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Norden Structures on Tangent Bundle of a Finsler Manifold

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Dedicated to Academician Radu Miron on his 80th Birthday

Abstract. Let J be an almost complex structure and G a pseudo - Riemannian metric. The pair (J, G) is called a Norden structure if $G(J\cdot, J\cdot) + G(\cdot, \cdot) = 0$. In §1 we construct such a structure on the tangent bundle TM of a Finsler manifold. Then we deform J to $\widetilde{J}$ and G to $\widetilde{G}$ in a natural way and so we provide a four parameters family of Norden structures on TM (§2). We study the integrability condition of $\widetilde{J}$ in §3 as well as the condition $\nabla\widetilde{J} = 0$ when ∇ is the Levi-Civita connection of G. It comes out that each of these conditions lead to strong restrictions on $\widetilde{J}$ on Finsler manifold M (Theorems 3.2 and 4.2).

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1 Introduction

Let M be a smooth i.e. C^∞ $n-$dimensional manifold and $\tau : TM \to M$ its tangent bundle. If (x^i), $i, j, .. = 1, 2, ...n$ are local coordinates on M, we take (x^i, y^i) as local coordinates on TM, where $x^i = x^i \circ \tau$ and y^i are the coordinates of a vector $v \in T_xM$ in the natural frame $\partial_i := \frac{\partial}{\partial x^i}\big|_x$.

The assignment $u \to V_uTM = \ker \tau_{x,u}$, $u \in TM$ defines a smooth distribution, called vertical, on TM. It is locally spanned by $\dot{\partial}_i := \frac{\partial}{\partial y^i}$, hence it is integrable. Its leaves through $u \in TM$ are T_xM, $x = \pi(u)$.

A nonlinear connection N is a distribution $u \to H_uTM$, called horizontal, which is supplementary to the vertical distribution, that is

$$T_uTM = H_uTM \oplus V_uTM, u \in TM. \tag{1.1}$$

The horizontal distribution is usually given by a local basis $\delta_i = \partial_i - N_i^j(x, y)\dot{\partial}_j$, where the functions $(N_i^j(x, y))$ have a special behavior when a change of local coordinates on TM is performed. If it happens that $N_j^i(x, y) = \Gamma^i_{j\ k}(x)y^k$, then $\left(\Gamma^i_{j\ k}(x)\right)$ are the local coefficients of a linear connection on M.

The decomposition (1.1) suggests to consider on TM

- an almost complex structure J ($J^2 = -I,$ the Kronecker tensor) given by

$$J(\delta_i) = \dot{\partial}_i, \qquad J(\dot{\partial}_i) = -\delta_i \tag{1.2}$$

- an almost product structure P ($P^2 = I$) given by

$$P(\delta_i) = \delta_i, \qquad P(\dot{\partial}_i) = -\dot{\partial}_i \tag{1.3}$$

- an almost product structure Q ($Q^2 = I$) given by

$$Q(\delta_i) = \dot{\partial}_i, \qquad Q(\dot{\partial}_i) = \delta_i \tag{1.4}$$

The vertical lifts of functions f, of vector field $X = X^i\partial_i$ and 1-forms $\omega = \omega_i dx^i$ from M to the tangent bundle TM are given by

$$f^V = f \circ \tau, \qquad X^V = (X^i \circ \tau)\dot{\partial}_i, \qquad \omega^V = (\omega_i \circ \tau)dx^i.$$

With respect to a nonlinear connection N, the horizontal lifts of functions f, of vector fields X and 1-forms ω from M to TM are given by

$$f^H = 0, \qquad X^H = (X^i \circ \tau)\delta_i, \qquad \omega^H = (\omega_i \circ \tau)\delta y^i,$$

where $\delta y^i = dy^i + N^i_j(x,y)dx^j$. The basis $(dx^i, \delta y^i)$ is dual to the basis $(\delta_j, \dot{\partial}_j)$. Then the vertical and horizontal lifts of arbitrary tensor fields are defined assuming that

$$(S \otimes T)^V = S^V \otimes T + S \otimes T^V, (S \otimes T)^H = S^V \otimes T^H + S^H \otimes T^V.$$

We may redefine J, P, Q as follows

$$J(X^H) = X^V, \quad J(X^V) = -X^H, \tag{1.2'}$$

$$P(X^H) = X^H, \quad P(X^V) = -X^V, \tag{1.3'}$$

$$Q(X^H) = X^V, \quad Q(X^V) = X^H, \quad \forall X \in \mathcal{X}(M). \tag{1.4'}$$

Here $\mathcal{X}(M)$ denotes the $\mathcal{F}(M)-$module of vector fields on M.

We set $T_0M = TM \setminus \{(x,0), x \in M\}$.

