LIBERTAS MATEMATICA, vol XXVII (2007)

The n-Almost Contact 2-π Structures on the Dual Bundle of the 2-Osculator Bundle

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Dedicated to Academician Radu Miron on his 80th Birthday

Abstract. We introduce and investigate the concept of n-almost contact 2-π structure on the dual bundle of the 2-osculator bundle. Such structures clearly exist in the natural case when they are given (2.2) and (3.1).

In the theorems 2.2 and 3.2 some properties of it are given and we point out one given by (2.2) and (3.1). We search also the conditions of normality for this structure.

Introduction

The theory of 2-π structures on the tangent bundle TM or higher order tangent bundle $T^{(k)}M$ was developed in the papers [1], [2], [3], [4].

The natural $(k-1)n$-almost contact 2-π structure in Lagrange spaces of order k was investigated in the paper [1].

We have studied the notion of almost contact 2-π structures, the linear connection compatible with them and the conditions of normality.

The metrical case was also considered.

In the present paper, we solve similar problems for the n-almost contact 2-π structure in the dual bundle of the osculator bundle of order 2.

To this aim we take into account the theory of the dual bundle of the 2-osculator bundle, excelent presented in the book [7].

We study mainly the case of the natural n-almost contact 2-π structure, showing the existence such a structure and establishing the normality conditions. Then we consider a Riemannian n-almost contact 2-π structure establishing some geometrical properties of it.

Some formulae and fundamental results are taken from the book [7].

1 Preliminaries

Starting from the 2-osculator bundle (Osc^2M, π, M) , identified with 2-tangent bundle (T^2M, π, M) we introduce a new differentiable bundle $(T^{*2}M, \pi^*, M)$ called the dual bundle of 2-osculator bundle (or 2-cotangent bundle), where the total space $T^{*2}M$ is the fibered product

$$T^{*2}M = TM \times_M T^{*}M, \tag{1.1}$$

and π^{*} is the natural projection on M.

The local canonical coordinates on the manifold $T^{*2}M$ of a point $u = (x, y, p) \in T^{*2}M$ are (x^i, y^i, p_i) . The indices i,j,h,... run over set {1,2,...,n}.

The rule of change of the local coordinates is as follows.

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \ \det\left\| \frac{\partial \widetilde{x}^i}{\partial x^j} \right\| \neq 0 \\ \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j, \ \widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i} p_j \end{cases} \tag{1.2}$$

and there exists a tangent structure $J : \chi(T^{*2}M) \to \chi(T^{*2}M)$ defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \cdot J\left(\frac{\partial}{\partial y^i}\right) = 0, \ J\left(\frac{\partial}{\partial p_i}\right) = 0 \tag{1.3}$$

On $T^{*2}M$ there exist the vertical distribution W_1, W_2.

These distributions are integrable. There exists also on $T^{*2}M$, the Liouville vector field Γ and Liouville covector field Γ^{*} defined by

$$\Gamma = y^i \frac{\partial}{\partial y^i}, \ \Gamma^{*} = p_i \frac{\partial}{\partial p_i} \tag{1.4}$$

We have the following properties:

$$rank J = 2n, \ Im J = W_1, \ Ker J = W_1 \oplus W_2.$$

J is a integrable structure.

A dual semispray on $T^{*2}M$ is a vector field S on $T^{*2}M$ with the property $JS = \overset{1}{\Gamma}$. It follows that S has the form

$$S = y^i \frac{\delta}{\delta x^i} + 2\xi^i \frac{\partial}{\partial y^i} + f_i \frac{\partial}{\partial p_i}. \tag{1.5}$$

The functions $(\xi^i(x, y, p), f_i(x, y, p))$ are called the local coefficients of the dual semispray S. Setting $W_1 = J(N)$, $W_2 = J(W_1)$ for any nonlinear connection N one obtains the following decomposition in direct sum

$$T_u T^{*2}M = N(u) \oplus W_1(u) \oplus W_2(u). \tag{1.6}$$

From now, the main geometrical objects on $T^{*2}M$ will be reported to the direct sum (1.6) of vector spaces.

