

Generalized Beil Metric on Finsler Spaces

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Dedicated to Academician Radu Miron on his 80th Birthday

Abstract. The geometry of generalized Lagrange metrics (GL-metrics for short) i.e. depending on point and on velocity in that point is now well-known. A differentiable manifold endowed with such GL-metric is called a generalized Lagrange space, GL-space for brevity, cf. [6]). Any Finsler space is a particular GL-space.

This paper deals with a generalization of some notions which appear in [1, 3]. Here, to a given Finsler metric is associated a special GL-metric. The latter is called the generalized Beil metric. The generalized Beil metrics provide, as particular cases, all the GL-metrics studied in [1, 6]. The applications of these particular GL-metrics to Physics and Biology justify the study of this kind of GL-metric.

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1 Introduction

The tangent bundle of a differentiable manifold gives a natural framework for the geometry of generalized Lagrange spaces. A systematical description of these spaces, as well as of the Finsler spaces and of the Lagrange spaces is presented for instance in [6]. These spaces are most interesting for their applications in the theoretical physics. Having the geometry of the Finsler spaces as the basis geometry, the so called Beil metrics have been defined and studied in [2]. Another facts concerning this kind of metrics can be seen in the papers [1, 3]. In this paper the generalized Beil metrics on Finsler spaces are introduced. The theory of the gauge generalized Beil metrics as well as their applications to unified field theory will be developed elsewhere.

2 Distinguished geometrical objects on TM

Let M be a real, finite-dimensional ($\dim M = n$), differentiable manifold of C^∞ -class and (TM, π, M) its tangent bundle. Since a point of TM is a vector (x, y) at the point x of the base manifold M , a coordinate system $x = (x^i)$ on M induces a canonical coordinate system

$(x, y) = (x^i, y^i)$ in TM . Furthermore, a local coordinate transformation on TM is given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, x^2, \dots, x^n), \\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \det \left\| \frac{\partial \tilde{x}^i}{\partial x^j} \right\| \neq 0. \end{cases} \quad (1)$$

Let $\ker \pi$ be denoted by V which is the vertical distribution on TM , that is, $(x, y) \in TM \rightarrow V_{(x,y)} \subset T_{(x,y)}TM$. Denoting by $\partial_i = \frac{\partial}{\partial x^i}$, $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ the local natural basis of the module of vector fields $\chi(TM)$, it is easily seen that the set $\left\{ \dot{\partial}_i \right\}_{i=1, \dots, n}$ is a local basis of the vertical distribution V .

A nonlinear connection N on TM is a C^∞ -distribution given by $(x, y) \in TM \rightarrow N_{(x,y)} \subset T_{(x,y)}TM$ such that $T_{(x,y)}TM = N_{(x,y)} \oplus V_{(x,y)}$. N is called a horizontal distribution on TM . If the coefficients of the nonlinear connection N are denoted by $N_j^i(x, y)$, then the set

$$\left\{ \delta_i := \partial_i - N_j^i(x, y) \dot{\partial}_j \right\}_{i=1, \dots, n},$$

is a local basis of the horizontal distribution N . Therefore $\left\{ \delta_i, \dot{\partial}_i \right\}_{i=1, \dots, n}$ is the local basis of $F(TM)$ -module of vector fields $\chi(TM)$, adapted to the distributions N and V . This basis is called the adapted basis. The dual basis of the adapted basis is given by

$$\left\{ dx^i, \delta y^i := dy^i + N_j^i(x, y) dx^j \right\}_{i=1, \dots, n}.$$

The Lie brackets of the vector fields from the adapted basis $\left\{ \delta_i, \dot{\partial}_i \right\}_{i=1, \dots, n}$ are as follows

$$\begin{cases} [\delta_j, \delta_h] = R_{jh}^i \cdot \dot{\partial}_i, \quad R_{jh}^i = \frac{\delta N_j^i}{\delta x^h} - \frac{\delta N_h^i}{\delta x^j}, \\ [\delta_j, \dot{\partial}_h] = \dot{\partial}_h N_j^i \cdot \dot{\partial}_i. \end{cases} \quad (2)$$