A function $F : TM \to \mathbb{R}_+, (x,y) \to F(x,y)$ is called a Finsler function if

(*i*) It is smooth on T_0M and only continuous on the set $\{(x,0), x \in M\}$,

(*ii*) It is positively homogeneous of degree 1 with respect to y i.e $F(x, \lambda y) = \lambda F(x,y)$ for every real number $\lambda > 0$,

(*iii*) The matrix with the entries $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is such that the quadratic form $g_{ij}(x,y)\xi^i\xi^j, (\xi) \in \mathbb{R}^n,$ has constant signature.

The pair $F^n = (M, F)$ is called a Finsler manifold.

The functions $(g_{ij}(x,y))$ behave in such a way that they define a pseudo - Riemannian g metric in the vertical bundle $\bigcup\limits_{u\in T_0M} V_uT_0M \to T_0M$ by the formula

$$g(u)(\dot\partial_i,\dot\partial_j) = g_{ij}(x,y), \quad u=(x,y)\in T_0M.$$

For any Finsler manifold $F^n=(M,F)$ there exists a nonlinear connection called the Cartan nonlinear connection which is completely determined by F. It is simply given by

$$N^i_j(x,y) = \frac{\partial G^i}{\partial y^j}, \tag{1.5}$$

$$4G^i(x,y) = g^{ik}\left(\frac{\partial^2F^2}{\partial y^k\partial x^h}y^h - \frac{\partial F^2}{\partial x^k}\right). \tag{1.6}$$

We shall refer only to it in the following.

In the presence of this nonlinear connection, the pseudo - Riemannian metric g can be written as $g(u)=g_{ij}\delta y^i\otimes\delta y^j$ and we may also consider a pseudo - Riemannian metric in the horizontal bundle $\bigcup\limits_{u\in T_0M} H_uT_0M \to T_0M$ in the form $g_{ij}(x,y)dx^i\otimes dx^j$. These can be added in order to provide a pseudo - Riemannian metric on TM

$$g_{ij}(x,y)dx^i\otimes dx^j + g_{ij}(x,y)\delta y^i\otimes\delta y^j$$

which generalizes the Sasaki metric.

Besides this we may define a metric of signature (n,n) given by

$$G(X^H,Y^H)=0,\ G(X^V,Y^V)=0,\ G(X^H,Y^V)=g_{ij}(x,y)X^iY^j \text{ for} \tag{1.7}$$

$X=X^i\partial_i, Y=Y^j\partial_j$ any vector fields on M.

Using (1.7) and (1.2′) one easily checks

Proposition 1.1. *The pair (J,G) is a Norden structure on T_0M, that is we have*

$$G(JA,JB)+G(A,B)=0 \textit{ for } A,B\in\mathcal{X}(T_0M). \tag{1.8}$$

2 A deformation of the Norden structure (J,G)

We consider on T_0M the vector fields $C=y^i\dot\partial_i$ and $S=y^i\delta_i$ and we define a tensor field $\widetilde{J}$ as follows

$$\begin{aligned}\widetilde{J}(X^H) &= \alpha(F^2)X^V+\beta(F^2)G(X^H,C)C,\\ \widetilde{J}(X^V) &= \gamma(F^2)X^H+\delta(F^2)G(X^V,S)S,\end{aligned} \tag{2.1}$$

where $\alpha,\beta,\gamma,\delta:\mathrm{Im}(F^2)\subset R\to R$ are smooth functions.

Lemma 2.1. *$\widetilde{J}^2 = -I$ if and only if*

$$\alpha\gamma = -1,\ \alpha\delta + \beta(\gamma + \delta F^2) = 0. \tag{2.2}$$

We say that $\widetilde{J}$ is a deformation of J.

Now we deform G in the form $\widetilde{G}$ given by

$$\begin{aligned} &\widetilde{G}(X^H, Y^H) = 0, \quad \widetilde{G}(X^V, Y^V) = 0, \\ &\widetilde{G}(X^H, Y^V) = a(F^2)G(X^H, Y^V) + b(F^2)G(X^H, C)G(Y^V, S), \end{aligned} \tag{2.3}$$

where $y_i = g_{ij}(x, y)y^j$ and $a, b : \mathrm{Im}(F^2) \subset R \to R$ are smooth functions satisfying $a > 0$ and $a + bF^2 > 0$.

Lemma 2.2 *Let be $\widetilde{J}$ verifying (2.2). Then the pair $(\widetilde{J}, \widetilde{G})$ is a Norden structure on T_0M.*

Proof. It suffices to check that

$$\widetilde{G}(\widetilde{J}X^H, \widetilde{J}Y^V) + \widetilde{G}(X^H, Y^V) = 0 \tag{$*$}$$

holds for every $X, Y \in \mathcal{X}(M)$.