We denote by

$$\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\right) (i = 1, ..., n) \tag{1.7}$$

a local adapted basis to $N_,W_1, W_2$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^i} + N_{ij} \frac{\partial}{\partial p_j}. \tag{1.8}$$

The functions $N_i^j(x, y, p)$ and N_{ij} are the coefficients of the nonlinear connection N. The dual basis of the adapted basis (1.6) is given by

$$(\delta x^i, \delta y^i, \delta p_i) \tag{1.9}$$

where

$$\delta x^i = dx^i;\ \delta y^i = dy^i + N_j^i dx^j,\ \delta p_i = dp_i - N_{ji} dx^j. \tag{1.9$'$}$$

Definition 1.1. A linear connection D on $T^{*2}M$ is called an N-linear connection if:
(i) D preserves by parallelism $N_,W_1, W_2$.
(ii) The 2-tangent structure J is absolute paralel with respect to D.
(iii) The presymplectic structure $\Theta = dp_i \wedge dx^i$ is absolute paralel with respect to D.

An N-linear connection D can be uniquely represented, in the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\right)$ in the following form:

$$\begin{cases} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = H_{ij}^k \frac{\delta}{\delta x^k}, D_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} = H_{ij}^k \frac{\partial}{\partial y^k}, D_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial p_i} = -H_{kj}^i \frac{\partial}{\partial p_k} \\ D_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} = C_{ij}^k \frac{\delta}{\delta x^k}, D_{\frac{\delta}{\delta y^j}} \frac{\partial}{\partial y^i} = C_{ij}^k \frac{\partial}{\partial y^k}, D_{\frac{\delta}{\delta y^j}} \frac{\partial}{\partial p_i} = -C_{kj}^i \frac{\partial}{\partial p_k} \\ D_{\frac{\partial}{\partial p_j}} \frac{\delta}{\delta x^i} = C_i^{kj} \frac{\delta}{\delta x^k}, D_{\frac{\delta}{\delta p_j}} \frac{\partial}{\partial y^i} = C_i^{kj} \frac{\partial}{\partial y^k}, D_{\frac{\delta}{\delta p_j}} \frac{\partial}{\partial p_i} = -C_k^{ij} \frac{\partial}{\partial p_k} \end{cases} \tag{1.10}$$

The functions

$$D\Gamma(N) = (H_{jk}^i, C_{jk}^i, C_i^{jk}) \tag{1.11}$$

are called the coefficients of the N-linear connection D.

2 Natural n-almost contact 2-πstructures on the dual bundle of the 2-osculator bundle

We define the notion of k-almost contact 2-π structures.

Definition 2.1. A k-almost contact 2-π structures is a system $\left(F, \xi_1, ..., \xi_k, \eta^1, ..., \eta^k\right)$ where F is a tensor field of type (1.1) on $E = \widetilde{T}^{*2}M = T^{*2}M - \{0\}, \xi_1, ..., \xi_k$ are the linear independent vector fields and $\eta^1, ..., \eta^k$ are 1-forms such that we have:

$$\begin{cases} F^2(X) = -\lambda^2 X + \lambda^2 \sum\limits_{a=1}^{n} \eta^a(X)\xi_a, \forall X \in \chi(T^{*2}M) \\ F(\xi_a) = 0, \eta^a(\xi_b) = \delta_b^a \quad (a, b = 1, ..., k) \end{cases} \tag{2.1}$$

where λ are complex number $\{1, -1, i, -i\}$.

Let us consider the $\mathcal{F}(E)$-linear mapping $F : \chi(T^{*2}M) \to \chi(T^{*2}M)$ defined on adapted basis to the direct decomposition (1.1), by

$$F\left(\frac{\delta}{\delta x^i}\right) = -\lambda\frac{\partial}{\partial y^i},\ F\left(\frac{\partial}{\partial y^i}\right) = \lambda\frac{\delta}{\delta x^i},\ F\left(\frac{\partial}{\partial p_i}\right) = 0 \tag{2.2}$$

We have

Theorem 2.1. *The mapping F has the following properties:*
1° F is globally defined on $T^{\widetilde{*2}}M$.