A vector $X \in \chi(TM)$ is uniquely expressed in the form $X = X^H + X^V$, $X^H \in N$, $X^V \in V$, and a 1-form ω can be written as $\omega = \omega^H + \omega^V$, where $\omega^H(X^V) = 0$ and $\omega^V(X^H) = 0$. A tensor field $t \in T_s^r(TM)$ is called a distinguished tensor field (shortly a d-tensor field) if its local components are transformed according to the law of transformation of the tensor fields on the basis manifold M . For instance, X^H , X^V are d-vectors fields and ω^H , ω^V are d-fields of 1-forms. Introduce the $F(TM)$ -linear mapping $J : \chi(TM) \rightarrow \chi(TM)$,

$$J\dot{\partial}_i = \dot{\partial}_i, \quad J\delta_i = 0.$$

A linear connection D on TM is called distinguished connection (shortly d-connection) if D preserves by parallelism the horizontal distribution N and vertical distribution V , respectively. Moreover, a linear connection D on TM is called N-linear connection if D is a d-connection and preserves by parallelism the almost tangent structure J . When the nonlinear connection N is given, N-linear connection D on TM is completely determined by its coefficients

$$\begin{cases} D_{\delta_j} \delta_i = L_{ij}^h \cdot \delta_h, & D_{\delta_j} \dot{\partial}_i = L_{ij}^h \cdot \dot{\partial}_h, \\ D_{\dot{\partial}_j} \delta_i = C_{ij}^h \cdot \delta_h, & D_{\dot{\partial}_j} \dot{\partial}_i = C_{ij}^h \cdot \dot{\partial}_h. \end{cases} \quad (3)$$

The system of the coefficients of D will be denoted by $D\Gamma(N) = (L_{ij}^h, C_{ij}^h)$. When N is fixed, it is denoted by $D\Gamma = (N_j^i : L_{ij}^h, C_{ij}^h)$.

3 Finsler spaces. GH-metrics

The modern formulation of the notion of a Finsler space is due to R. Miron [5, 6]. Briefly, following [5, 6], some aspects concerning the geometry of the Finsler spaces and their generalizations are presented below.

3.1 The notion of Finsler space

Let M be a real, n -dimensional smooth manifold, let (TM, π, M) be the tangent bundle and let $T_0M := TM \setminus \{0\}$. A Finsler space is a pair $F^n = (M, F(x, y))$ such that the following axioms hold.

- (F1) F is a real function on T_0M and continuous on the null section of the projection π .
- (F2) F is positive on T_0M .
- (F3) F is positively 1-homogeneous with respect to the velocity y^i .
- (F4) The Hessian of F^2 , with elements

$$\gamma_{ij}(x, y) = \frac{1}{2} \cdot \dot{\partial}_i \dot{\partial}_j F^2,$$

is positively-defined.

It follows that $\gamma_{ij}(x, y)$ is a symmetric and nonsingular d-tensor field of (0,2)-type. Let $\gamma^{jk}(x, y)$ be the reciprocal of $\gamma_{ij}(x, y)$, i.e. $\gamma_{ij} \cdot \gamma^{jk} = \delta_i^k$. Denoting $y_i = \gamma_{ij} \cdot y^j$, it is easily seen that

$$F^2 = \gamma_{ij} \cdot y^i \cdot y^j = \gamma^{ij} \cdot y_i \cdot y_j = y_i \cdot y^i.$$

The Christoffel symbols of $\gamma_{ij}(x, y)$ are given by

$$\gamma_{jk}^i = \frac{1}{2} \cdot \gamma^{ih} \cdot (\partial_k \gamma_{jh} + \partial_j \gamma_{hk} - \partial_h \gamma_{jk}).$$

The following contraction by y^i

$$\gamma_{00}^i = \gamma_{jk}^i \cdot y^j y^k,$$

will be used below. The canonical nonlinear connection $\overset{(0)}{N}$ of the Finsler space F^n is given by the following coefficients

$$\overset{(0)}{N}_j^i = \frac{1}{2} \cdot \dot{\partial}_j \gamma_{00}^i. \quad (4)$$