For calculation it is useful to notice that $G(X^H, Y^V) = G(X^V, Y^H)$, $G(C, S) = F^2$ as well as

$$G(X^H, C)G(Y^V, S) = G(X^V, S)G(Y^V, S) = G(X^H, C)G(Y^H, C).$$

Making use of $\alpha\gamma = -1$, the left side of (*) reduces to the expression $(a + bF^2)$ $(\alpha\delta + \beta\gamma + \beta\delta F^2)G(X^H, C)G(Y^V, S)$ which vanishes because of the second equation (2.2), q.e.d.

Note that in fact by Lemma 2.2 we have obtained a family of Norden structures on T_0M. This family depends on four parameters, let say α, β, a, b subject to the condition $\alpha \neq 0$, $a + bF^2 > 0$. Then $\gamma = -\frac{1}{\alpha}$, $\delta = \frac{\beta}{\alpha(\alpha+\beta F^2)}$. Indeed, by the first equation (2.2) α and γ should be different from zero and then necessarily follows $\alpha + \beta F^2 \neq 0$ and $\gamma + \delta F^2 \neq 0$.

It is to be expected that in so large set of Norden structures to find a great variety of such structures as for instance complex Norden structures ($\widetilde{J}$ integrable) or Kähler - Norden structures ($\nabla\widetilde{J} = 0$ for ∇ the Levi - Civita connection of $\widetilde{G}$) and so on. Unhappy our deformations of J and G are not very generous in this respect. As we shall see below, there are very few complex Norden structures as well as Kähler - Norden structures in the family $(\widetilde{J}, \widetilde{G})$ and only in some restrictive conditions on the Finsler space F^n.

3 On the integrability of $\widetilde{J}$

The Nijenhuis tensor field of $\widetilde{J}$ is

$$N_{\widetilde{J}}(A, B) = [\widetilde{J}A, \widetilde{J}B] - [A, B] - \widetilde{J}[\widetilde{J}A, B] - \widetilde{J}[A, \widetilde{J}B], \tag{3.1}$$

$A, B \in \mathcal{X}(T_0M)$.

We choosed to evaluate $N_{\widetilde{J}}$ in the adapted frame $(\delta_i, \dot\partial_i)$. To this aim we write $\widetilde{J}$ in the form

$$\begin{aligned} \widetilde{J}(\delta_i) &= A_i^k \dot\partial_k, \qquad & A_i^k &= \alpha\delta_i^k + \beta y_i y^k, \\ \widetilde{J}(\dot\partial_i) &= B_i^k \dot\partial_k, \qquad & B_i^k &= \gamma\delta_i^k + \delta y_i y^k, \end{aligned} \tag{3.2}$$

and we shall use the following formulae for brackets

$$\begin{aligned} [\delta_i, \delta_j] &= R_{ij}^k \dot\partial_k, \qquad & R_{ij}^k &= \delta_j N_i^k - \delta_i N_j^k, \\ [\delta_i, \dot\partial_j] &= G_{ji}^k \dot\partial_k, \qquad & G_{ji}^k &= \dot\partial_j N_i^k = G_{ij}^k, \\ & [\dot\partial_i, \dot\partial_j] = 0. \end{aligned} \tag{3.3}$$

By a direct calculation one gets

$$\begin{aligned} N_{\widetilde{J}}(\delta_i, \delta_j) &= \overset{1}{N}{}^{k}_{ij}\, \delta_k + \overset{2}{N}{}^{k}_{ij} \dot\partial_k, \\ N_{\widetilde{J}}(\delta_i, \dot\partial_j) &= \overset{3}{N}{}^{k}_{ij}\, \delta_k + \overset{4}{N}{}^{k}_{ij}\, \dot\partial_k, \\ N_{\widetilde{J}}(\dot\partial_i, \dot\partial_j) &= \overset{5}{N}{}^{k}_{ij}\, \delta_k + \overset{6}{N}{}^{k}_{ij}\, \dot\partial_k, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \overset{1}{N}{}^{k}_{ij} &= B_h^k(\delta_j A_i^h - \delta_i A_j^h + G_{js}^h A_i^s - G_{is}^h A_j^s), \\ \overset{2}{N}{}^{k}_{ij} &= A_i^h \dot\partial_h A_j^k - A_j^h \dot\partial_h A_i^k - R_{ij}^k, \\ \overset{3}{N}{}^{k}_{ij} &= A_i^h \dot\partial_h B_j^k + B_h^k \dot\partial_j A_i^h - A_h^k B_j^s R_{is}^h, \\ \overset{4}{N}{}^{k}_{ij} &= B_j^h \delta_h A_i^k + A_h^k \delta_i B_j^h + A_i^s B_j^h G_{sh}^k + G_{ij}^k, \\ \overset{5}{N}{}^{k}_{ij} &= B_i^h \delta_h B_j^k - B_j^h \delta_h B_i^k + (G_{ih}^s B_j^h - G_{jh}^s B_i^h) B_s^k, \\ \overset{6}{N}{}^{k}_{ij} &= B_i^h B_j^s R_{hs}^k + A_h^k(\dot\partial_j B_i^h - \dot\partial_i B_j^h). \end{aligned} \tag{3.5}$$

