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KST Variational Statements on Pseudometric Structures

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Dedicated to Academician Radu Miron on his 80th Birthday

Abstract. A refinement of the variational result in Kada, Suzuki and Takahashi [Math. Japonica, 44 (1996), 381-391] is given via pseudometric variational statements. The connections with some related contributions in Suzuki [J. Math. Anal. Appl., 253 (2001), 440-458] or Lin and Du [J. Math. Anal. Appl., 323 (2006), 360-370] are also investigated. Further, an application is given to variational principles involving the semigroup metrics introduced by Tataru [J. Math. Anal. Appl., 163 (1992), 345-392].

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1 Introduction

Let (M, d) be a complete metric space; and $\varphi : M \to R \cup \{\infty\}$, some function with

$$\varphi \text{ is inf-proper } (\mathrm{Dom}(\varphi) \neq \emptyset \text{ and } \inf[\varphi(M)] > -\infty) \tag{1.1}$$

$$\varphi \text{ is } d\text{-lsc on } M \ (\liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever } x_n \xrightarrow{d} x). \tag{1.2}$$

The following 1974 statement in Ekeland [7] (referred to as Ekeland's variational principle) is well known.

Theorem E. *Let the precise conditions hold. Then*
a) for each $u \in \mathrm{Dom}(\varphi)$ *there exists* $v = v(u) \in \mathrm{Dom}(\varphi)$ *with*

$$d(u, v) \leq \varphi(u) - \varphi(v) \ (\text{hence } \varphi(u) \geq \varphi(v)) \tag{1.3}$$

$$\begin{array}{l} v \text{ is E-variational (modulo } (d, \varphi)\text{):} \\ d(v, x) > \varphi(v) - \varphi(x), \quad \text{for all } x \in M \setminus \{v\}. \end{array} \tag{1.4}$$

aa) if $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ *fulfill* $\varphi(u) - \varphi_* \le \rho$, *then (1.3) gives*

$$(\varphi(u) \ge \varphi(v) \text{ and}) \quad d(u, v) \le \rho. \tag{1.5}$$

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the 1979 paper by Ekeland [8] for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of Theorem E were proposed. For example, the abstract (order) one starts from the fact that, with respect to the (quasi-)order

$$(x, y \in M) \ x \le y \text{ iff } d(x, y) + \varphi(y) \le \varphi(x)$$

the point $v \in M$ appearing in (1.4) is *maximal*; so that, Theorem E is nothing but a variant of the Zorn-Bourbaki maximality principle [4]; see also Taskovic [23]. The dimensional way of extension refers to the support space (R) of Codom(φ) being substituted by a (topological or not) vector space. An account of the results in this area is to be found in the 2003 monograph by Goepfert, Riahi, Tammer and Zălinescu [10, Ch 3]; see also Isac [12], Rozoveanu [20] and Turinici [25]. Finally, the metrical one consists in the conditions imposed to the ambient metric over M being relaxed. The basic 1996 result in this direction obtained by Kada, Suzuki and Takahashi [13] may be stated as follows. By a *pseudometric* over M we shall mean any map $(x, y) \vdash e(x, y)$ from $M \times M$ to $R_+ := [0, \infty[$. Suppose that we fixed such an object; which in addition is *triangular* $[e(x, z) \le e(x, y) + e(y, z), \forall x, y, z \in M]$. We say that it is a *w-distance* (modulo d) over M provided

(W1) $y \vdash e(x, y)$ is d-lsc on M (see above), $\forall x \in M$

(W2) e is strongly d-sufficient: for each $\varepsilon > 0$, there exists $\delta > 0$ such that: $e(z, x), e(z, y) \le \delta \Longrightarrow d(x, y) \le \varepsilon$.

Theorem KST. *Let the conditions in Theorem E be admitted; and e be some w-distance (modulo d) over M. Then*

b) For each $u \in \mathrm{Dom}(\varphi)$, *there exists an E-variational (modulo* (e, φ)*)* $v = v(u) \in \mathrm{Dom}(\varphi)$ *with* $\varphi(u) \ge \varphi(v)$

bb) For each $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ *with* $e(u, u) = 0$, $\varphi(u) \le \varphi_* + \rho$ *there exists an E-variational (modulo* (e, φ)*)* $v = v(u, \rho) \in \mathrm{Dom}(\varphi)$ *with* $\varphi(u) \ge \varphi(v)$ *and* $e(u, v) \le \rho$.

In particular, when $e = d$, these regularity conditions hold; and Theorem KST includes the local version of Theorem E based upon (1.5). The relative form of the same, based upon (1.3) also holds, but indirectly; see Bao and Khanh [3] for details. Note that the (rather involved) authors' argument relies on the nonconvex minimization theorem in Takahashi [22]. It is our aim in the present exposition to show (in Section 3) that a simplification of this is possible. As an argument for its usefulness, we show that the variational principles in Suzuki [21] or Lin and Du [18] are obtainable from such an approach. The basic tool of our investigations is a lot of pseudometric variational statements (given in Section 2) deductible in fact by the original Ekeland's argument. Finally, some applications of the obtained results are given (in Section 4) to variational principles involving (semigroup) Tataru metrics [24]. Further aspects will be delineated elsewhere.

