1}[B(\Gamma(x)) + (\varphi(x) - \varphi_*)] - \Gamma(x), \quad x \in M. \tag{3.3}$$

This may be viewed as an "explicit" formula; its "implicit" version is

$$\varphi(x) = \varphi_* + [B(\Gamma(x) + \psi(x)) - B(\Gamma(x))], \quad x \in M. \tag{3.4}$$

By the admitted hypotheses (and Lemma 1), it is not hard to see that

$$\varphi(x) = \infty \text{ iff } \psi(x) = \infty \quad \text{(hence Dom}(\varphi) = \text{Dom}(\psi)).$$

This, in particular, says that ψ is inf-proper too. Further, let $(\le)$ be a quasi-order on M; and $(\ge)$, its dual. From the definitions/properties above

$$\varphi \text{ is } (\ge)\text{-lsc if and only if } \psi \text{ is } (\ge)\text{-lsc.} \tag{3.5}$$

The verification is immediate; so we do not give details. The last property of (φ, ψ) to be discussed here is of variational nature. Precisely, let $(\triangleleft)$ be either of the relations $\{<, \le\}$.

Lemma 2. *Under these conventions,*

$$b(\Gamma(x))d(x, y) + \varphi(y) \triangleleft \varphi(x) \Longrightarrow d(x, y) + \psi(y) \triangleleft \psi(x). \tag{3.6}$$

Proof. Let the points $x, y \in M$ be as in the premise of this implication; without loss, one may assume that (in addition) $x, y \in \text{Dom}(\varphi)$ (hence $x, y \in \text{Dom}(\psi)$). By Lemma 1 (ii) and the implicit formula (3.4), this yields

$$B(\Gamma(x) + d(x, y)) - B(\Gamma(x)) \triangleleft$$
$$[B(\Gamma(x) + \psi(x)) - B(\Gamma(x))] - [B(\Gamma(y) + \psi(y)) - B(\Gamma(y))];$$

or equivalently (by a simple re-arrangement)

$$B(\Gamma(x)+d(x,y))+B(\Gamma(y)+\psi(y))-B(\Gamma(y)) \lhd B(\Gamma(x)+\psi(x)).$$

On the other hand, non-expansiveness condition (3.2) gives $\Gamma(x)+d(x,y) \geq \Gamma(y)$; so, by Lemma 1 (iii)

$$B(\Gamma(x)+d(x,y)+\psi(y))-B(\Gamma(x)+d(x,y)) \leq B(\Gamma(y)+\psi(y))-B(\Gamma(y)).$$

A simple combination with the previous relation yields

$$B(\Gamma(x)+d(x,y)+\psi(y)) \lhd B(\Gamma(x)+\psi(x)).$$

It suffices now taking Lemma 1 (i) into account to get the conclusion. ■

In particular, the non-expansiveness condition (3.2) holds under

$$\Gamma(x)=d(a,x), \quad x \in M, \quad \text{for some } a \in M. \tag{3.7}$$

Note that, in such a case Lemma 2 includes directly the statement in Park and Bae [21]. Further aspects may be found in Suzuki [24].

4 Main results

With these information at hand, we may now return to the questions of the introductory part. Let M be some nonempty set. Consider a quasi-order $(\leq)$ and a metric $d : M \times M \to R_+$ in such a way that

$$(\geq) \text{ is self-closed and } d \text{ is } (\geq)\text{-complete.} \tag{4.1}$$

Further, let $b : R_+ \to R_+$ stand for a normal function (in the sense of (1.6)+(1.7)); and $\Gamma : M \to R_+$ be non-expansive (cf. (3.2)). Finally, pick some function $\varphi : M \to R \cup \{\infty\}$ according to (1.1)+(2.6); and let $\psi = \psi(B, \Gamma; \varphi)$ stand for its associated via (3.3)/(3.4) function. Remember that, by the remarks in Section 3, $\mathrm{Dom}(\varphi) = \mathrm{Dom}(\psi)$ (hence ψ is inf-proper too); and (2.6) is true for ψ as well. The main result of this exposition is

Theorem 1. *Let $u \in \mathrm{Dom}(\varphi)$ be given. Then,*
b) there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with

$$\begin{aligned} &u \geq v,\ \varphi(u) \geq \varphi(v),\ d(u,v) \leq \psi(u)-\psi(v); \\ &\text{hence } d(u,v) \leq B^{-1}[B(\Gamma(u))+(\varphi(u)-\varphi_*)]-\Gamma(u) \end{aligned} \tag{4.2}$$

$$\begin{aligned} &d(v,x) > \psi(v)-\psi(x),\ \forall\, x \in M(v,\geq) \setminus \{v\} \\ &(\text{hence } b(\Gamma(v))d(v,x) > \varphi(v)-\varphi(x)),\ \forall\, x \in M(v,\geq) \setminus \{v\}) \end{aligned} \tag{4.3}$$

bb) If $u \in \mathrm{Dom}(\varphi), \rho > 0$ satisfy $\varphi(u)-\varphi_* \leq B(\Gamma(u)+\rho)-B(\Gamma(u))$, the above evaluation (4.2) gives (2.9).