We notice that the matrices A_i^j and B_i^j are invertible and we have

$$A_i^k B_k^j = -\delta_i^j. \tag{3.6}$$

By (3.6) it follows

$$(\delta_h A_i^k) B_k^j + A_i^k \delta_h(B_k^j) = 0. \tag{3.7}$$

Using (3.6) and (3.7) one finds

$$B_r^i B_s^j\, \overset{1}{N}{}^{k}_{ij} = \overset{5}{N}{}^{k}_{rs}\,. \tag{3.8}$$

It follows

Lemma 3.1 $\overset{1}{N}{}^{k}_{ij} = 0$ *if and only if* $\overset{5}{N}{}^{k}_{ij} = 0$.

Similarly one gets

$$\overset{6}{N}{}_{ij}^{k} = -B_i^r B_j^s \overset{2}{N}{}_{rs}^{k}\ . \tag{3.9}$$

Hence we have

Lemma 3.2 *$\overset{2}{N}{}_{ij}^{k} = 0$ if and only if $\overset{6}{N}{}_{ij}^{k} = 0$.*

Going further we need more formulae from the geometry of F^n. We shall assume that F^n is endowed with the Cartan metrical connection (F^i_{jk}, C^i_{jk}) and we shall denote as usual by $_{|i}$ and $|_i$ and $h-$ and $v-$covariant derivative with respect to it. Then we have (for details see [1])

$$\begin{aligned} &F^2_{|k} := \delta_k F^2 = 0, F^2|_k := \dot{\partial}_k F^2 = 2y_k, \\ &y^i_{|k} = 0 \Leftrightarrow \delta_k \cdot y^i = -F^i_{jk}y^j \Leftrightarrow F^i_{jk}y^j = N^i_k, \\ &y_{i|j} = 0 \Leftrightarrow \delta_i \cdot y_j = F^k_{ij}y_k, \dot{\partial}_k y_i = g_{ik}, y^s G^h_{js} = N^h_j \end{aligned} \tag{3.10}$$

We set $t = \frac{1}{2}F^2$ and denote by α', β'... the derivatives with respect to t. We have

$$\delta_k \alpha = 0, \dot{\partial}_k \alpha = \alpha' y_k, \tag{3.11}$$

and similarly for $\beta, \gamma, ...$

Now we are going to compute $\overset{1}{N}{}_{ij}^{k}, ..., \overset{6}{N}{}_{ij}^{k}$. First, we have $\overset{1}{N}{}_{ij}^{k} = B_h^k(\delta_j A_i^h + G^h_{js}A^s_i - i/j)$, where by i/j we denoted the previous terms with i changed to j and j changed to i. Then using (3.10) and (3.11) we get

$$\delta_j A_i^h + A_i^s G^h_{js} - i/j = \beta(\delta_j y_i)y^h + \beta y_i \delta_j y^h + \alpha G^h_{ji} + \beta y_i N^h_j - i/j =$$

$$= \beta(F^k_{ij}y_k y^h - y_i F^h_{sj} + y_i N^h_j) - i/j = 0.$$

Thus $\overset{1}{N}{}_{ij}^{h} = 0$ and by the Lemma 3.1, $\overset{5}{N}{}_{ij}^{k} = 0$.

Next,

$$\overset{4}{N}{}_{ij}^{k} = B^h_j(\delta_h A^k_i + A^s_i G^k_{sh}) + A^k_h \delta_i B^h_j + G^k_{ij} =$$

$$= B^h_j(\beta F^s_{ih}y_s y^k + \alpha G^k_{ih}) + \delta A^k_h(F^s_{ij}y_s y^h - y_j N^h_i) + G^k_{ij} =$$

$$= G^k_{ij}(\alpha\gamma + 1) + y^k y_s F^s_{ij}(\alpha\delta + \beta\gamma + \beta\delta F^2) = 0$$

by (2.2).

We continue with $\overset{2}{N}{}_{ij}^{k}$. We get

$$(\alpha\delta^h_i + \beta y_i y^h)(\alpha' y_h \delta^k_j + \beta' y_h y_j y^k + \beta g_{jh} y^k + \beta y_j \delta^k_h) - i/j =$$

$$= \alpha'(\alpha + \beta F^2) y_i \delta^k_j + \alpha\beta y_j \delta^k_i - i/j.$$

Hence

$$\overset{2}{N}{}_{ij}^{k} = (\alpha\alpha' - \alpha\beta + \alpha'\beta F^2)(y_i\delta^k_j - y_j\delta^k_i) - R^k_{ij}. \tag{3.12}$$

By Lemma 3.2, $\overset{6}{N}{}_{ij}^{k}$ vanishes together with $\overset{2}{N}{}_{ij}^{k}$ so we need not its explicit form. However, we give it at follows: $\overset{6}{N}{}_{ij}^{k} = \gamma^2 R^k_{ij} - \gamma\delta(y_i R^k_j - y_j R^k_i) + \alpha(\delta - \gamma')(y_i\delta^k_j - y_j\delta^k_i)$, for $R^k_i := R^k_{ih} y^h$. One can check directly that $\overset{2}{N}{}_{ij}^{k} = 0 \Leftrightarrow \overset{6}{N}{}_{ij}^{k} = 0$.