2 Pseudometric variational statements

Let M be some nonempty set. Remember that, by a *pseudometric* over it we shall mean any map $e : M \times M \to R_+$. Suppose that we fixed such an object; which in addition is *triangular* (cf. Section 1); and let $\varphi : M \to R \cup \{\infty\}$ be some inf-proper function (cf. (1.1)). For an easy reference, we shall formulate the basic regularity conditions involving our data. Call the sequence (x_n) in M, *strongly e-asymptotic* when the series $\sum_n e(x_n, x_{n+1})$ converges (in R). Further, let the *e-Cauchy* property of this object be the usual one: $\forall \delta > 0, \exists n(\delta)$, such that $n(\delta) \leq p < q \Longrightarrow e(x_p, x_q) \leq \delta$. The generic relation below is clear (by the triangular property of e)

$$\text{(for each sequence) strongly } e\text{-asymptotic} \Longrightarrow e\text{-Cauchy;} \tag{2.1}$$

but the converse is not in general true. Nevertheless, in many conditions involving *all* such objects, this is retainable. A concrete example is to be constructed under the lines below. Let us introduce an *e-convergence* structure over M by: $x_n \xrightarrow{e} x$ iff $e(x_n, x) \to 0$ as $n \to \infty$. We consider the regularity condition

$$\begin{array}{l}(e,\varphi) \text{ is weakly descending complete: for each strongly} \\ e\text{-asymptotic sequence } (x_n) \text{ in } \mathrm{Dom}(\varphi) \text{ with } (\varphi(x_n)) \text{ descending} \\ \text{there exists } x \in M \text{ with } x_n \xrightarrow{e} x \text{ and } \lim_n \varphi(x_n) \geq \varphi(x).\end{array} \tag{2.2}$$

By (2.1) above, it includes its (stronger) counterpart

$$\begin{array}{l}(e,\varphi) \text{ is descending complete: for each } e\text{-Cauchy sequence} \\ (x_n) \text{ in } \mathrm{Dom}(\varphi) \text{ with } (\varphi(x_n)) \text{ descending there exists } x \in M \\ \text{with the properties } x_n \xrightarrow{e} x \text{ and } \lim_n \varphi(x_n) \geq \varphi(x).\end{array} \tag{2.3}$$

A remarkable fact to be added is that the reciprocal inclusion also holds:

Lemma 1. *We have (2.2) $\Longrightarrow$ (2.3); hence (2.2) $\Longleftrightarrow$ (2.3).*

Proof. Assume that (2.2) holds; and let (x_n) be an e-Cauchy sequence in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending. By the very definition of this property, there must be a strongly e-asymptotic subsequence $(y_n = x_{i(n)})$ of (x_n) with $(\varphi(y_n))$ descending. This, along with (2.2), yields an element $y \in M$ fulfilling $y_n \xrightarrow{e} y$ and $\lim_n \varphi(y_n) \geq \varphi(y)$. It is now clear (by the choice of (x_n)) that the point y has all the desired in (2.3) properties. ∎

Now, let $(\nabla = \nabla_\varphi)$ stand for the relation (over M)

$$(x, y \in M) \ x \nabla y \text{ iff } e(x,y) + \varphi(y) \leq \varphi(x). \tag{2.4}$$

This is clearly transitive ($x\nabla y, y \nabla z \Longrightarrow x \nabla z$); but not in general reflexive ($x \nabla x$ may be false for certain $x \in M$). So, for $u \in M$, it is possible that $M(u, \nabla) := \{x \in M; u \nabla x\}$ be empty. If this does not hold ($M(u,\nabla) \neq \emptyset$), we say that u is *∇-starting*; i.e.,

$$e(u,x) + \varphi(x) \leq \varphi(u), \quad \text{for at least one } x \in M; \tag{2.5}$$

or, more precisely, *starting* (modulo (e,φ)); for example, this holds under

$$u \text{ is } \textit{reflexive} \text{ (modulo } e\text{):} \;\; e(u,u)=0. \tag{2.6}$$

Finally, call $v \in \mathrm{Dom}(\varphi)$, BB-*variational* (modulo (e,φ)) provided

$$e(v,x) \le \varphi(v)-\varphi(x) \Longrightarrow \varphi(v)=\varphi(x) \; [\Longrightarrow e(v,x)=0]. \tag{2.7}$$

(The introduced terminology is related to the methods in Brezis and Browder [5]; see also Altman [1], Anisiu [2] or Kang and Park [14]). For a non-trivial concept, we must take v as starting (modulo (e,φ)); since otherwise, (2.7) is vacuously fulfilled. Some basic properties of such points are collected in

Lemma 2. *Suppose that $v \in \mathrm{Dom}(\varphi)$ is BB-variational (modulo (e,φ)). Then, the following are true*

$$e(v,x) \ge \varphi(v)-\varphi(x), \text{ for all } x \in M \tag{2.8}$$

$$e(v,x) > \varphi(v)-\varphi(x), \text{ for each } x \in M \text{ with } e(v,x)>0. \tag{2.9}$$

Proof. The latter part is clear, by definition; so, it remains to establish the former one. Assume this would be false: $e(v,x) < \varphi(v)-\varphi(x)$, for some $x \in M$. This, along with (2.7), yields $\varphi(v)=\varphi(x)$; wherefrom $0 \le e(v,x) < 0$, contradiction; hence the claim. ■

We may now state a useful pseudometric variational principle.