Proof. Denote for simplicity $M_u = \{x \in M; u \geq x, \varphi(u) \geq \varphi(x)\}$ (where u is the above one). By (2.6) (and the choice of $(\leq)$), M_u is $(\geq)$-closed; hence d is $(\geq)$-complete on M_u.

Let again ψ stand for the restriction to M_u of the function $\psi = \psi(B, \Gamma; \varphi)$. By the remarks we just made, Proposition 4 is applicable to $(M_u, \leq; d)$ and ψ. So, for the starting point $u \in M_u$, there exists another one $v = v(u)$ in M_u with the properties (2.7)+(2.8). The former of these gives (4.2), by the definition of M_u and $d(u, v) \leq \psi(u)$. Moreover, if our data are like in the premise of *bb)* then (by (4.2))

$$d(u, v) \leq B^{-1}[B(\Gamma(u)) + (\varphi(u) - \varphi_*)] - \Gamma(u) \leq \\ B^{-1}(B(\Gamma(u) + \rho)) - \Gamma(u) = \rho;$$

and the conclusion (2.9) is clear. On the other hand, the latter of these properties yields (4.3) via Lemma 2. The proof is thereby complete. ∎

Now, Theorem 1 includes Proposition 4, to which it reduces when $b(t) = 1, t \in R_+$ (hence B= the identity, $\psi = \varphi - \varphi_*$). The reciprocal inclusion also holds, by the argument above; so, Theorem 1 is logically equivalent with Proposition 4. Note that such a conclusion cannot be deduced under the way described in Bao and Khanh [4]; because the ψ-localizing evaluation (4.2) is not accessible there. On the other hand, when $(\leq)$ is the trivial quasi-order on M, Theorem 1 reduces to the "functional" variational principle in Turinici [31]. But, as precise there, this result includes the one in Zhong [33] (Theorem Z) when Γ is taken as in (3.7); hence, so does our statement. The question of the reciprocal implication being also retainable is open; we conjecture that the answer is positive. Finally, by the remarks in Section 2, our statement is also reducible to the Brezis-Browder ordering principle [5]. For a direct proof of this (in the trivial quasi-order setting) we refer to Ray and Walker [22].

It is to be stressed here the special role of (4.2) in these developments. For practical reasons, it is natural to ask under which conditions may this local relation be written in the simpler way of (4.3) (with opposite inequality sign). An appropriate answer may be given under the lines below. Let $\{P, Q\}$ be a *partition* of M (i.e.: $P, Q \neq \emptyset$, $P \cap Q = \emptyset$, $P \cup Q = M$); we term it Γ-*separated*, when

$$\Gamma(x) \leq \Gamma(y), \forall x \in Q, \forall y \in P \text{ [i.e.: } \sup_{x \in Q} \Gamma(x) \leq \inf_{y \in P} \Gamma(y) \text{].}$$

Theorem 2. *Let $\{P, Q\}$ be a Γ-separated partition of M and $z \in P \cap \mathrm{Dom}(\varphi)$ be such that*

$$\text{z is $(b, \Gamma; \varphi, \geq)$-maximal in P:} \\ b(\Gamma(z))d(z, x) > \varphi(z) - \varphi(x), \text{ for all } x \in P(z, \geq) \setminus \{z\} \tag{4.4}$$

$$\text{z is not $(b, \Gamma; \varphi, \geq)$-maximal in M:} \\ b(\Gamma(z))d(z, x) \leq \varphi(z) - \varphi(x), \text{ for some } x \in M(z, \geq) \setminus \{z\}. \tag{4.5}$$

There exists then $v = v(z) \in Q \cap \mathrm{Dom}(\varphi)$ in such a way that (4.3) (the second half) holds, as well as

$$z \geq v,\ b(\Gamma(z))d(z, v) \leq \varphi(z) - \varphi(v) \text{ (hence } \varphi(z) \geq \varphi(v)\text{).} \tag{4.6}$$