By a tedious computation we find

$$\begin{gathered}\overset{3}{N}{}_{ij}^{k} = y_i\delta^k_j(\alpha\gamma' + \beta\gamma + \beta\gamma' F^2) + y_j\delta^k_i(\alpha\delta + \gamma\alpha') + \\ + y_i y_j y^k[\beta\delta - \beta\gamma' + (\alpha\delta + \beta\gamma + 2t\beta\delta)'] + R^k_{ij} - \alpha\delta y_j(y^s R^k_{is}) - \\ - \beta\gamma y^k(y_h R^h_{ij}) - \beta\delta y^s(y^h R^h_{is}).\end{gathered}$$

This simplifies using (2.2) and

Lemma 3.1 $y_h R^k_{ij} = 0$.

Proof. We have $y_h R^h_{ij} = y_h\ R^h_{k\ ij} y^k = y^s g_{sh} R^h_{k\ ij} y^k = y^s R_{ks\ ij} y^k = 0$, because of $R_{sk\ ij} = -R_{ks\ ij}$. Here R^h_{kij} is the $(hh)-$curvature of the Cartan connection.

Therefore,

$$\begin{aligned}\overset{3}{N}{}_{ij}^{k} = {} & R^k_{ij} - \alpha\delta y_j R^k_i + y_i\delta^k_j(\alpha\gamma' + \beta\gamma + \beta\gamma' F^2) + y_j\delta^k_i(\alpha\delta + \gamma\alpha') + \\ & + y_i y_j y^k \beta(\delta - \gamma').\end{aligned} \tag{3.13}$$

The following assertion is now clear.

Theorem 3.1. *The almost complex structure* $\widetilde{J}$ *is integrable if and only if* $\overset{2}{N}{}_{ij}^{k} = 0$ *and* $\overset{3}{N}{}_{ij}^{k} = 0$.

We continue by distinguishing two cases.

Case 1: $\delta = 0 \Leftrightarrow \beta = 0$.

In this case we get

$$\overset{2}{N}{}_{ij}^{k} = -R^k_{ij} + \alpha'\alpha(y_i\delta^k_j - y_j\delta^k_i) \tag{3.12'}$$

$$\overset{3}{N}{}_{ij}^{k} = R^k_{ij} - \gamma\alpha'(y_i\delta^k_j - y_j\delta^k_i). \tag{3.13'}$$

It follows that $\overset{2}{N}{}_{ij}^{k} = 0 = \overset{3}{N}{}_{ij}^{k}$ implies $(y_i\delta^k_j - y_j\delta^k_i)(\alpha - \gamma)\alpha' = 0$. As $\alpha \neq \gamma$ and $y_i\delta^k_j - y_j\delta^k_i$ never vanishes it results $\alpha' = 0$, hence $\alpha = c$(constant)$\neq 0$. Then $R^k_{ij} = 0$. Conversely, $R^k_{ij} = 0$ and $\alpha = c$ implies $N_{\widetilde{J}} = 0$.

Then $\widetilde{J}$ reduces to the almost complex structure J_c given by

$$\begin{cases} J_c(\delta_i) = c\dot{\partial}_i, \\ J_c(\dot{\partial}_i) = -\frac{1}{c}\delta_i. \end{cases} \tag{3.14}$$

Case 2: $\delta \neq 0$.

In this case $\overset{2}{N}{}^{k}_{ij}$ and $\overset{3}{N}{}^{k}_{ij}$ can be put into the forms

$$\overset{2}{N}{}^{k}_{ij}= \frac{\beta}{\gamma\delta}(\delta-\gamma)(y_i\delta^k_j - y_j\delta^k_i) - R^k_{ij}, \tag{3.12$''$}$$

$$\overset{3}{N}{}^{k}_{ij}= R^k_{ij} - \alpha\delta y_j R^k_i + \frac{\beta\gamma}{\delta}(\delta-\gamma')y_i\delta^k_j - \frac{\delta-\gamma'}{\gamma}y_j\delta^k_i + \beta(\delta-\gamma')y_iy_jy^k. \tag{3.13$''$}$$