Theorem 1. *Let the general conditions upon (e,φ) be accepted; as well as (2.2)/(2.3). Then, for each starting (modulo (e,φ)) $u \in \mathrm{Dom}(\varphi)$ there exists a BB-variational (modulo (e,φ)) $v=v(u) \in \mathrm{Dom}(\varphi)$ with*

$$e(u,v) \le \varphi(u)-\varphi(v) \text{ (hence } \varphi(u) \ge \varphi(v)\text{)}. \tag{2.10}$$

Proof. Let (∇) stand for the transitive relation (2.4); and $u \in \mathrm{Dom}(\varphi)$ be as in the statement. By definition, $M(u,\nabla) \ne \emptyset$; hence $u \nabla u_0$, for some $u_0 \in \mathrm{Dom}(\varphi)$. Put $n=0$. If the alternative below holds

$$u_n \text{ is BB-variational (modulo } (e,\varphi)) \;\; \text{[see above]} \tag{2.11}$$

we are done, with $v=u_n$. Otherwise, one has (for $n=0$)

$$u_n \text{ is not BB-variational (modulo } (e,\varphi)). \tag{2.12}$$

In this case, the relations below hold (for $n=0$)

$$S_n := M(u_n,\nabla) \ne \emptyset, \quad \inf[\varphi(S_n)] < \varphi(u_n); \tag{2.13}$$

wherefrom, by definition (again for $n=0$)

$$\exists u_{n+1} \in S_n: \; \varphi(u_{n+1}) \le (1/2)(\varphi(u_n)+\inf[\varphi(S_n)]) < \varphi(u_n). \tag{2.14}$$

Now, like before, the couple (2.11)+(2.12) (with $n+1$ in place of n) comes into discussion, etc. The only point to be clarified is that of the alternative (2.12) taking place for all n. Then, a sequence (u_n) in $M(u,\nabla)$ may be found so as (cf. (2.14))

$$e(u_n,u_m) \leq \varphi(u_n) - \varphi(u_m), \text{ whenever } n < m. \tag{2.15}$$

The sequence $(\varphi(u_n))$ is (strictly) descending and bounded below in R; hence a Cauchy one. This, along with (2.15), tells us that (u_n) is an e-Cauchy sequence in $\mathrm{Dom}(\varphi)$. Putting these together, it follows via (2.3) that there must be some $v \in M$ with

$$u_n \xrightarrow{e} v \text{ and } \lambda := \lim_n \varphi(u_n) \geq \varphi(v). \tag{2.16}$$

The second half of this gives $v \in \mathrm{Dom}(\varphi)$; since $(u_n) \subseteq \mathrm{Dom}(\varphi)$. On the other hand, fix some rank n. By (2.15) (and the triangular property of e)

$$e(u_n,v) \leq e(u_n,u_m) + e(u_m,v) \leq \varphi(u_n) - \varphi(u_m) + e(u_m,v), \forall m > n.$$

This, along with (2.16), yields by a limit process (relative to m)

$$e(u_n,v) \leq \varphi(u_n) - \lim_m \varphi(u_m) \leq \varphi(u_n) - \varphi(v) \quad \text{(i.e.: } u_n \nabla v\text{)}. \tag{2.17}$$

Let $x \in M(v,\nabla)$ be arbitrary fixed. By (2.17) (and the definition of (∇))

$$\varphi(u_n) \geq \varphi(v) \geq \varphi(x), \forall n; \quad \text{hence } \lambda \geq \varphi(v) \geq \varphi(x).$$

On the other hand, (2.14) yields

$$\varphi(u_{n+1}) \leq (1/2)[\varphi(u_n) + \varphi(x)], \forall n; \ \text{ hence } \lambda \leq \varphi(x) \leq \varphi(v).$$

This gives $\varphi(v) = \varphi(x)$; which (by the arbitrariness of x) tells us that v is BB-variational (modulo (e,φ)). The proof is complete. ■

Now, the regularity condition (2.2) holds under

$$\begin{aligned}&(e,\varphi) \text{ is weakly complete:}\\ &\text{for each strongly } e\text{-asymptotic sequence } (x_n) \text{ in } \mathrm{Dom}(\varphi)\\ &\text{there exists } x \in M \text{ with } x_n \xrightarrow{e} x \text{ and } \liminf_n \varphi(x_n) \geq \varphi(x).\end{aligned} \tag{2.18}$$

For example, this is retainable whenever

$$\begin{aligned}&e \text{ is weakly complete:}\\ &\text{each strongly } e\text{-asymptotic sequence is } e\text{-convergent}\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\varphi \text{ is weakly } e\text{-lsc: } \liminf_n \varphi(x_n) \geq \varphi(x) \text{ whenever the}\\ &\text{strongly } e\text{-asymptotic sequence } (x_n) \text{ fulfills } x_n \xrightarrow{e} x.\end{aligned} \tag{2.20}$$

In particular, when e is (in addition) *reflexive* $[e(x,x) = 0, \forall x \in M]$, Theorem 1 includes the variational principle in Tataru [24]. The question of the converse inclusion being also true remains open; we conjecture that the answer is positive.