3.2 The canonical connection of a Finsler space

Let $D\Gamma \left(\overset{(0)}{N} \right) = \left(L_{jk}^i, C_{jk}^i \right)$ be a $\overset{(0)}{N}$ -linear connection of the Finsler space F^n in which $\overset{(0)}{N}$ is the canonical nonlinear connection, with the coefficients N_j^i given by (4). The h- and v-derivatives of the fundamental tensor $\gamma_{ij}(x, y)$ of the space F^n are expressed by

$$\begin{cases} \gamma_{ij|k} = \delta_k \gamma_{ij} - \gamma_{sj} \cdot L_{ik}^s - \gamma_{is} \cdot L_{jk}^s, \\ \gamma_{ij}^{\cdot|k} = \dot{\partial}_k \gamma_{ij} - \gamma_{sj} \cdot C_{ik}^s - \gamma_{is} \cdot C_{jk}^s. \end{cases} \quad (5)$$

In a Finsler space F^n there exists a unique $\overset{(0)}{N}$ -linear connection $D \overset{(0)}{\Gamma} \left(\overset{(0)}{N} \right) = \left(\overset{(0)}{L}_{jk}, \overset{(0)}{C}_{jk} \right)$

satisfying the axioms (A), (B) and (C) given below.

(A) $\overset{(0)}{N}$ is the canonical nonlinear connection of the space F^n .

(B) The equations $\gamma_{ij|k} = 0$ and $\gamma_{ij}|_k = 0$ hold with respect to $D \overset{(0)}{\Gamma} \left(\overset{(0)}{N} \right)$.

(C) $D \overset{(0)}{\Gamma} \left(\overset{(0)}{N} \right)$ is h- and v- torsion free.

Moreover, the connection $D \overset{(0)}{\Gamma} \left(\overset{(0)}{N} \right)$ has the coefficients given by the generalized Christoffel symbols

$$\begin{cases} \overset{(0)}{L}_{jk} = \frac{1}{2} \cdot \gamma^{is} \cdot (\delta_j \gamma_{sk} + \delta_k \gamma_{js} - \delta_s \gamma_{jk}), \\ \overset{(0)}{C}_{jk} = \frac{1}{2} \cdot \gamma^{is} \cdot (\dot{\partial}_j \gamma_{sk} + \dot{\partial}_k \gamma_{js} - \dot{\partial}_s \gamma_{jk}). \end{cases} \quad (6)$$

3.3 The spaces GL^n . GL -metrics

A generalized Lagrange space is a pair $GL^n = (M, g_{ij}(x, y))$, where M is a real, n -dimensional, smooth manifold and $g_{ij}(x, y)$ is a d-tensor field of $(0, 2)$ -type, symmetric, nondegenerate and of constant signature on T_0M . The tensor $g_{ij}(x, y)$ is called the fundamental (or metric) tensor of the space GL^n , or, shortly a GL -metric.

A Lagrange space $L^n = (M, L(x, y))$ is a particular case of a generalized Lagrange space $GL^n = (M, g_{ij}(x, y))$ in the sense that its fundamental tensor derived from a regular Lagrange function $L : TM \rightarrow R$. More precisely, a Lagrange space is a pair $L^n = (M, L(x, y))$ where M is a real, n -dimensional, smooth manifold and L is a function on TM having the following properties

(L1) $L : (x, y) \in TM \rightarrow L(x, y) \in R$ is differentiable on the manifold T_0M and it is continuous on the null section of $\pi : TM \rightarrow M$.

(L2) The Hessian of L (with respect to the velocity y^i) is nondegenerate:

$$g_{ij}(x, y) = \frac{1}{2} \cdot \dot{\partial}_i \dot{\partial}_j L, \text{rank } \|g_{ij}(x, y)\| = n, \forall (x, y) \in T_0M.$$

(L3) The d-tensor field $g_{ij}(x, y)$ has constant signature on T_0M .

One says that a GL -metric $g_{ij}(x, y)$ is provided by a regular Lagrangian if there exists a Lagrangian $L : TM \rightarrow R$ such that

$$g_{ij}(x, y) = \frac{1}{2} \cdot \dot{\partial}_i \dot{\partial}_j L.$$

A necessary condition that a generalized Lagrange space be reducible to a Lagrange one is that the d-tensor $C^{ijk} := \frac{1}{2} \cdot \dot{\partial}_k g_{ij}$ be totally symmetric.