These forms suggest to consider firstly the case when $\delta=\gamma'$. Then $\overset{2}{N}{}^{k}_{ij}=0$ is equivalent with $R^k_{ij}=0$ and then $\overset{3}{N}{}^{k}_{ij}=0$ follows. Conversely, if $\delta=\gamma'$ and $R^k_{ij}=0$, both $\overset{2}{N}{}^{k}_{ij}$ and $\overset{3}{N}{}^{k}_{ij}$ vanish, hence $N_{\widetilde{J}}=0$. For $\delta=\gamma'$, the almost complex structure $\widetilde{J}$ reduces to J_γ given by

$$\begin{array}{l} J_\gamma(\delta_i) = \left(-\frac{1}{\gamma}\delta^k_i + \frac{\gamma'}{\gamma(\gamma+\gamma' F^2)}y_iy^k\right)\dot{\partial}_k \\ J_\gamma(\dot{\partial}_i) = (\gamma\delta^k_i + \gamma' y_iy^k)\delta_k \text{ with } \gamma\neq 0 \end{array} \tag{3.15}$$

For $\gamma=-\frac{1}{c}$ $(c\neq 0)$ we get J_c. Thus J_c is included in the family of the almost complex structures J_γ.

Summarizing the above analysis we obtain

Theorem 3.2. *Let $F^n=(M,F)$ be a Finsler space and let be H the horizontal distribution on T_0M defined by the nonlinear Cartan connection of F^n. The almost complex structures J_γ given by (3.15) are integrable if and only if the distribution H is integrable.*

Remark 3.1. That J_1 is integrable if and only if the distribution H is integrable follows also from Proposition 3.3, Ch. VII of [1], noticing that for Finsler space $G^i_{jk}=G^i_{kj}$.

Now let us assume that $\delta\neq\gamma'$. If $\overset{2}{N}{}^{k}_{ij}=0$, then $R^k_{ij}=\frac{\beta}{\gamma\delta}(\delta-\gamma')(y_i\delta^k_j-y_j\delta^k_i)$ and $R^k_i=\frac{\beta}{\gamma\delta}(\delta-\gamma')(y_iy^k-F^2\delta^k_i)$. Inserting these in the equation $\overset{3}{N}{}^{k}_{ij}=0$, after some simplifications we arrive at the condition

$$\beta(1+\gamma^2)y_i\delta^k_j + \delta(\alpha\beta F^2-1)y_j\delta^k_i + \beta(\gamma-\alpha)y_iy_jy^k = 0. \tag{3.16}$$

Contracting it by y^j, it results

$$[\beta(1+\gamma^2)+\beta F^2(\gamma-\alpha)]y_iy^k + \delta F^2(\alpha\beta F^2-1)\delta^k_i = 0.$$

Since $\operatorname{rank}(y_iy^k)=1$ and $\operatorname{rank}(\delta^k_i)=n$, this equation holds if and only if we have

$$\begin{cases} F^2(\gamma-\alpha)+1+\gamma^2=0, \\ \alpha\beta F^2-1=0. \end{cases} \tag{3.17}$$

This system together with (2.2) gives a system of four equations with the unknown $\alpha,\beta,\gamma,\delta$. It is compatible with the solutions

$$\alpha=\frac{\Gamma}{F^2}, \qquad \beta=\frac{1}{\Gamma_0}, \qquad \delta=\gamma\frac{\Gamma^2-F^2}{\Gamma_0^2F^2},$$

for $\Gamma = \gamma^2 + F^2\gamma + 1$ and γ a solution of the equation $\gamma^3 + 2t\gamma^2 + \gamma + 2t = 0$, ($t = \frac{1}{2}F^2$). But with this solution, (3.16) becomes $y_i(F^2\delta_j^k - y_jy^k) = 0$ which never holds. Thus the only cases for $N_{\widetilde{J}} = 0$ are those found in the above.

4 The Kähler condition $\nabla\widetilde{J} = 0$

Now we search, among the Norden structures $(\widetilde{J}, G)$, for those that satisfy the condition $\nabla\widetilde{J} = 0$, where ∇ denotes the Levi - Civita connection of G. They will called Kähler - Norden structures.

To this aim we shall compute $\nabla\widetilde{J}$ in the adapted frame $(\delta_i, \dot{\partial}_i)$.