Let us now return to our initial setting. An interesting completion of Theorem 1 is the following. Let the concept of E-*variational* (modulo (e,φ)) point be that of (1.4) (with e in place of d). This is stronger than the concept of BB-variational (modulo (e,φ)) element introduced via (2.7). To get a corresponding form of Theorem 1 involving such points we have to impose (in addition to (2.2)/(2.3))

$$e \text{ is } \textit{transitively sufficient } [e(z,x)=e(z,y)=0 \Longrightarrow x=y]. \tag{2.21}$$

Theorem 2. *Let the precise conditions be in force. Then, for each starting (modulo (e,φ)) $u \in \mathrm{Dom}(\varphi)$ there exists an E-variational (modulo (e,φ)) $w = w(u) \in \mathrm{Dom}(\varphi)$ with the property (2.10).*

Proof. Let $u \in \mathrm{Dom}(\varphi)$ be taken as in the statement. By Theorem 1, we have promised a BB-variational (modulo (e,φ)) $v = v(u) \in \mathrm{Dom}(\varphi)$ with the property (2.10). If v is E-variational (modulo (e,φ)) we are done (with $w = v$); so, it remains the alternative of v fulfilling the opposite property:

$$v\nabla w \text{ (hence } \varphi(v)=\varphi(w)\text{), for some } w \in M \setminus \{v\}.$$

In this case, w is our desired element. Assume not: $w\nabla y$, for some $y \in M$, $y \neq w$. By the preceding relation, $v\nabla y$ (hence $\varphi(v)=\varphi(y)$). Summing up, $v\nabla w$, $v\nabla y$ and $\varphi(v)=\varphi(w)=\varphi(y)$; wherefrom (by (2.21) and the definition of (∇)) $w=y$, contradiction. This ends the argument. ■

As before, the regularity condition (2.3) holds under

$$\begin{aligned}&(e,\varphi) \text{ is complete: for each } e\text{-Cauchy sequence } (x_n) \subseteq \mathrm{Dom}(\varphi)\\ &\text{there exists } x \in M \text{ with } x_n \xrightarrow{e} x \text{ and } \liminf_n \varphi(x_n) \geq \varphi(x).\end{aligned} \tag{2.22}$$

For example, this is retainable whenever

$$e \text{ is complete: each } e\text{-Cauchy sequence is } e\text{-convergent} \tag{2.23}$$

$$\begin{aligned}&\varphi \text{ is } e\text{-lsc: } \liminf_n \varphi(x_n) \geq \varphi(x), \text{ whenever}\\ &\text{the } e\text{-Cauchy sequence } (x_n) \text{ fulfills } x_n \xrightarrow{e} x.\end{aligned} \tag{2.24}$$

On the other hand, (2.21) holds whenever e is *sufficient* $[e(x,y)=0 \Longrightarrow x=y]$. Note that, in such a case, Theorem 2 becomes the variational principle in Turinici [26]. Another interesting choice is e=(standard) metric over M; when, Theorem 2 includes directly Ekeland's variational principle [7,8] (Theorem E). Further aspects were delineated in Dancs, Hegedus and Medvegyev [6]; see also Fang [9] and Hamel [11, Ch 4].

3 Extended KST principles

We are now in position to get an appropriate answer to the questions in Section 1. Let (M,d) be a complete metric space; and $e : M \times M \to R_+$, a triangular pseudometric over M. We shall say that this object is a KST-*metric* (modulo d) provided

$$\begin{aligned}&e \text{ is Cauchy subordinated to } d\text{:}\\ &\text{each } e\text{-Cauchy sequence is } d\text{-Cauchy (hence } d\text{-convergent)}\end{aligned} \tag{3.1}$$

$$e \text{ is Cauchy } d\text{-lsc in the second variable: } (y_n) \text{ is } e\text{-Cauchy and } y_n \xrightarrow{d} y \text{ imply } \liminf_n e(x, y_n) \geq e(x, y), \forall x \in M. \tag{3.2}$$

Further, let $\varphi : M \to R \cup \{\infty\}$ be some inf-proper, d-lsc function (cf. (1.1)+(1.2)). The following auxiliary fact will be useful for us.

Lemma 3. *Assume that e is some KST-metric (modulo d) over M. Then, (e, φ) is complete (in the sense of (2.22)); hence (a fortiori), descending complete (in the sense of (2.3)).*

Proof. Let (x_n) be some e-Cauchy sequence in $\mathrm{Dom}(\varphi)$. From (3.1), (x_n) is d-Cauchy; so, by completeness, $x_n \xrightarrow{d} y$ as $n \to \infty$, for some $y \in M$. We claim that this is our desired point for (2.22). In fact, let $\gamma > 0$ be arbitrary fixed. By the choice of (x_n), there exists $k = k(\gamma)$ so that $e(x_p, x_m) \leq \gamma$, for each $p \geq k$ and each $m > p$. Passing to limit upon m one gets (via (3.2) and the relation above) $e(x_p, y) \leq \gamma$, for each $p \geq k$; and since $\gamma > 0$ was arbitrarily chosen, $x_n \xrightarrow{e} y$. This, along with (1.2), yields the needed conclusion. ∎

Now, combining these with Theorem 1 (and Lemma 2), we derive

Theorem 3. *Let e be some KST-metric (modulo d) and φ be inf-proper, d-lsc. Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with the properties (2.8)-(2.9) and (2.10).*

An interesting problem to be posed is that of getting a corresponding form of this result involving E-variational (modulo (e, φ)) points. The appropriate answer to this is obtainable via Theorem 2. Call the almost pseudometric $e : M \times M \to R_+$, a *strong* KST-metric (modulo d) when it is a KST-metric (modulo d) and fulfills (2.21).