Proof. Denote $M[z] = \{x \in M; z \geq x, b(\Gamma(z))d(z, x) \leq \varphi(z) - \varphi(x)\}$. By the hypotheses about $(\leq)$ and φ, $M[z]$ is $(\geq)$-closed; hence d is $(\geq)$-complete over it. In addition (by (4.5) and the choice of $\{P, Q\}$), $M[z]$ has at least two elements; precisely

$$z \in M[z] \subseteq \mathrm{Dom}(\varphi), \quad \emptyset \neq M[z] \setminus \{z\} \subseteq Q. \tag{4.7}$$

Let again ψ stand for the restriction to $M[z]$ of the function $\psi = \psi(B, \Gamma; \varphi)$ (introduced via (3.3/(3.4)). By the developments of the preceding section, Proposition 4 is applicable to $(M[z], \leq; d)$ and ψ. So, for the starting point $z \in M[z]$ there exists $v = v(z) \in M[z]$ fulfilling (2.7)+(2.8) (with z in place of u). This gives (4.6) (by the very definition of $M[z]$); so, it remains to establish that (4.3) (the second half) holds too. Suppose that the mentioned property would be false; i.e.,

$$b(\Gamma(v))d(v, y) \leq \varphi(v) - \varphi(y), \quad \text{for some } y \in M(v, \geq) \setminus \{v\}. \tag{4.8}$$

By (4.7) one has $v \in \{z\} \cup Q$; wherefrom $\Gamma(v) \leq \Gamma(z)$ (cf. the choice of $\{P, Q\}$ and $\{z\}$). This, along with (4.8), yields (via (1.6)) $b(\Gamma(z))d(v, y) \leq \varphi(v) - \varphi(y)$; so that (combining with (4.6))

$$(z \geq y \text{ and}) \; b(\Gamma(z))d(z, y) \leq \varphi(z) - \varphi(y); \quad \text{that is: } y \in M[z].$$

The working hypothesis now gives (via Lemma 2)

$$d(v, y) \leq \psi(v) - \psi(y), \quad \text{for some } y \in M[z](v, \geq) \setminus \{v\}.$$

This, however, contradicts (2.8) for the precise data. Hence, (4.8) cannot be true; and the claim follows. As a direct consequence, one has $v \in M[z] \setminus \{z\} \subseteq Q$ (cf. (4.5) and (4.7)); wherefrom, all is clear. ■

In particular, when $(\leq)$ is the trivial quasi-order on M, this result is just the one in Turinici [op. cit.].

5 Metric functional coercivity

Let X be some nonempty set. Take a quasi-order $(\leq)$ over it; as well as a metric $d : X \times X \to R_+$. For an easy reference we list the basic hypotheses to be used. The former of these is just (4.1); this is related to the results in Section 4 being applicable here. And the latter of these may be written as

$$\{y \in X(x, \geq); 0 < d(x, y) < \rho\} \neq \emptyset, \quad \forall x \in X, \forall \rho > 0; \tag{5.1}$$

the reason of imposing it is "differential" in nature (see below). Further, let us take some couple of maps $\Gamma : X \to R_+$, $F : X \to R \cup \{\infty\}$ with

$$\Gamma \text{ is } d\text{-nonexpansive, } \sup[\Gamma(X)] = \infty \text{ and } F \text{ is inf-proper, } (\geq)\text{-lsc.} \tag{5.2}$$

Denote, for each $\sigma > 0$

$$X_\Gamma(\sigma) = \{x \in X; \Gamma(x) \geq \sigma\}, \; m(\Gamma, F)(\sigma) = \inf[F(X_\Gamma(\sigma))].$$

The subsets $\{X_\Gamma(\sigma); \sigma > 0\}$ are nonempty closed (in X). Moreover, the map $\sigma \vdash m(\Gamma, F)(\sigma)$ is nondecreasing from $R^0_+ =]0, \infty[$ to $R \cup \{\infty\}$; wherefrom

$$\liminf_{\Gamma(u) \to \infty} F(u) := \sup_{\sigma > 0} m(\Gamma, F)(\sigma) \; [= \lim_{\sigma \to \infty} m(\Gamma, F)(\sigma)]$$

exists, as an element of $R \cup \{\infty\}$, in view of

$$F_* \leq m(\Gamma, F)(\sigma) \leq \alpha(\Gamma, F) := \liminf_{\Gamma(u)\to\infty} F(u) \leq \infty, \quad \forall \sigma > 0 \tag{5.3}$$