Firstly, we write the equation defining ∇:

$$2G(\nabla_U V, W) = UG(V,W) + VG(U,W) - WG(U,V) + G([U,V],W)-$$

$$-G([U,W],V) - G([V,W],U), U, V, W \in \mathcal{X}(T_0M)$$

for $U = \delta_i, \dot{\partial}_i$, $V = \delta_j, \dot{\partial}_j$, $W = \delta_k, \dot{\partial}_k$ in all needed combinations and we find

$$\begin{aligned} \nabla_{\delta_i}\delta_j &= \left(F_{ij}^k + \tfrac{1}{2}g^{kh}g_{ij\|h}\right)\delta_k + \rho_{ij}^k\dot{\partial}_k, \\ \nabla_{\delta_i}\dot{\partial}_j &= \left(F_{ij}^k - \tfrac{1}{2}g^{kh}g_{ih\|j}\right)\dot{\partial}_k, \\ \nabla_{\dot{\partial}_i}\delta_j &= \tfrac{1}{2}g^{kh}(g_{ih\|j} - g_{ij\|h})\dot{\partial}_k, \\ \nabla_{\dot{\partial}_i}\dot{\partial}_j &= C_{ij}^k\dot{\partial}_k, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} F_{ij}^k &= \tfrac{1}{2}g^{kh}(\delta_i g_{jh} + \delta_j g_{ih} - \delta_h g_{ij}), \\ C_{ij}^k &= \tfrac{1}{2}g^{kh}(\dot{\partial}_i g_{jh} + \dot{\partial}_j g_{ih} - \dot{\partial}_h g_{ij}) = \tfrac{1}{2}g^{kh}\dot{\partial}_i g_{jh}, \\ g_{ij\|k} &= \delta_k g_{ij} - G_{ik}^h g_{hj} - G_{jk}^h g_{hi}, \\ \rho_{ji}^k &= \tfrac{1}{2}g^{kh}(R_{kij} - R_{jik} - R_{ijk}), R_{kij} := R_{ij}^h g_{kh}. \end{aligned} \tag{4.2}$$

We introduce also the notations

$$\begin{aligned} L_{ji}^k &= F_{ij}^k + \tfrac{1}{2}g^{kh}g_{ij\|h}, \\ H_{ji}^k &= F_{ij}^k - \tfrac{1}{2}g^{kh}g_{ih\|j}, \\ D_{ji}^k &= \tfrac{1}{2}g^{kh}(g_{ih\|j} - g_{ij\|h})\dot{\partial}_k \end{aligned} \tag{4.3}$$

and thus we get

$$\begin{aligned} \nabla_{\delta_i}\delta_j &= L_{ji}^k\delta_k + \rho_{ji}^k\dot{\partial}_k, & \nabla_{\delta_i}\dot{\partial}_j &= H_{ji}^k\dot{\partial}_k, \\ \nabla_{\dot{\partial}_i}\delta_j &= D_{ji}^k\dot{\partial}_k, & \nabla_{\dot{\partial}_i}\dot{\partial}_j &= C_{ji}^k\dot{\partial}_k. \end{aligned} \tag{4.4}$$

Then we use $(\nabla_U\widetilde{J})V = \nabla_U\widetilde{J}V - \widetilde{J}(\nabla_U V), U, V \in \mathcal{X}(T_0M)$ in order to find

$$\begin{aligned} (\nabla_{\delta_i}\widetilde{J})\delta_j &= -\rho_{ij}^h B_h^k\delta_k + A_{ij}^k\dot{\partial}_k, \\ (\nabla_{\dot{\partial}_i}\widetilde{J})\dot{\partial}_j &= B_{ij}^k\delta_k + B_j^h\rho_{ih}^k\dot{\partial}_k, \\ (\nabla_{\dot{\partial}_i}\widetilde{J})\delta_j &= -D_{ji}^h B_h^k\delta_k + U_{ij}^k\dot{\partial}_k, \\ (\nabla_{\dot{\partial}_i}\widetilde{J})\dot{\partial}_j &= V_{ij}^k\delta_k + B_j^h D_{hi}^k\dot{\partial}_k, \end{aligned} \tag{4.5}$$

where we have put

$$\begin{aligned} A^k_{ij} &= \delta_i A^k_j + A^h_j H^k_{hi} - A^k_h L^h_{ij}, \\ B^k_{ij} &= \delta_i B^k_j + B^h_j L^k_{hi} - B^k_h H^h_{ji}, \\ U^k_{ij} &= \dot{\partial}_i A^k_j + A^h_j C^k_{ih}, V^k_{ij} = \dot{\partial}_i B^k_j - C^k_{ij} B^k_h. \end{aligned} \tag{4.6}$$

Using $A^i_k B^k_j = -\delta^i_j$ it comes out that $A^k_{ij} = A^k_h B^h_{is} A^s_j$ and $U^k_{ij} = A^k_h V^h_{is} A^s_j$. Thus we have

Lemma 4.1. *A^k_{ij} and B^k_{ij} simultaneously vanish. So do U^k_{ij} and V^k_{ij}.*

Based on (3.5) and Lemma 3.1 it follows

Theorem 4.1. *$\nabla \widetilde{J} = 0$ if and only if*

$$\rho^k_{ij} = 0, D^k_{ij} = 0, A^k_{ij} = 0, U^k_{ij} = 0. \tag{4.7}$$

Are the conditions (4.7) compatible? Recall that $(\widetilde{J}, G)$ is a Norden structure, that is we have (2.2). We shall see that the answer is yes only for a class of very particular Finsler space.