4 Generalized Beil metrics on tangent bundle

Let a , b and c three smooth real functions on T_0M such that $a(x, y) > 0$, $b(x, y) \geq 0$ and $c(x, y) \geq 0$ for every $(x, y) \in T_0M$. Let $\omega_i(x)$ be the local components of a 1-form on the base manifold M . Denote $L^2 := \gamma^{ij}\omega_i\omega_j \geq 0$ and $M := \omega_i y^i$. Consider the matrix having the entries

$$g_{ij}(x, y) = a(x, y)\gamma_{ij}(x, y) + b(x, y)y_i y_j + c(x, y)\omega_i\omega_j, \quad (7)$$

where $\gamma_{ij}(x, y)$ is a Finsler metric. The following notations

$$A = \frac{1}{a}, \quad B = -\frac{b}{a} \cdot \frac{a + cL^2}{a^2 + abF^2 + acL^2 + bc(F^2L^2 - M^2)},$$

$$C = -\frac{c}{a} \cdot \frac{a + bF^2}{a^2 + abF^2 + acL^2 + bc(F^2L^2 - M^2)},$$

$$D = \frac{bc}{a} \cdot \frac{M^2}{a^2 + abF^2 + acL^2 + bc(F^2L^2 - M^2)},$$

are useful; here F^2 is the square of the Finsler function corresponding to the Finsler metric $\gamma_{ij}(x, y)$. For a real smooth function $\varphi = \varphi(x, y)$ the following notations $\varphi_k = \delta_k\varphi$, $\varphi^k = \gamma^{kh} \cdot \varphi_h$, $\dot{\varphi}_k = \dot{\partial}_k \varphi$ and $\dot{\varphi}^k = \varphi^{kh} \cdot \dot{\varphi}_h$ will be used.

Proposition 4.1 *The matrix $g_{ij}(x, y)$ in (7) is a positively defined GL-metric.*

Proof. The symmetry is obvious. The tensorial character of $g_{ij}(x, y)$ follows from the tensorial character of $\gamma_{ij}(x, y)$ and $y_i y_j$. Take $\omega^j := \gamma^{jt}\omega_t$ and

$$g^{jk} = A\gamma^{jk} + By^j y^k + C\omega^j \omega^k + D(\omega^j y^k + y^j \omega^k).$$

Clearly, $g_{ij} \cdot g^{jk} = \delta_i^k$. It is easily seen that $\det \|g_{ij}(x, y)\| \neq 0$ and $a^2 + abF^2 + acL^2 + bc(F^2L^2 - M^2) > 0$. Let $(\xi^i)_{i=1, n} \in R^n$. From

$$g_{ij}\xi^i\xi^j = a\gamma_{ij}\xi^i\xi^j + b(\gamma_{im}\xi^i y^m)(\gamma_{jn}\xi^j y^n) + c(\gamma_{im}\xi^i \omega^m)(\gamma_{jn}\xi^j \omega^n), \quad (8)$$

it follows that $g_{ij}(x, y)$ is positively defined. \square

From now on, the GL-metric (7) will be called the generalized Beil metric on tangent bundle.

Remark 4.2 *When $\gamma_{ij}(x, y)$ has only a constant signature then $g_{ij}(x, y)$ has a constant signature only on some subsets of T_0M .*

The function $E = g_{ij}(x, y)y^i y^j$ is called the absolute Lagrange energy of the GL-metric $g_{ij}(x, y)$. A simple computation leads to

Proposition 4.3 *The absolute Lagrange energy of the generalized Beil metric (7) is $E = (a + bF^2) \cdot F^2 + cM^2$.*

A GL -metric is said to be regular if its absolute Lagrange energy is a regular Lagrangian, that is, the matrix with the entries $\frac{1}{2} \cdot \dot{\partial}_i \dot{\partial}_j E$ is nonsingular. In general, the Beil metric (7) is not regular. In the special case a, b and c do not depend on y it follows

$$\frac{1}{2} \cdot \dot{\partial}_i \dot{\partial}_j E = (a + 2bF^2) \cdot \gamma_{ij} + b \cdot y_i y_j + c \cdot \omega_i \omega_j,$$

which is a generalized Beil metric. Therefore, in this case the Beil metric (7) is regular. The following result deals with the problem of reducibility of a generalized Beil metric.