2° F is a tensor field on $T^{\widetilde{*2}}M$ *of type (1.1)*

3° $KerF = W_2,\ ImF = N \oplus W_1$.

4° $rank\|F\| = 2n$.

5° $F^3 + \lambda^2 F = 0$.

Proof. 1° Taking into account of the rule

$$\frac{\delta}{\delta x^i} = \frac{\partial\widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta\widetilde{x}^j}\frac{\partial}{\partial y^i} = \frac{\partial\widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial\widetilde{y}^j};\ \frac{\partial}{\partial p_i} = \frac{\partial x^i}{\partial\widetilde{x}^j}\frac{\partial}{\partial\widetilde{p}_j} \tag{2.3}$$

of transformations of the adapted basis, we have that $\frac{\delta\widetilde{x}^i}{\partial x^j}F\left(\frac{\delta}{\partial x^i}\right) = -\lambda\frac{\partial\widetilde{x}^i}{\partial x^j}\frac{\partial}{\partial y^i}$ implies $F\left(\frac{\delta\widetilde{x}^i}{\delta\widetilde{x}^j}\right) = -\lambda\frac{\partial}{\partial\widetilde{y}^j}$.

Also, $\frac{\partial\widetilde{x}^i}{\partial x^j}F\left(\frac{\partial}{\partial y^i}\right) = \lambda\frac{\partial\widetilde{x}^i}{\partial x^j}\frac{\delta}{\delta x^i}$ and $\frac{\partial\widetilde{x}^j}{\partial x^i}F\left(\frac{\partial}{\partial p_i}\right) = 0$, lead to $F\left(\frac{\partial}{\partial\widetilde{y}^j}\right) = \lambda\frac{\delta}{\delta x^j}$ and $F\left(\frac{\partial}{\partial\widetilde{p}_i}\right) = 0$.

2° F is $\mathcal{F}(\Gamma^{\widetilde{*2}}M)$- linear mapping from $\chi(T^{\widetilde{*2}}M)$ to $\chi(T^{\widetilde{*2}}M)$.

3° $F\left(\frac{\partial}{\partial p_i}\right) = 0$ implies $F_{1W_1}$is trivial and $F(N \oplus W_1 \oplus W_2) = N \oplus W_1$.

4° Evidently, by means of 3^o.

5° $F^2\left(\frac{\delta}{\delta x^i}\right) = F\left(-\lambda\frac{\partial}{\partial y^i}\right) = -\lambda^2\frac{\delta}{\delta x^i};\ F^3\left(\frac{\delta}{\delta x^i}\right) = -\lambda^2 F\left(\frac{\delta}{\delta x^i}\right) = \lambda^3\frac{\partial}{\partial y^i}$
$F\left(\frac{\delta}{\delta x^i}\right) = -\lambda\frac{\partial}{\partial y^i};\to \left(F^3 + \lambda^2 F\right)\frac{\delta}{\delta x^i} = 0.$
$F^2\left(\frac{\partial}{\partial y^i}\right) = F\left(\lambda\frac{\delta}{\delta x^i}\right) = -\lambda^2\frac{\partial}{\partial y^i};\ F^3\left(\frac{\partial}{\partial y^i}\right) = -\lambda^2 F\left(\frac{\partial}{\partial y^i}\right) = \lambda^3\frac{\delta}{\delta x^i}$
$\to \left(F^3 + \lambda^2 F\right)\frac{\partial}{\partial y^i} = 0.$

Evidently $\left(F^3 + \lambda^2 F\right)\left(\frac{\partial}{\partial p_i}\right) = 0$. q.e.d.

Let us consider a local basis $(\xi_a)\,(a = 1, 2, ..., n)$ of the distribution W_2 and (η^a) its dual.