The equation $U^k_{ij} = 0$ is equivalent with the equation

$$\alpha' y_i \delta^k_j + \beta' y_i y_j y^k + \beta g_{ij} y^k + \beta y_j \delta^k_i + \alpha C^k_{ij} = 0, \tag{4.8}$$

because $y^h C^k_{ih} = 0$.

In (4.8) we change i and j and subtract the obtained equation from (4.8). It results

$$(\alpha' - \beta)(y_i \delta^k_j - y_j \delta^k_i) = 0.$$

But $y_i \delta^k_j - y_j \delta^k_i$ never vanish. Hence, necessarily $\alpha' = \beta$. Thus (4.8) takes the form

$$\beta(y_i \delta^k_j + y_j \delta^k_i) + \beta g_{ij} y^k + \beta' y_i y_j y^k + \alpha C^k_{ij} = 0. \tag{4.8$'$}$$

We multiply this by y^j and we get

$$(2\beta + \beta' F^2) y_i y^k + \beta F^2 \delta^k_i = 0.$$

As $\mathrm{rank}(y_i y^k) = 1$ and $\mathrm{rank}(\delta^k_i) = n$, this equation hold if and only if $\beta = 0$. Returning to (4.8$'$), we find that necessarily $C^k_{ij} = 0$, ($\alpha \neq 0$!).

The form of C^k_{ij} in (4.2) says us that if $C^k_{ij} = 0$, then the metrical tensor field (g_{ij}) does not depend on y. In the other words, the Finsler space F^n reduces to a (pseudo-)Riemannian one. In this case, the coefficients (N^i_j) of the Cartan nonlinear connection given by (cf. (2.2), Ch. VIII, [1]):

$$N^i_j(x, y) = \frac{1}{2} \dot{\partial}_j (\gamma^i_{rs}(x) y^r y^s),$$

reduce to

$$N^i_j(x, y) = \gamma^i_{jk}(x) y^k,$$

where $(\gamma^i_{jk}(x))$ are the Christoffel symbols constructed with $(g_{ij}(x))$.

Then $R^k_{ij} = r^k_{hij} y^h$ and so $R_{kij} = r_{hkij} y^h$, where $r^k_{h\ ij}(x)$ is the curvature tensor of $(g_{ij}(x))$.

The condition $\rho^k_{ij} = 0$ is equivalent with the equation

$$R_{kij} - R_{jik} - R_{ijk} = 0.$$

This takes the form

$$r_{h\ kij} - r_{h\ jik} - r_{h\ ijk} = 0,$$

and, by the Bianchi identity, it reduces to $r_{hij\ k} = 0$.

Thus the condition $\rho^k_{ij} = 0$ holds if the (pseudo-)Riemannian metric $(g_{ij}(x))$ is locally flat.

The form of $(N^i_j(x,y))$ says us that $G^k_{ij} = \gamma^k_{ij}(x)$ and then $g_{ij\|k}$ is nothing but the covariant derivative of $g_{ij}(x)$, hence $g_{ij\|k} = 0$. It follows $D^k_{ij} = 0$. We notice also that

$$L^k_{ji} = \gamma^k_{ji}, H^k_{ji} = \gamma^k_{ji}. \tag{4.9}$$

Using (4.9) one easily checks that $A^k_{ij} = 0$. For $\beta = 0$ we have also $\delta = 0$ and from $\alpha' = \beta$ it follows $\alpha = c$(constant). Thus the almost product structure $\widetilde{J}$ reduces to J_c given by

$$J_c(\delta_i) = c\dot{\partial}_i, \qquad J_c(\dot{\partial}_i) = \frac{1}{c}\delta_i. \tag{4.10}$$

Summarizing up we have

Theorem 4.2. *The only Kähler - Norden structures among the Norden structures* $(\widetilde{J}, G)$ *are the pairs* (J_c, G), $0 \neq c \in \mathbb{R}$, *and this only if the Finsler space* F^n *reduces to a locally flat Riemannian space.*

Corollary 4.1. *Let* (M, g) *be a Riemannian space. The Norden structure* (J, G) *on* TM *is Kähler - Norden if and only if the Riemannian space* (M, g) *is locally flat.*

The Theorem 4.2 gives merely a negative result: for a Finsler space F^n no one of the deformations $\widetilde{J}$ paired with G gives a Kähler - Norden structure. The same holds if $\widetilde{J}$ are paired with $\widetilde{G}$.

References

[1] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Pub. FTPN 59, (1994).

[2] Zbigniew Olszak, *On almost complex structures with Norden metrics on tangent bundles.* Period. Math. Hung. 51, No. 2, 59-74 (2005).