Theorem 4. *Assume that e is some strong KST-metric (modulo d). Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists $w = w(u) \in \mathrm{Dom}(\varphi)$ fulfilling (1.4) (with (e, w) in place of (d, v)) and (2.10).*

In the following, we shall give some particular cases of our statements.

a) By the very definition of this concept (cf. Section 1), any w-distance (modulo d) is a strong KST-metric (modulo d). Hence the variational statement in Kada, Suzuki and Takahashi [13] (subsumed to Theorem KST) is a particular case of Theorem 4. But, as explicitly stated by these authors, their contribution extends the one due to T. H. Kim, E. S. Kim and S. S. Shin [16]; hence so does our statement. This also shows that any recursion to the nonconvex minimization theorem in Takahashi [22] is avoidable in such developments. Some related facts may be found in Ume [27]; see also G. M. Lee, B. S. Lee, J. S. Yung and S. S. Chang [17]. For a number of structural aspects we refer to Khanh [15] and Nemeth [19].

b) Let (M, d) be a complete metric space; and $e : M \times M \to R_+$ be a triangular pseudometric over M. According to Suzuki [21], we say that this object is a *τ-distance* (modulo d) over M when there exists a function $\eta = \eta(e)$ from $M \times R_+$ to R_+ with the properties

(T1) $t \vdash \eta(x, t)$ is increasing and $\lim_{t \to 0} \eta(x, t) = 0 = \eta(x, 0), \forall x \in M$

(T2) $\lim_n[\sup\{\eta(z_n, e(z_n, y_m)); m \geq n\}] = 0$ and $y_n \xrightarrow{d} y$ imply $\liminf_n e(x, y_n) \geq e(x, y)$, for each $x \in M$

(T3) $\lim_n[\sup\{e(x_n, y_m); m \geq n\}] = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$

(T4) $\lim_n \eta(z_n, e(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, e(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Clearly, any w-distance is a τ-distance (modulo d); just take $\eta(x, t) = t, \forall x \in M, \forall t \in R_+$. On the other hand, any τ-distance (modulo d) is transitively sufficient. In fact, let $x, y, z \in M$ be such that $e(z, x) = e(z, y) = 0$. By (T1), we have $\eta(z, e(z, x)) = \eta(z, e(z, y)) = 0$; and this, added to (T4), gives $d(x, y) = 0$ (hence $x = y$); wherefrom (2.21) holds. Finally, (3.2) is also retainable here; because we have (cf. Suzuki [op. cit.]):

Lemma 4. *Let the precise conventions be in use. Then, each τ-distance (modulo d) is necessarily Cauchy d-lsc in the second variable (cf. (3.2)).*

Proof. Let $\eta = \eta(e)$ stand for the associated to e map fulfilling (T1)-(T4). Call the sequence (x_n) in M, (η, e)-*Cauchy* provided

$$\lim_n[\sup\{\eta(z_n, e(z_n, x_m)); m \geq n\}] = 0, \quad \text{for some } (z_n) \subseteq M. \tag{3.3}$$

By [21, Lemma 3], the generic inclusion holds

$$\text{[for each sequence] } e\text{-Cauchy} \Longrightarrow (\eta, e)\text{-Cauchy}. \tag{3.4}$$

On the other hand, note that (T2) may be written as

(T2*) (y_n) is (η, e)-Cauchy and $y_n \xrightarrow{d} y$ imply $\liminf_n e(x, y_n) \geq e(x, y)$, for each $x \in M$,

Combining these gives the conclusion we want. ■

Now, by simply adding this to Theorem 4, one gets the following variational statement. Let (again) $\varphi : M \to R \cup \{\infty\}$ be as in (1.1)+(1.2).

Theorem 5. *Assume that e is some τ-distance (modulo d); which is Cauchy subordinated to d (cf. (3.1)). Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists an E-variational (modulo (e, φ)) $w = w(u) \in \mathrm{Dom}(\varphi)$ with the property (2.10).*

Formally, this result is strictly included in Suzuki's [op. cit., Proposition 8]; which (in addition) asserts that no Cauchy subordination property like (3.1) is needed so as to retain the conclusion above. Precisely, by [21, Lemma 1], the generic inclusion is established

$$\text{[for each sequence]} \quad (\eta, e)\text{-Cauchy} \Longrightarrow d\text{-Cauchy}; \tag{3.5}$$

and this, in combination with (3.4), yields (3.1). So, it is legitimate to ask of why did we state our result separately. To answer this, note that the author's verification of (3.5) runs as follows (in our notations):

"Assume that (x_n) is (η, e)-Cauchy; i.e., (3.3) holds. By (T4), we have

$$\lim_n [\sup\{d(x_{n+a}, x_{n+b}); 0 \leq a < b\}] = 0; \tag{3.6}$$

which means that (x_n) is d-Cauchy".