(where $F_* := \inf[F(X)]$). When $\alpha(\Gamma, F) = \infty$, the functional F will be referred to as Γ-*coercive*. It is our aim in the following to get sufficient conditions in order that such a property be attained. These, as a rule, require a differential setting (relative to F). Denote, for each $u \in \mathrm{Dom}(F)$,

$$|\nabla_{(\leq)}|F(u) = \max\{0, \nabla_{(\leq)}F(u) := \limsup_{\substack{x\leq u \\ x\to u}} \frac{F(u) - F(x)}{d(u,x)}\}. \tag{5.4}$$

This object may be viewed as a quasi-order version of the one introduced in 1980 by DeGiorgi, Marino and Tosques [8]; we shall term it the strong $(\leq, d)$-*slope* of F at u. The usefulness of its amorph version ($(\leq) = X \times X$) for the critical point theory was underlined in the 1993 paper by Corvellec, DeGiovanni and Marzocchi [7]. Here, we shall establish that the concept (5.4) is the natural tool for our (functional) coercivity question. Take some normal function $b : R_+ \to R_+$ (cf. (1.6)+(1.7)). The following asymptotic type statement is a basic step for the desired answer.

Theorem 3. *Suppose that*

$$\alpha(\Gamma, F) < \infty \quad \textit{(hence (cf. (5.3)) } \alpha(\Gamma, f) \textit{ is finite).} \tag{5.5}$$

There exists then a sequence (v_n) *in* $\mathrm{Dom}(F) \cap \Gamma^{-1}(R^0_+)$ *with*

$$\Gamma(v_n) \to \infty \textit{ (hence } \Gamma(y_n) \to \infty\textit{, for each subsequence } (y_n) \textit{ of } (v_n)\textit{)} \tag{5.6}$$

$$F(v_n) \to \alpha(\Gamma, F) \textit{ and } \frac{1}{b(\Gamma(v_n))}|\nabla_{(\leq)}|F(v_n) \to 0 \textit{ as } n \to \infty. \tag{5.7}$$

Proof. There are two steps to be passed.

(i) Let ε be arbitrary fixed in $]0, 1/3[$. By the remarks involving (5.3), there exists $r(\varepsilon)$ with

$$r(\varepsilon) \geq 1/\varepsilon, \text{ and } m(\Gamma, F)(r) > \alpha(\Gamma, F) - \varepsilon^2, \forall r \geq r(\varepsilon). \tag{5.8}$$

Having this precise, we claim that

$$\begin{aligned} &\text{there exists } v_\varepsilon \in X \text{ with } \Gamma(v_\varepsilon) \geq r(\varepsilon) \text{ and} \\ &|F(v_\varepsilon) - \alpha(\Gamma, F)| \leq \varepsilon^2, \ \frac{1}{b(\Gamma(v_\varepsilon))}|\nabla_{(\leq)}|F(v_\varepsilon) \leq \varepsilon. \end{aligned} \tag{5.9}$$

To this end, fix $s > r(\varepsilon)$; as well as some (p, q) with $q > p > s$ and

$$B(q) - B(p) \geq 1; \text{ or equivalently, } \int_p^q b(\tau)d\tau \geq 1.$$

(This is evidently possible, by the normality of b). From the definition of the Riemann integral, there exists a finite system of points $(r_n; 0 \le n \le k)$ (where $k \ge 3$) with $p = r_k < r_{k-1} < ... < r_1 < r_0 = q$ (division of $[p, q]$) in such a way that

$$|\int_p^q b(\tau)d\tau - \sum_{i=0}^{k-1} b(r_i)(r_i - r_{i+1})| < \varepsilon;$$

wherefrom (by the decreasing property of b)

$$\int_p^q b(\tau)d\tau = \sum_{i=0}^{k-1} \int_{r_{i+1}}^{r_i} b(\tau)d\tau \ge \sum_{i=0}^{k-1} b(r_i)(r_i - r_{i+1}) > \int_p^q b(\tau)d\tau - \varepsilon. \tag{5.10}$$

Without loss, one may complete this finite system up to some sequence (r_n) with $r_{k+1} = s$, (r_n) is strictly descending and $\lim_n (r_n) = r(\varepsilon)$. Denote for simplicity $X_n = X_\Gamma(r_n)$ (hence $X_n \subseteq \text{int}[X_{n+1}]$), for all n. By (5.8) (and the definition of these quantities)

$$\alpha(\Gamma, F) - \varepsilon^2 < m(\Gamma, F)(r_0) < \alpha(\Gamma, F) + \varepsilon^2; \quad \text{wherefrom}$$

$$m(\Gamma, F)(r_0) \le F(u_{-1}) < \alpha(\Gamma, F) + \varepsilon^2, \quad \text{for some } u_{-1} \in X_0 \cap \text{Dom}(F).$$