Proposition 4.4 *The generalized Beil metric (7) is provided by a Lagrangian if and only if*

$$\begin{aligned} & \dot{a}_k \gamma_{ij} - \dot{a}_i \gamma_{kj} + \left(\dot{b}_k y_i - \dot{b}_i y_k \right) \cdot y_j \\ & + b \cdot (\gamma_{jk} y_i - \gamma_{ji} y_k) + \left(\dot{c}_k \omega_i - \dot{c}_i \omega_k \right) \omega_j = 0. \end{aligned} \quad (9)$$

Proof. From previous section it follows that a generalized Beil metric $g_{ij}(x, y)$ is a L -metric if and only if $\dot{\partial}_k g_{ij} = \dot{\partial}_i g_{kj}$. A direct calculation leads to (9). \square

Remark 4.5 *The equation (9) holds in very few circumstances. In fact, the Beil metric (7) with arbitrary a, b and c provides a large class of Beil metrics which are not reducible to L -metrics. The next result gives two examples.*

Proposition 4.6 *Let $\dim M = n > 1$.*

a) *If a, b and c do not depend on y then the generalized Beil metric (7) is not reducible to a L -metric.*

b) *The Beil metric*

$$g_{ij}(x, y) = e^{2\Phi(x, y)} \cdot \gamma_{ij}(x, y), \quad (10)$$

is not reducible to a L -metric when Φ depends on y .

Proof. a) The relation (9) reduces to $\gamma_{jk} y_i - \gamma_{ji} y_k = 0$. Multiplying this by γ^{ij} it follows $(1 - n)y_k = 0$. Since the last relation is false for $n > 1$, (9) does not hold.

b) For $b = c = 0$ and $a(x, y) = e^{2\Phi(x, y)}$, (9) leads to $\dot{\Phi}_k \gamma_{ij} - \dot{\Phi}_i \gamma_{kj} = 0$. Multiplying this relation by γ^{ij} one obtains $(n - 1) \dot{\Phi}_k = 0$ which is false when Φ depends on y . Therefore (9) is false. \square

4.1 Particular cases

A. If $a = e^{2\Phi(x)}$, $b = c = 0$ and $\gamma_{ij}(x, y) = \gamma_{ij}(y)$ then the ecological Beil metric

$$g_{ij}(x, y) = e^{2\Phi(x)} \gamma_{ij}(y), \quad (11)$$

is obtained.

B. For $a = 1$, $b = 1 - \frac{1}{n^2(x, y)}$ and $c = 0$ it follows

$$g_{ij}(x, y) = \gamma_{ij}(x, y) + \left(1 - \frac{1}{n^2(x, y)} \right) y_i y_j, \quad (12)$$

that is the Beil metric of the Relativistic Optics. This n means the refractive index of medium. The cases n constant or n depending on x only are remarkable ones.

C. Let $a(x, y)$ be $\alpha(F^2(x, y))$, let $b(x, y)$ be $\beta(F^2(x, y))$ and let $c(x, y)$ be $\delta(F^2(x, y))$ with $\alpha, \beta, \delta : R_+ \rightarrow R_+^*$. Then (7) becomes

$$g_{ij} = \alpha(F^2) \gamma^{ij} + \beta(F^2) y_i y_j + \delta(F^2) \omega_i \omega_j. \quad (13)$$

Concerning this Beil metric the following result holds.