However, this last relation is not in general true. In fact, the only way of using (T4) in such reasonings is the one below. Let $a, b \in N$ be such that $(0 \leq)a < b$. By (3.3),

$$\lim_n \eta(z_n, e(z_n, x_{n+a})) = 0, \ \lim_n \eta(z_n, e(z_n, x_{n+b})) = 0.$$

So, by (T4), one gets a relation like

$$\lim_n d(x_{n+a}, x_{n+b}) = 0, \ \text{ for all such } (a, b).$$

But, from this, (3.6) [hence, a fortiori (3.5)] cannot be derived (in general). So, the proof of Lemma 1 in Suzuki [21] hangs in the air; and this tells us that Theorem 5 may be not retainable for general τ-distances (modulo d). Logically speaking, it is possible that Suzuki's conclusion be eventually available; i.e., Theorem 5 be retainable without imposing (3.1). But, *at this very moment*, the opoposite alternative is equally probable. In fact, due to the complicated nature of such objects, this seems to be the most realistic position; further aspects will be delineated elsewhere.

c) Let (M, d) be a complete metric space; and $e : M \times M \to R_+$ be a triangular pseudometric over M. According to Lin and Du [18] we say that it is a *τ-function* (modulo d) provided (2.21) holds and

(F1) $x \in M$, $y_n \to y$ and $e(x, y_n) \leq M$, $\forall n$ (for some $M = M(x) > 0$) imply $e(x, y) \leq M$

(F2) $\lim_n [\sup\{e(x_n, x_m); m > n\}] = 0$ and $\lim_n e(x_n, y_n) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Clearly, any w-distance is a τ-function (modulo d); cf. [18, Remark 2.1]. On the other hand, by definition, any τ-function (modulo d) is transitively sufficient (cf. (2.21)). Finally, (3.2) is also retainable here, via (F1). Hence, by simply combining with Theorem 4, one gets a coresponding version of Theorem 5 for such objects; we do not give details.

As before, the obtained result is deductible from the one in Lin and Du [18, Theorem 2.1]; which asserts that no Cauchy subordination property like (3.1) is needed so as to retain the conclusion above. In fact, call the sequence (x_n), *almost e-Cauchy* when $\lim_n [\sup\{e(x_n, x_m); m > n\}] = 0$. By definition,

$$\text{[for each sequence] } \ e\text{-Cauchy} \Longrightarrow \text{almost } e\text{-Cauchy}.$$

On the other hand, [18, Lemma 2.1] tells us that

$$\text{[for each sequence] } \ \text{almost } e\text{-Cauchy} \Longrightarrow d\text{-Cauchy}. \tag{3.7}$$

Combining with the above gives (3.1); and the claim follows. ■

So, the question arises of why did we state such a result separately. To answer this, note that the authors' verification of (3.7) runs as follows (in our notations):

"Assume that (x_n) is almost e-Cauchy. Putting $(y_n = x_{n+1}; n \in N)$ we have $\lim_n e(x_n, y_n) = 0$. This, along with (F2) yields

$$d(x_n, x_{n+1}) \to 0; \text{ hence } (x_n) \text{ is } d\text{-Cauchy".}$$

However, the last inference seems to be not in general true; because the written relation is just necessary for the d-Cauchy property. But, from this, (3.7) cannot be derived (in general). Hence, the proof of Lemma 2.1 in Lin and Du [op. cit.] hangs in the air; and this tells us that the corresponding version of Theorem 5 may be not retainable for general τ-functions (modulo d). As before, it is possible that Lin-Du's conclusion be eventually available; i.e.: the precise version of Theorem 5 be retainable without imposing (3.1). But, *at this very moment*, the opposite alternative is equally probable. Further aspects will be delineated elsewhere.

4 Application (Tataru metrics)

Let (M, d) be a complete metric space. By a *nonexpansive continuous semigroup* over M we shall mean any map $(t, x) \vdash S(t, x) = S(t)x$ from $R_+ \times M$ to M with

(S1) $S(0)x = x$ and $t \vdash S(t)x$ is continuous on M, $\forall x \in M$

(S2) $S(t + s)x = S(t)S(s)x$, for each $t, s \in R_+$ and each $x \in M$

(S3) $d(S(t)x, S(t)y) \leq d(x, y)$, for each $t \in R_+$ and each $x, y \in M$.

Assume that we fixed such an object. Denote ($\forall t \in R_+$, $\forall x, y \in M$)

$$\Delta(t; x, y) = t + d(S(t)x, y); \quad \Gamma(x; t) = \sup\{d(S(s)x, x); 0 \leq s \leq t\}. \tag{4.1}$$

According to Tataru [24], we construct the pseudometric $e = e_S$ over M as

$$e(x, y) = \inf\{\Delta(t; x, y); t \in R_+\}, \quad x, y \in M. \tag{4.2}$$

Note that (by the very definition (4.1))

$$\Delta(t; x, y) \geq t, \forall t \in R_+; \quad \text{hence} \lim_{t\to\infty} \Delta(t; x, y) = \infty.$$

This, along with (S1), tells us that the level sets

$$L(\Delta(.; x, y); \theta) = \{t \in R_+; \Delta(t; x, y) \leq e(x, y) + \theta\}, \theta > 0$$

are compacts of R_+; wherefrom, the infimum in (4.2) is effectively attained

$$e(x, y) = \Delta(r; x, y), \text{ for some } r = r(x, y) \text{ in } [0, e(x, y)]. \tag{4.3}$$

The basic properties of this object are as follows:

Lemma 5. *The pseudometric e is triangular reflexive and sufficient (cf. Section 2); moreover*

$$y \vdash e(x,y) \text{ is } d\text{-nonexpansive (hence continuous), for each } x \in M \tag{4.4}$$

$$e(x,y) \leq d(x,y) \leq e(x,y) + \Gamma(x; e(x,y)), \quad \text{for each } x,y \in M. \tag{4.5}$$

