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# On a Result of Y. Yang

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*Dedicated to Academician Radu Miron on his 80th Birthday*

This note deals with a result of Y. Yang [2], related to the almost periodicity of solutions to nonlinear parabolic equations of the form

$$u_t = \Delta u + f(t, x, u), \quad t \in \mathbb{R},\ x \in \Omega, \tag{1}$$

where $u : \mathbb{R} \times \Omega \to \mathbb{R}$, the domain $\Omega \subset \mathbb{R}^n$ being bounded and with boundary $\partial\Omega$ smooth enough. $\Delta u = \operatorname{div}(\operatorname{grad} u)$ is the Laplacian in $n$ dimensions, since $x = \operatorname{col}(x_1, x_2, \ldots, x_n)$. The nonlinearity $f(t, x, u)$ is defined on $\mathbb{R} \times \overline{\Omega} \times \mathbb{R}$, real valued, and satisfies other conditions to be specified below.

The Dirichlet type problem related to (1) consists in finding a solutions of (1) in $\mathbb{R} \times \Omega$, such that

$$u\big|_{\partial\Omega} = 0, \quad t \in \mathbb{R}. \tag{2}$$

Other types of boundary value data could be also considered (Neumann, Newton-radiation laws etc.).

The result of Y. Yang [2] we want to present and discuss below, can be formulated as follows:

**Theorem.** *Consider the problem* (1), (2), *under the following assumptions:*

*1) The domain $\Omega \subset \mathbb{R}^n$ is bounded, and its boundary $\partial\Omega$ is such that the Dirichlet problem*

$$\Delta u + \lambda u = 0, \quad u\big|_{\partial\Omega} = 0, \tag{3}$$

*has a sequence of eigenvalues $\{\lambda_k\}$,*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

*2) The function $f : \mathbb{R} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, and $L^2(\Omega)$ almost periodic in $t$, uniformly with respect to $u \in \mathbb{R}$.*

*3) $f$ is of class $C^{(1)}$ with respect to $u \in \mathbb{R}$, and $f_u = \dfrac{\partial f}{\partial u}$ verifies one of the inequalities:*

$$f_u \le \mu < \lambda_1 \quad in \quad \mathbb{R} \times \overline{\Omega} \times \mathbb{R}, \tag{4}$$

*or, for a fixed* $k \geq 1$*,*

$$\lambda_k < p_k \leq f_u \leq q_k < \lambda_{k+1} \quad in \quad \mathbb{R} \times \overline{\Omega} \times \mathbb{R}, \tag{5}$$

*where* $\mu$*,* $p_k$ *and* $q_k$*,* $k \geq 1$*, are some real numbers.*

*Then, if* $u = u(t,x)$*,* $t \in \mathbb{R}$*,* $x \in \Omega$*, is a solutions of* (1)*,* (2)*, satisfying also*

$$\int_\Omega u^2(t,x)\,dx \leq M < \infty, \quad t \in \mathbb{R}, \tag{6}$$

*then* $u(t,x)$ *is Bohr almost periodic as a map from* $\mathbb{R}$ *into* $L^2(\Omega,\mathbb{R})$*.*

The proof provided in Y. Yang [2] has a gap, and the scope of this note is to show that in case of (5), one must also add the assumption

$$\lambda_{k+1} - \lambda_k > 1. \tag{7}$$

In the proof of Theorem in [2], one find the (equivalent) statement that for each $k$ in (5), there must be an $m = m(k)$, $p_k < m < q_k$, such that

$$\max\{(m-\lambda_k)(m-p_k), (m-\lambda_k)(q_k-m), (\lambda_{k+1}-m)(m-p_k), (\lambda_{k+1}-m)(q_k-m)\} > 1. \tag{8}$$

Or, in case $\lambda_{k+1}-\lambda_k < 1$, each of the eight parentheses in (8) is smaller than 1. Therefore, such $m$ cannot exist in this case. A similar situation occurs when $\lambda_{k+1} - \lambda_k = 1$. Hence, (1) represents a necessary condition.

We can prove now that (7) is a sufficient condition for the existence of $m$, such that (8) be true. Indeed, let us first observe that $p_k$ can be chosen as close as we want from $\lambda_k$, without loss of generality on the validity of (5). A similar remark for $q_k$, which can be chosen as close as we want from $\lambda_{k+1}$.

We can prove that (7) is sufficient for the existence of $m$, as requested above. If one writes $(m-\lambda_k)(m-p_k) > 1$, then it amounts to the quadratic inequality

$$m^2 - (\lambda_k + p_k)\,m + \lambda_k p_k - 1 > 0. \tag{9}$$

The greatest root of the associated equation is

$$r_k = \frac{1}{2}(\lambda_k + p_k) + \sqrt{\frac{1}{4}(p_k - \lambda_k)^2 + 1}, \tag{10}$$

and we obviously have

$$p_k < r_k < p_k + 1 < q_k, \tag{11}$$

when $q_k - p_k > 1$.

Based on the above observation about the possibility of choosing $p_k$ and $q_k$ within $(\lambda_k, \lambda_{k+1})$, from (7) we derive immediately that $q_k - p_k > 1$ is a possible choice.

Therefore, when (7) takes place, changing, if necessary, $p_k$ and $q_k$ within interval $(\lambda_k, \lambda_{k+1})$, we can assure the existence of $m = m(k)$, such that (8) be satisfied. More precisely (8) or (9) hold for values of $m$ between $r_k$ given by (10), and $q_k$. Any number in this (open) interval, can serve as $m$ for which (8) or (9) holds true.

It is now clear that to the hypotheses 1), 2) and 3) above, one must add the following:

*4) The consecutive eigenvalues $\lambda_k$ and $\lambda_{k+1}$ must satisfy*

$$\lambda_{k+1} - \lambda_k > 1.$$

**Remark 1.** The condition $\lambda_{k+1} - \lambda_k > 1$ is not exceedingly restrictive. The distribution of eigenvalues of the Laplace operator depends, among other things, of the properties of $\Omega$ and $\partial\Omega$. In case $n = 1$, which means that $\lambda_k$ are the eigenvalues of the problem $y'' + \lambda y = 0$, $y(0) = y(l) = 0$, and $l < \pi$, there is no restriction at all. Moreover $\lambda_{k+1} - \lambda_k \to \infty$ as $k \to \infty$. But for larger $l$, one finds some restrictions.

**Remark 2.** We have to consider the fact that the way of proof may introduce the restriction (7). Hence, a problem arising from our discussion is whether this extra condition may be dropped by using another method of proof.

**Remark 3.** The existence of a solution to the problem (1), (2) is still open. It seems adequate to the method of fixed point (Banach), especially due to the fact that an equivalent form of condition (8) looks very much as a condition of contaction (see [2]). Likely, a different approach should be used when (5) is substituted by (4).

**References:**

1. C. Corduneanu, *Almost periodic oscillations and waves* (to appear).

2. Y. Yang, *Almost periodic solutions to nonlinear parabolic equations.* Bull. Australian Math. Soc., 38 (1988).