Proposition 4.7 *The Beil metric (13) is reducible to a L -metric if and only if $\beta = 2\alpha'$ and $\delta' = 0$.*

Proof. For the Beil metric (13) the relation (9) is equivalent to $(\beta - 2\alpha') \cdot (\gamma_{jk} y_i - \gamma_{ji} y_k) + 2\delta' (y_k \omega_i - y_i \omega_k) \omega_j = 0$, or $\beta = 2\alpha'$ and $\delta' = 0$ since $\gamma_{jk} y_i - \gamma_{ji} y_k \neq 0$ and $\gamma^{rs} \omega_r \omega_s = \gamma^{rs} \omega_s \omega_r$. \square

Without loss of generality suppose that $\beta = 2\alpha'$ and $\delta = 1$. The generalized Beil metric (13) becomes

$$g_{ij}(x, y) = \alpha(F^2(x, y)) \gamma_{ij}(x, y) + 2\alpha'(F^2(x, y)) y_i y_j + \omega_i \omega_j, \quad (14)$$

which is in fact a L -metric. Looking for L in the form $L = \Lambda(F^2) + M^2$, $\Lambda : R_+ \rightarrow R_+^*$ it is easily seen that $\Lambda' = \alpha$. Thus the metric

$$g_{ij}(x, y) = \Lambda'(F^2(x, y)) \gamma_{ij}(x, y) + 2\Lambda''(F^2(x, y)) y_i y_j + \omega_i \omega_j, \quad (15)$$

with $\Lambda'(t) + 2t\Lambda''(t) > 0$ is a L -metric provided by the Lagrangian $L = \Lambda(L^2) + M^2$.

5 The canonical connection of $g_{ij}(x, y)$

The geometry of the Beil metric (7) is naturally connected with the geometry of $F^n = (M, F(x, y))$. The geometrical objects associated to $g_{ij}(x, y)$ can be expressed using similar ones for F^n . The Cartan connection of F^n is

$$D \overset{(0)}{\Gamma} \left(\overset{(0)}{N} \right) = \left(\overset{(0)}{N}_j; \overset{(0)}{L}_{jk}, \overset{(0)}{C}_{jk} \right),$$

where $\overset{(0)}{N}_j$ is given by (4) and $\overset{(0)}{L}_{jk}, \overset{(0)}{C}_{jk}$ are given by (6). A similar connection for the Beil metric $g_{ij}(x, y)$ is now considered. Let $D\Gamma \left(\overset{(0)}{N} \right) = \left(\overset{(0)}{N}_j; \overset{(0)}{L}_{jk}^i, \overset{(0)}{C}_{jk}^i \right)$ be the $\overset{(0)}{N}$ -linear connection given by

$$\begin{cases} \overset{(0)}{L}_{jk}^i = \frac{1}{2} \cdot g^{is} \cdot (\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}), \\ \overset{(0)}{C}_{jk}^i = \frac{1}{2} \cdot g^{is} \cdot (\dot{\partial}_j g_{sk} + \dot{\partial}_k g_{js} - \dot{\partial}_s g_{jk}). \end{cases} \quad (16)$$

This ${}^{(0)}N$ -linear connection is h- and v- metrical, and is h- and v- torsion free, respectively. Moreover, when ${}^{(0)}N$ is fixed, $D\Gamma\left({}^{(0)}N\right)$ is the unique ${}^{(0)}N$ -linear connection with these properties and it is called the canonical connection of $g_{ij}(x, y)$. Clearly, the canonical connection of $g_{ij}(x, y)$ can be expressed in terms of the Cartan connection of C^n .

6 Conclusions

1. In this paper a new GL-metric is introduced. In the case $c = 0$ this metric was intensively studied in [2, 3]. The nature of the function $c = c(x, y)$ implies the fact that our metric (7) is not reducible to the case $c = 0$.
2. The last term in (7) appears as a "gauge" component of the metric. It will be used elsewhere in order to determine a "good" metric for the Kaluza-Klein theory on the total space of the tangent bundle;
3. Using the same arguments it is possible to study the general case determined by the following metrical structure

$$g_{ij}(x, y) = a\gamma_{ij}(x, y) + \sum_{t=1}^k a^{(t)}(x, y)B_{i(t)}(x, y)B_{j(t)}(x, y),$$

where $\gamma_{ij}(x, y)$ are the local components of a Finsler metric, $a^{(t)}(x, y)$ are smooth real functions on T_0M and $B_{i(t)}(x, y)$ are the local coefficients of some 1-forms on T_0M .

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