Proof. Let $x,y,z \in M$ be arbitrary fixed. For each $t,s \geq 0$ we have

$$e(x,z) \leq t+s+d(S(t+s)x,z) \leq t+d(S(t+s)x,S(s)y)+$$
$$s+d(S(s)y,z) \leq \Delta(t;x,y)+\Delta(s;y,z) \text{ (cf. (S2)+(S3))}.$$

Passing to infimum over (t,s) yields the triangular property. The sufficiency is clear, by (4.3) above. Further, take a triple $x,y_1,y_2 \in M$. We have (by definition)

$$\Delta(t;x,y_2) - d(y_1,y_2) \leq \Delta(t;x,y_1) \leq \Delta(t;x,y_2) + d(y_1,y_2), \quad \forall t \in R_+;$$

and, from this, (4.4) follows by taking the infimum over t. The left part of (4.5) is deductible via $\Delta(0;x,y) = d(x,y)$; in particular, this tells us that e is reflexive. For the right part of the same, note that

$$d(x,y) \leq \Delta(t;x,y) + d(S(t)x,x) \leq \Delta(t;x,y) + \Gamma(x;e(x,y)), \quad \forall t \in [0,e(x,y)].$$

It will suffice taking (4.3) into account to get the desired conclusion. ■

Summing up, $e = e_S$ has all the properties of a metric, except the symmetry; however, we shall term it the *Tataru metric* associated to $(t,x) \vdash S(t)x$. A natural question to be posed is that of the variational principles in Sections 2 and 3 be applicable to such objects. Technically speaking, two possible alternatives may occur in this discussion.

A) The first alternative refers to the ambient metric d appearing at the implicit level only; i.e., to Theorem 2 being applicable to our data. The following auxiliary statement will clarify this (cf. Tataru [op. cit.]):

Lemma 6. *Let the general conditions above be accepted. Then, the Tataru metric $e = e_S$ is weakly complete, in the sense of (2.19).*

Proof. Let (x_n) be a strongly e-asymptotic sequence in M; i.e., in such a way that $\sum_n e(x_n,x_{n+1}) < \infty$. By the representation formula (4.3), there must be a sequence (t_n) in R_+ fulfilling

$$\sum_n \Delta(t_n;x_n,x_{n+1}) = \sum_n t_n + \sum_n d(S(t_n)x_n,x_{n+1}) < \infty.$$

Put $(s_n = \sum_{k\geq n} t_k; n \in N)$; it is a decreasing and zero convergent sequence in R_+. By the preceding evaluation (and (S2)+(S3))

$$\sum_n d(S(s_n)x_n, S(s_{n+1})x_{n+1}) \leq \sum_n d(S(t_n)x_n,x_{n+1}) < \infty.$$

The sequence $(y_n = S(s_n)x_n); n \in N)$ is strongly d-asymptotic; hence d-Cauchy; wherefrom (by completeness)

$$S(s_n)x_n \xrightarrow{d} y \text{ as } n \to \infty, \text{ for some } y \in M.$$

This, along with

$$e(x_n, y) \leq s_n + d(S(s_n)x_n, y), \quad n = 0, 1, ...,$$

proves that $x_n \xrightarrow{e} y$; hence the conclusion. ■

Now, a simple combination with Theorem 2 yields the variational result we are looking for. Let the function $\varphi : M \to R \cup \{\infty\}$ be inf-proper (cf. (1.1)).

Theorem 6. *Suppose that (in addition) φ is weakly e-lsc (cf. (2.20)). Then, for each $u \in \mathrm{Dom}(\varphi)$ there exists $v \in \mathrm{Dom}(\varphi)$ in such a way that (1.3)+(1.4) are retainable, with e in place of d.*

Note that this result is implicit in Tataru [24]; but the reasoning we used here is different from the one appearing in the quoted paper.

B) The second alternative refers to the ambient metric d being explicitly considered; i.e., to Theorem 4 being applicable to our data. Note that, by Lemma 5, the Tataru metric $e = e_S$ is both transitively sufficient (cf. (2.21)) and Cauchy d-lsc in the second variable (cf. (3.2)). So, by simply combining with Theorem 4, one gets the following variational statement. Let the function $\varphi : M \to R \cup \{\infty\}$ be as in (1.1)+(1.2).

Theorem 7. *Assume that $e = e_S$ is (in addition) Cauchy subordinated to d (cf. (3.1)). Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists an E-variational (modulo (e, φ)) $w = w(u) \in \mathrm{Dom}(\varphi)$ with the property (2.10).*

It is to be noted that this "relative" type statement cannot be identified with the "absolute" Tataru's variational principle (Theorem 6); because the weak e-lsc condition (2.20) imposed there is not comparable with the d-lsc condition (1.2) used above.