Put also $\varphi = (1/\varepsilon)F$, $\psi = \psi(B, \Gamma; \varphi)$ (the associated by (3.3)/(3.4) function). Take $i = 0$; and note that, by the imposed conditions, Theorem 1 is applicable to the data $[(M = X_i, \le; d)$; $\varphi =$ as before; $u = u_{i-1}]$. So, for the starting point $u_{i-1} \in X_i \cap \text{Dom}(F)$ there exists $u_i \in X_i \cap \text{Dom}(F)$ with

$$u_{i-1} \ge u_i, F(u_{i-1}) \ge F(u_i), d(u_{i-1}, u_i) \le \psi(u_{i-1}) - \psi(u_i) \tag{5.11}$$

$$\begin{array}{l} u_i \text{ is } (b, \Gamma; \varphi, \ge)\text{-maximal in } X_i\text{:} \\ \varepsilon b(\Gamma(u_i))d(u_i, x) > F(u_i) - F(x), \quad \text{for all } x \in X_i(u_i, \ge) \setminus \{u_i\}. \end{array} \tag{5.12}$$

If the alternative below is true (for $i = 0$)

$$\text{either } \Gamma(u_i) > r_i \text{ or } u_i \text{ is } (b, \Gamma; \varphi, \ge)\text{-maximal in } X_{i+1} \tag{5.13}$$

then (5.11)+(5.12) give directly (for $i = 0$)

$$\Gamma(u_i) \ge r_i, \ \frac{1}{b(\Gamma(u_i))}|\nabla_{(\le)}|F(u_i) \le \varepsilon;$$

(since u_i is interior and maximal in either X_i or X_{i+1}) as well as (see above)

$$m(\Gamma, F)(r_i) \le F(u_i) \le F(u_{i-1}) < \alpha(\Gamma, F) + \varepsilon^2;$$

wherefrom, (5.9) is clear, for $v_\varepsilon = u_i$. Otherwise, the alternative below holds (for $i = 0$)

$$\Gamma(u_i) = r_i \text{ and } u_i \text{ is not } (b, \Gamma; \varphi, \ge)\text{-maximal in } X_{i+1}. \tag{5.14}$$

This allows us applying Theorem 2 to the data $[(M = X_{i+1}, \leq; d); P = X_i,\ \varphi =$ as before, $z = u_i]$. So, for the point $u_i \in X_i \cap \mathrm{Dom}(F)$ there must be $u_{i+1} \in (X_{i+1} \setminus X_i) \cap \mathrm{Dom}(\varphi)$ fulfilling (5.12) (with $i+1$ in place of i) and

$$u_i \geq u_{i+1}, \varepsilon b(\Gamma(u_i))d(u_i, u_{i+1}) \leq F(u_i) - F(u_{i+1}). \tag{5.15}$$

Now, like before, the couple of alternatives (5.13)+(5.14) (with $i+1$ in place of i) comes into discussion, etc. We now claim that the alternative (5.14) cannot be realized at each step. For, if this happens, a sequence $(u_n) \subseteq \mathrm{Dom}(F)$ may be obtained according to

$$\Gamma(u_n) = r_n,\ u_n \geq u_{n+1},\ \varepsilon b(r_n)d(u_n, u_{n+1}) \leq F(u_n) - F(u_{n+1}), \text{ for all } n.$$

The latter of these yields via (5.8)

$$\alpha(\Gamma, F) - \varepsilon^2 < m(\Gamma, F)(r_n) \leq F(u_n) \leq F(u_0) \leq \alpha(\Gamma, F) + \varepsilon^2, \quad \forall n.$$

This, along with the former of these, leads us to an evaluation like

$$\sum_{i=0}^{k-1} b(r_i)(r_i - r_{i+1}) = \sum_{i=0}^{k-1} b(r_i)|\Gamma(u_i) - \Gamma(u_{i+1})| \leq \sum_{i=0}^{k-1} b(r_i)d(u_i, u_{i+1})$$
$$\leq \frac{1}{\varepsilon}\sum_{i=0}^{k-1}(F(u_i) - F(u_{i+1})) = \frac{1}{\varepsilon}(F(u_0) - F(u_k)) \leq 2\varepsilon.$$

But then, a simple combination with (5.10) tells us that

$$1 \leq \int_p^q b(\tau)d\tau < \sum_{i=0}^{k-1} b(r_i)(r_i - r_{i+1}) + \varepsilon \leq 3\varepsilon; \text{ contradiction.}$$

Consequently, the alternative (5.13) must be realized at a certain step $h \leq k$; wherefrom (cf. the arguments above) the claim (5.9) holds with $v_\varepsilon = u_h$.