Theorem 2.2. *The set* $\{F, \xi_a, \eta^a\}$, *determines an n-almost contact 2-π structure on* $\widetilde{T^{*2}}M$. *Indeed, one easly checs that*

$$\begin{cases} F(\xi_a) = 0,\ \eta^a(\xi_b) = \delta^a_b \\ F^2(X) = -\lambda^2 X + \lambda^2\sum\limits_{a=1}^{n}\eta^a(X)\xi_a,\ \forall X \in \chi(E). \end{cases} \tag{2.4}$$

From the last formulae we deduce

$$\eta^a \circ F = 0, \quad (a = 1, ..., n) \tag{2.4'}$$

Definition 2.2. The structure $\{F, \xi_a, \eta^a\}$ is said to be normal if

$$N_F(X,Y) + \sum_{a=1}^{n} d\eta^a(X,Y)\xi_a = 0 \tag{2.5}$$

where N_F is the Nijenhuis tensor of F and is given by

$$N_F(X,Y) = [FX, FY] + F^2[X,Y] - F[FX,Y] - F[X,FY] \tag{2.6}$$

Theorem 2.3. *The n-almost contact 2-π structure $\{F, \xi_a, \eta^a\}$ is normal if and only if, the following equations holds:*

$$N_F(X,Y) + \sum_{a=1}^{n} d\left(\delta p_i\right)(X,Y)\frac{\partial}{\partial p_i} = 0, \forall X, Y \in \chi(T^{*2}M) \tag{2.7}$$

Indeed with $\xi_a = \frac{\partial}{\partial p_a}(a = 1, ...n)$. and dual basis (δp_a) , the condition (2.5) becomes the condition (2.7).

Now we shall express the equations (2.7) by means of d-tensor fields, using the expression of the exterior differentials in the adapted basis.

We have:

Lemma 2.4. [7] *The exterior differentials of the 1-forms $\left(\delta x^i, \delta y^i, \delta p_i\right)$ are given by*

$$\begin{cases} d(\delta x^i) = 0 \\ d(\delta y^i) = \{\frac{1}{2}\ \overset{R}{(1)}{}^{\ i}_{jm}\, dx^m + \overset{B}{(1)}{}^{\ i}_{jm}\, \delta y^m + \overset{B}{(1)}{}^{\ im}_{j}\, \delta p_m\} \wedge dx^j \\ d(\delta p_i) = \{\frac{1}{2}\ \overset{R}{(2)}_{ijm}\, dx^m + \overset{B}{(2)}_{ijm}\, \delta y^m + \overset{B}{(2)}{}^{\ m}\ \delta p_m\} \wedge dx^j \end{cases} \tag{2.8}$$

and

Theorem 2.5. [7] *The Lie brackets of the vector fields of the adapted basis are given by*

$$\begin{cases} \left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^h}\right] = \overset{R}{(1)}{}^{\ i}_{jh}\ \frac{\partial}{\partial y^i} + \overset{R}{(1)}_{jh}\ \frac{\partial}{\partial p_i} \\ \left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^h}\right] = \overset{B}{(1)}{}^{\ i}_{jh}\ \frac{\partial}{\partial y^i} + \overset{B}{(2)}_{ijh}\ \frac{\partial}{\partial p_i} \\ \left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial p_h}\right] = \overset{B}{(1)}{}^{\ ih}_{j}\ \frac{\partial}{\partial y^i} + \overset{B}{(2)}{}^{\ h}_{ij}\ \frac{\partial}{\partial p_i} \\ \left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^h}\right] = \left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial p_h}\right] = \left[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_h}\right] = 0 \end{cases} \tag{2.9}$$

where

$$\begin{cases} \overset{R}{(1)}{}^{\ i}_{jh} = \frac{\delta}{\delta x^h}N^i_j - \frac{\delta}{\delta x^j}N^i_h, \overset{R}{(2)}_{ijh} = \frac{\delta}{\delta x^h}N_{hi} - \frac{\delta}{\delta x^j}N_{ji} \\ \overset{B}{(1)}{}^{\ i}_{jh} = \frac{\partial}{\partial y^h}N^i_j \qquad \overset{B}{(2)}_{ijh} = -\frac{\partial}{\partial y^h}N_{ji} \\ \overset{B}{(1)}{}^{\ ih}_{j} = \frac{\partial}{\partial p_h}N^i_j \qquad \overset{B}{(2)}{}^{\ h}_{ij} = -\frac{\partial}{\partial p_h}N_{ji} \end{cases} \tag{2.10}$$