Now, it is natural to ask under which conditions upon the ambient semigroup S, the regularity condition (3.1) is obtainable. Two posible answers to this must be considered.

b1) The former of these is founded by the developments in Section 3. Precisely, we have (cf. Suzuki [21]):

Lemma 7. *Let the general conditions above be accepted. Then, the Tataru metric $e = e_S$ is a τ-distance (modulo d).*

Proof. For the moment, e is triangular (cf. Lemma 5). Define further

$$\eta(x, t) = t + \Gamma(x; t), \ x \in M, t \in R_+ \text{ (where } \Gamma \text{ is that of (4.1))}. \tag{4.6}$$

We show that (T1)-(T4) hold with such a map. This is clearly the case with (T1); further, (T2) is assured via (4.4). On the other hand,

$$\eta(y, t) \leq 2e(x, y) + \eta(x, t), \quad \forall x, y \in M, \quad \forall t \in R_+; \tag{4.7}$$

and, from this (T3) follows. In fact, by (4.3), $e(x,y) = r + d(S(r)x, y)$, for some $r \in [0, e(x,y)]$. This yields, for each $s \in [0,t]$,

$$t + d(S(s)y, y) \leq t + d(S(s)y, S(s+r)x) + d(S(s+r)x, S(r)x) +$$
$$d(S(r)x, y) \leq 2d(S(r)x, y) + [t + d(S(s)x, x)] \leq 2e(x,y) + t + \Gamma(x;t);$$

wherefrom, (4.7) is deductible by passing to supremum over $s \in [0,t]$. Finally, note that (4.5) may be written as (cf. our convention)

$$e(x,y) \leq d(x,y) \leq \eta(x, e(x,y)), \quad x, y \in M.$$

As a consequence of this, we have the generic relation

$$\text{(for each } (z_n),\ (x_n)\text{): } \lim_n \eta(z_n, e(z_n, x_n)) = 0 \Longrightarrow \lim_n d(z_n, x_n) = 0;$$

wherefrom, (T4) is clear, by the metrical character of d. This completes the argument. ■

Summing up, Theorem 5 is applicable to these data; and the product of this is just Theorem 7 above.

The obtained result is deductible from the one in Suzuki [op. cit., Corollary 2]; which asserts that no Cauchy subordination property like (3.1) is needed so as to retain the conclusion above. But, as precise in Section 3, this assertion is not acceptable (with his argument). In addition (see above) the result in question cannot be identified with Tataru's variational principle (Theorem 6); in contradiction to the claim in Suzuki [21].

b2) The second answer to the posed question is to be given as follows. Call the initial semigroup $(t,x) \vdash S(t)x$, *boundedly uniformly continuous at origin* provided

$$S(t)x \to x \text{ as } t \to 0, \text{ uniformly on bounded parts of } M. \tag{4.8}$$

Lemma 8. *Assume that (in addition) (4.8) holds. Then, the Tataru metric $e = e_S$ fulfills (3.1), in the stronger sense*

$$\textit{each } e\textit{-Cauchy sequence in } M \textit{ is } d\textit{-Cauchy, and vice versa.} \tag{4.9}$$

Proof. Let (x_n) be an e-Cauchy sequence in M. We have to verify that it is a d-Cauchy one; or equivalently (as $(x,y) \vdash d(x,y)$ is symmetric)

$$\text{for each } \gamma > 0 \text{ there exists } p = p(\gamma) \text{ such that } p < n \Longrightarrow d(x_p, x_n) < \gamma.$$

Assume by contradiction that this is not true:

$$\exists \gamma > 0 \text{ such that: } \forall p, \exists q \text{ such that } p < q, d(x_p, x_q) \geq \gamma. \tag{4.10}$$

It follows that a subsequence $(y_n = x_{i(n)})$ of (x_n) may be constructed with

$$(y_n) \text{ is } e\text{-Cauchy and } d(y_n, y_{n+1}) \geq \gamma, \text{ for all } n. \tag{4.11}$$

The first half of this relation yields (by the triangular property of e) $\mu := \sup\{e(y_i, y_j); i < j\} < \infty$ (i.e.: (y_n) is e-bounded). We claim that the d-version of this property also holds; i.e., (y_n) is d-bounded too. In fact, we have (by virtue of (4.5))

$$d(y_0, y_n) \leq e(y_0, y_n) + \Gamma(y_0; e(y_0, y_n)) \leq \mu + \Gamma(y_0, \mu), \text{ for all } n;$$

and this proves our claim. From the obtained fact we deduce (via (4.8)) that, for the number $\gamma > 0$ appearing in (4.10) there must be some β in $]0, \gamma/2[$ so that $\Gamma(y_n; \beta) \leq \gamma/2$, for each n. In addition, (4.5) gives

$$d(y_n, y_{n+1}) \leq e(y_n, y_{n+1}) + \Gamma(y_n; e(y_n, y_{n+1})), \text{ for all } n. \tag{4.12}$$

On the other hand, (4.11) (the first half) tells us that there must be some rank $n(\beta)$ with $e(y_n, y_{n+1}) \leq \beta$, for all $n \geq n(\beta)$; and this yields (via (4.12))

$$d(y_n, y_{n+1}) \leq \beta + \Gamma(y_n; \beta) \leq \beta + \frac{\gamma}{2} < \gamma, \text{ for each } n \text{ as before;}$$

in contradiction with (4.11) (the second half). Consequently, the working assumption (4.10) cannot be accepted; and conclusion follows. ■

Now, by simply adding this to Theorem 4, we derive the following variational result for such (triangular) pseudometrics. Let again $\varphi : M \to R \cup \{\infty\}$ be some function as in (1.1)+(1.2).

Theorem 8. *Assume that, in addition, (4.8) holds. Then, conclusions of Theorem 7 are holding.*

Finally, it would be useful to determine whether the extra condition (4.8) is necessary as well [or, equivalently: to what extent is (3.1) retainable without such a requirement]. We conjecture that the answer to this is negative; further aspects will be discussed elsewhere. For useful applications of such facts to Hamilton-Jacobi equations we refer to Tataru [op. cit.] and the references therein.

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