(ii) Let (ε_n) be a descending convergent to zero sequence in $]0, 1/3[$ and put $r_n = r(\varepsilon_n)$[=the quantity of (5.8)], $\forall n$. Note that, by this choice, $r_n \to \infty$ as $n \to \infty$. Moreover, the developments in **(i)** give us a sequence $(v_n = v_{\varepsilon_n})$ in X fulfilling

$$\Gamma(v_n) \geq r_n,\ |F(v_n) - \alpha(\Gamma, F)| \leq \varepsilon_n^2,\ \frac{1}{b(\Gamma(v_n))}|\nabla_{(\leq)}|F(u_n) \leq \varepsilon_n, \text{ for all } n.$$

But, from this, (5.6)+(5.7) are clear. The proof is thereby complete. ■

We are now in position to give the promised answer to our coercivity question. The "hybrid" condition below is needed:

$$\begin{aligned}&\text{each sequence } (x_n) \text{ in } \mathrm{Dom}(F) \cap \Gamma^{-1}(R_+^0) \text{ for which}\\ &(F(x_n)) \text{ converges and } \frac{1}{b(\Gamma(x_n))}|\nabla_{(\leq)}|F(x_n) \to 0\\ &\text{has a subsequence } (y_n) \text{ with } (\Gamma(y_n)) \text{ bounded;}\end{aligned} \tag{$(PS(\Gamma, b; \leq))_s$}$$

we call it, a standard *Palais-Smale condition* (modulo $(\Gamma, b; \leq)$) upon F.

Theorem 4. *Suppose that (in addition) F satisfies a standard Palais-Smale condition (modulo $(\Gamma, b; \leq)$). Then, F is necessarily Γ-coercive.*

Proof. If, by absurd, this cannot happen, the relation (5.5) must be true. By Theorem 3, we have promised a sequence (v_n) in $\mathrm{Dom}(F) \cap \Gamma^{-1}(R_+^0)$ with the properties (5.6)+(5.7). The latter of these and the imposed Palais-Smale condition tells us that (v_n) must have a subsequence (y_n) with $(\Gamma(y_n))$ bounded (in R_+). And, by the former of these, $\Gamma(y_n) \to \infty$. The obtained contradiction shows that (5.5) cannot be true; hence the conclusion. ■

An interesting question to be raised is that of the regularity conditions in (4.1)+(5.1) being the most general ones. An appropriate answer to this is available by the above methods; we shall discuss these facts elsewhere.

6 Normed aspects

A basic particular case of these developments corresponds to the ambient (metric) space X being, in addition, linear. So, assume that X is such an object. Let K be some (non-degenerate) convex cone of it $[K + K \subseteq K; \lambda K \subseteq K$, for each $\lambda \in R_+]$. Denote by $(\leq)$ the associated quasi-order (on X): $x \leq y$ iff $y - x \in K$. This, by the choice of K, is also linear ($x \leq y$ implies $x + a \leq y + a$, $\lambda x \leq \lambda y$, for each $a \in X$, $\lambda \in R_+$). On the other hand, $(-K)$ is also a convex cone of X; and its associated quasi-order is just $(\geq)$ (the dual of $(\leq)$). Let also $||.||$ stand for a norm over X; and denote by d the (standard) metric on X induced by it. Our basic condition is

$$K \text{ is } (\leq)\text{-closed and } d \text{ is } (\leq)\text{-complete.} \tag{6.1}$$

Note that, as a consequence of this (and the linearity of X) one has: $(-K)$ is $(\geq)$-closed and d is $(\geq)$-complete; wherefrom, (4.1) is fulfilled. Moreover (as K is non-degenerate)

$$\{y \in x - K; 0 < d(x, y) < \rho\} \neq \emptyset, \quad \text{for each } x \in X, \rho > 0; \tag{6.2}$$

and this shows that (5.1) holds too for our data. Finally, let the map $\Gamma : X \to R_+$ and the functional $F : X \to R \cup \{\infty\}$ be as in (5.2). By the remarks above, the differential operators in (5.4) are well defined. A basic fact to be added is that, by the linear setting we adopted, they may be "approximated" in terms of directional derivatives. This will necessitate some conventions. Denote, for each $u \in \mathrm{Dom}(F)$,

$$DF(u)(h) = \limsup_{t \to 0+} \frac{1}{t}(F(u) - F(u + th)), \quad h \in X. \tag{6.3}$$