Calculating the expressions from (2.7) for $\{X,Y\}=\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\}$ we obtain:

Theorem 2.6. *The n-almost contact 2-π structure $\{F,\xi_a,\eta^a\}$ is normal if and only if, the following equations hold:*

$$\begin{cases} \overset{B}{(1)}{}^{i}_{jk}-\overset{B}{(1)}{}^{i}_{kj}-\overset{R}{(1)}{}^{i}_{jk}=0 \quad \overset{B}{(2)}_{ijk}-\overset{B}{(2)}_{ikj}-\overset{R}{(2)}_{ijk}=0. \end{cases} \tag{2.12}$$

3 The metrical n-almost contact 2-π structures on the dual bundle of the 2-osculator bundle.

Let us consider an k-almost contact 2-π structure on $\widetilde{T^{*2}}M$ given by $(F,\xi_1,...,\xi_k,\eta^1,...,\eta^k)$ and G a Riemannian structure on the manifold $\widetilde{T^{*2}}M$.

Definition 3.1. A k-almost contact 2-π structure $(F,\xi_1,...,\xi_k,\eta^1,...,\eta^k)$ is called Riemannian if there exists a Riemannian metric G on such that

$$\begin{cases} G(FX,FY)=\lambda^2G(X,FY)\\ G(X,\xi_a)=\eta^a(X)\ (a=1,...,k)\end{cases} \tag{3.1}$$

We will prove that in $T^{*2}M$, there exists a lift G of the fundamental tensor field g_{ij}, such that G makes the natural n-almost contact 2-π structure $(F,\xi_1,...,\xi_n,\eta^1,...,\eta^n)$ a Riemannian n-almost contact 2-π structure.

Let us consider the fundamental tensor fields g_{ij} and h^{ij}. We construct using then the N-lift G given by

$$G=g_{ij}dx^i\otimes dx^j+g_{ij}\delta y^i\otimes\delta y^j+h^{ij}\delta p_i\otimes\delta p_j. \tag{3.2}$$

where $(dx^i,\delta y^i,\delta p_i)$ is the dual basis of the adapted basis for the nonlinear connection of the space $T^{*2}M$.

We notice that G is a Riemann structure on $T^{*2}M$.

We have

Theorem 3.1. *The N-lift G is covariant constant with respect to $D\Gamma(N)=\{H^i_{jk},C^i_{jk},C^{jk}_i\}$ i.e. the equation*

$$D_XF=0,D_XG=0 \tag{3.3}$$

are verified.

The distributions N,W_1,W_2 are mutual orthogonal with respect to G.

Now, we shall prove the following important result.

Theorem 3.2. *The pair (G,F), where G the Riemann structure is given by (3.2) and $(F,\xi_1,...,\xi_n,\eta^1,...,\eta^n)$ with (ξ_a) an ortogonal basis and η^a its dual basis, is a Riemannian natural n-almost contact 2-π structure.*

Proof. Indeed for $X\in\left\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i},\frac{\partial}{\partial p_i}\right\}$ any (ξ_a,η^a) satisfying the second equation (3.1) with $a=1,2,...,n$ we verify immediately the first condition (3.1) q.e.d.

Theorem 3.3. *The canonical metrical N-connection $D\Gamma(N)=(H^i_{jk},C^i_{jk},C^{jk}_i)$ is compatible with the Riemannian natural n-almost contact 2-π structure (G,F).*

Proof. Let us consider the metric N-connection with respect to G which has the coefficients given by the Christoffel generalized symbols.

$$\begin{cases} H^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right) \\ C^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s}\right) \\ C_i^{jk} = \frac{1}{2} g_{is}\left(\frac{\partial g^{sk}}{\partial p_j} + \frac{\partial g^{js}}{\partial p_k} - \frac{\partial g^{jk}}{\partial p_s}\right) \end{cases} \tag{3.4}$$

Taking into account that G and F are covariant constant with respect to $D\Gamma(N)$, the Theorem 3.3. holds.

Remark. The Riemannian natural structure (G, F) is normal if (2.12) and (3.4) hold.

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