This object always exists, as an element of $R \cup \{-\infty, \infty\}$; it will be referred to as the *h-directional* derivative of F at u. In addition, $h \vdash DF(u)(h)$ is positively homogeneous; but, in general, it is not sub-additive. Denote

$$|\Lambda_{(K)}|F(u) = \max[0, \Lambda_{(K)}F(u) := \sup\{DF(u)(h); h \in (-K) \cap X_{(1)}\}]. \tag{6.4}$$

(Here, $X_{(1)} = \{x \in X; ||x|| = 1\}$ is the boundary of the unit sphere in X). The connection between this operator and the one of (5.4) is stated in

Lemma 3. *For each* $u \in \mathrm{Dom}(F)$, *one has*

$$\Lambda_{(K)}F(u) \leq \nabla_{(\leq)}F(u); \quad \textit{hence} \quad |\Lambda_{(K)}|F(u) \leq |\nabla_{(\leq)}|F(u). \tag{6.5}$$

Proof. Let $h \in (-K) \cap X_{(1)}$ be arbitrary fixed. For each δ in $]0, \infty[$,

$$\sup_{0<t<\delta} \frac{F(u) - F(u + th)}{t} \leq \sup_{\substack{x \leq u \\ 0<d(x,u)<\delta}} \frac{F(u) - F(x)}{d(u, x)}.$$

Passing to infimum after δ yields $DF(u)(h) \leq \nabla_{(\leq)}F(u)$, for each $h \in (-K) \cap X_{(1)}$; wherefrom, taking the supremum over all such h, we are done. ■

Having these precise, let $b : R_+ \to R_+$ be some normal function (cf. (1.6)+(1.7)). We consider the (differential type) hybrid condition

$$\begin{array}{l}\text{each sequence } (x_n) \text{ in } \mathrm{Dom}(F) \cap \Gamma^{-1}(R^0_+) \text{ for which} \\ (F(x_n)) \text{ converges and } \dfrac{1}{b(\Gamma(x_n))}|\Lambda_{(K)}|F(x_n) \to 0 \\ \text{has a subsequence } (y_n) \text{ with } (\Gamma(y_n)) \text{ bounded.}\end{array} \qquad (PS(\Gamma, b; K))_d$$

and call it, a *directional* Palais-Smale condition (modulo $(\Gamma, b; K)$) upon F. Clearly,

$$(\mathrm{PS}(\Gamma, b; K))_d \Longrightarrow (\mathrm{PS}(\Gamma, b; \leq))_s \quad \text{(cf. Lemma 3).}$$

(Here, in the right hand side, $(\leq)$ is the induced by K quasi-order). By Theorem 4 we thus have (under the same general conditions)

Theorem 5. *Assume that (in addition) F satisfies a directional Palais-Smale condition (modulo $(\Gamma, b; K)$). Then, F is Γ-coercive (in the above precise sense).*

In particular, when $K = X$ and

$$F \text{ is Gateaux differentiable over } \mathrm{Dom}(F) \tag{6.6}$$

Theorem 5 is nothing but the coercivity result in Turinici [31]; which, in turn, extends Zhong's [33]. On the other hand, if $b = 1$ (hence B=the identity) Theorem 5 covers the order coercivity result in D. Motreanu, V. V. Motreanu and M. Turinici [18]. Hence, our developments may be viewed as a common extension of all these. Finally, it is worth noting that the monotone type coercivity principle in Motreanu and Turinici [19] is also accessible via such methods. Further aspects may be found in D. Motreanu, V. V. Motreanu and D. Paşca [17].

References

[1] M. Altman, *A generalization of the Brezis-Browder principle on ordered sets,* Nonlinear Analysis, 6 (1981), 157-165.

[2] M. C. Anisiu, *On maximality principles related to Ekeland's theorem,* Seminar Funct. Analysis Numer. Meth. (Faculty of Math. Research Seminars), Preprint No. 1 (8 pp), "Babeş-Bolyai" Univ., Cluj-Napoca (România), 1987.

[3] G. Bantaş and M. Turinici, *Mean value theorems via division methods,* An. Şt. Univ. "A. I. Cuza" Iaşi (S. I-a, Mat.), 40 (1994), 135-150.

[4] T. Q. Bao and P. Q. Khanh, *Are several recent generalizations of Ekeland's variational principle more general than the original principle?,* Acta Math. Vietnamica, 28 (2003), 345-350.

[5] H. Brezis and F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis,* Advances in Math., 21 (1976), 355-364.

[6] O. Cârjă and C. Ursescu, *The characteristics method for a first order partial differential equation,* An. Şt. Univ. "A. I. Cuza" Iaşi (S. I-a, Mat.), 39 (1993), 367–396.

[7] J. N. Corvellec, M. DeGiovanni and M. Marzocchi, *Deformation properties for continuous functionals and critical point theory,* Topol. Meth. Nonlin. Analysis, 1 (1993), 151-171.

[8] E. DeGiorgi, A. Marino and M. Tosques, *Problemi di evoluzione in spazi metrici e curve di massima pendenza,* Rend. Accad. Naz. Lincei (Serie 8), 68 (1980), 180-187.

[9] I. Ekeland, *On the variational principle,* J. Math. Analysis Appl., 47 (1974), 324-353.

[10] I. Ekeland, *Nonconvex minimization problems,* Bull. Amer. Math. Soc. (New Series), 1 (1979), 443-474.

[11] A. Goepfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces,* [Canad. Math. Soc. Books in Math. vol. 17], Springer, New York, 2003.

[12] A. Hamel, *Variational Principles on Metric and Uniform Spaces,* (Habilitation Thesis), Martin-Luther University, Halle-Wittenberg (Germany), 2005.

[13] G. Isac, *The Ekeland's principle and Pareto ε-efficiency,* in "Multi–Objective Programming and Goal Programming" (M.Tamiz ed.), pp. 148–163, L. Notes Econ. Math. Systems vol. 432, Springer, Berlin, 1996.

[14] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces,* Math. Japonica, 44 (1996), 381-391.

[15] B. G. Kang and S. Park, *On generalized ordering principles in nonlinear analysis,* Nonlinear Analysis, 14 (1990), 159-165.

[16] P. Q. Khanh, *On Caristi-Kirk's theorem and Ekeland's variational principle for Pareto extrema,* Bull. Polish Acad. Sci. (Math.), 37 (1988), 33-39.

[17] D. Motreanu, V. V. Motreanu and D. Paşca, *A version of Zhong's coercivity result for a general class of nonsmooth functionals,* Abstr. Appl. Analysis, 7 (2002), 601-612.

[18] D. Motreanu, V. V. Motreanu and M. Turinici, *Coerciveness property on quasi-ordered Banach spaces,* Nonlinear Funct. Analysis Appl., 7 (2002), 155-166.

[19] V. V. Motreanu and M. Turinici, *Coercivity properties for monotone functionals,* Note Mat., 21 (2002), 83-91.

[20] A. B. Nemeth, *A nonconvex vector minimization problem,* Nonlinear Analysis, 10 (1986), 669-678.

[21] S. Park and J. S. Bae, *On the Ray-Walker extension of the Caristi-Kirk fixed point theorem,* Nonlinear Analysis, 9 (1985), 1135-1136.

[22] W. O. Ray and A. Walker, *Mapping theorems for Gateaux differentiable and accretive operators,* Nonlinear Analysis, 6 (1982), 423-433.

[23] P. Rozoveanu, *Ekeland's variational principle for vector valued functions,* Math. Reports [St. Cerc. Mat.], 2(52) (2000), 351-366.

[24] T. Suzuki, *Generalized distance and existence theorems in complete metric spaces,* J. Math. Analysis Appl., 253 (2001), 440-458.

[25] A. Szaz, *An Altman type generalization of the Brezis-Browder ordering principle,* Math. Moravica, 5 (2001), 1-6.

[26] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings,* in "Fixed Point Theory and Applications" (J. B. Baillon, ed.), pp. 397-406, Pitman Res. Notes Math. vol. 252, Longman Sci. Tech., Harlow, 1991.

[27] D. Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms,* J. Math. Analysis Appl., 163 (1992), 345-392.

[28] M. Turinici, *Metric variants of the Brezis-Browder ordering principle,* Demonstr. Math., 22 (1989), 213-228.

[29] M. Turinici, *A monotone version of the variational Ekeland's principle,* An. Şt. Univ. "A. I. Cuza" Iaşi (S. I-a, Mat.) 36 (1990), 329-352.

[30] M. Turinici, *Minimal points in product spaces,* An. Şt. Univ. "Ovidius" Constanţa (Ser. Math.), 10 (2002), 109-122.

[31] M. Turinici, *Function variational principles and coercivity,* J. Math. Analysis Appl., 304 (2005), 236-248.

[32] M. Turinici, *Brezis-Browder principles in separable ordered sets,* Libertas Math., 26 (2006), 15-30.

[33] C. K. Zhong, *A generalization of Ekeland's variational principle and application to the study of the relation between the weak P.S. condition and coercivity,* Nonlinear Analysis, 29 (1997), 1421-1431.