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New Refinements of Cesaro's Inequality

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Abstract. In this paper we present new refinements for the Cesaro's inequality, namely if $a, b, c > 0$ then $(a+b)(b+c)(c+a) \geq 8abc$.

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1 Main Results

Theorem 1. *If $x, y, z > 0$, $n \geq 2$ and*

$$C_n(x,y,z) = \frac{1}{\sqrt{(x+y)(y+z)(z+x)}} \left(\frac{1}{3}\sqrt[n]{4xy(x+y)} + \frac{1}{3}\sqrt[n]{4yz(y+z)} + \frac{1}{3}\sqrt[n]{4zx(z+x)} \right)^{\frac{3n}{2}},$$

then $8xyz \leq C_n(x,y,z) \leq (x+y)(y+z)(z+x)$ and $C_n \geq C_k$ for all $n \geq k \geq 2$.

Proof.

$$\sum_{cyclic} \sqrt[n]{\frac{4yz}{(x+y)(x+z)}} = \sum_{cyclic} \sqrt[n]{\frac{2y}{x+y} \cdot \frac{2z}{x+z} \underbrace{1 \cdot 1 \cdot \cdots \cdot 1}_{(n-2)-time}} \leq \sum_{cyclic} \frac{\frac{2y}{x+y} + \frac{2z}{x+z} + n - 2}{n} = 3,$$

therefore

$$\sum_{cyclic} \sqrt[n]{4xy(x+y)} \leq 3\sqrt[n]{(x+y)(y+z)(z+x)}$$

but

$$\sum_{cyclic} \sqrt[n]{4xy(x+y)} \geq 3\sqrt[3n]{64x^2y^2z^2(x+y)(y+z)(z+x)}$$

so

$$\sqrt[3n]{64x^2y^2z^2(x+y)(y+z)(z+x)} \leq \frac{1}{3}\sum_{cyclic} \sqrt[n]{4xy(x+y)(y+z)} \leq \sqrt[n]{(x+y)(y+z)(z+x)},$$

after simple computation we obtain the desired result.

Application 1.1. In all triangle ABC holds

1) $32sRr \le C_n(a,b,c) \le 2s\left(s^2+r^2+2Rr\right)$
2) $8sr^2 \le C_n(s-a,s-b,s-c) \le 4sRr$
3) $8s^2r \le C_n(r_a,r_b,r_c) \le 4s^2R$
4) $\frac{8r}{s} \le C_n\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right) \le \frac{4R}{s}$
5) $\frac{8s}{r} \le C_n\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right) \le \frac{4sR}{r^2}$
6) $\frac{r^2}{2R^2} \le C_n\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right) \le \frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{32R^3}$
7) $\frac{s^2}{2R^2} \le C_n\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right) \le \frac{(4R+r)^3+s^2(2R+r)}{32R^3}$

which are new refinements for Euler's and Gerretsens inequalities.

Now, we introduce the following new means

$$D_1(x_1,x_2,\dots,x_n)=\frac{1}{n}\sum_{cyclic}\sqrt[3]{x_1x_2x_3},\quad D_2^m(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\sqrt[3]{C_m(x_1x_2x_3)},$$

$$D_3(x_1,x_2,\dots,x_n)=\frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)},\quad H(x_1,x_2,\dots,x_n)=\frac{n}{\sum\limits_{k=1}^{n}\frac{1}{x_k}},$$

$$G(x_1,x_2,\dots,x_n)=\sqrt[n]{\prod_{k=1}^{n}x_k},\quad A(x_1,x_2,\dots,x_n)=\frac{1}{n}\sum_{k=1}^{n}x_k$$

and $\overline{D_i}(x_1,x_2,\dots,x_n)=\frac{1}{D_i\left(\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}\right)}$, for which we obtain the following refinements:

Theorem 2.1. *If $x_k>0$ $(k=1,2,\dots,n)$ then*

$$H \le \overline{D_3} \le \overline{D_2^m} \le \overline{D_1} \le G \le D_1 \le D_2^m \le D_3 \le A$$

for all $m\ge 2$ and if $m\ge k\ge 2$ then $D_2^m \ge D_2^k$.

Proof. Using the Theorem 1 and the AM-GM inequality, we obtain

$$8x_1x_2x_3 \le C_m(x_1,x_2,x_3) \le (x_1+x_2)(x_2+x_3)(x_3+x_1) \le \frac{8}{27}(x_1+x_2+x_3)^3$$

or

$$2\sqrt[3]{x_1x_2x_3} \le \sqrt[3]{C_m(x_1,x_2,x_3)} \le \sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)} \le \frac{2}{3}(x_1+x_2+x_3),$$

therefore

$$G(x_1,x_2,\dots,x_n)=\sqrt[n]{\prod_{k=1}^{n}x_k} \le \frac{1}{n}\sum_{cyclic}\sqrt[3]{x_1x_2x_3}=D_1(x_1,x_2,\dots,x_n)\le$$

$$\le\frac{1}{2n}\sum_{cyclic}\sqrt[3]{C_m(x_1,x_2,x_3)}=D_2^m(x_1,x_2,\dots,x_n)\le\frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)}=$$

$$= D_3(x_1, x_2, \ldots, x_n) \le \frac{1}{n} \sum_{cyclic} \frac{x_1 + x_2 + x_3}{3} = A(x_1, x_2, \ldots, x_n).$$

If in this chan of inequalities we take $x_k \to \frac{1}{x_k}$ $(k = 1, 2, \ldots, n)$, then we obtain the others inequalities.

Now, we introduce the following new means:

$$D_4(x_1, x_2, \ldots, x_n) = \frac{1}{2} \prod_{cyclic} \sqrt[3n]{C_m(x_1, x_2, x_3)},$$

$$D_5(x_1, x_2, \ldots, x_n) = \frac{1}{2} \prod_{cyclic} \sqrt[3n]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)}$$

$$D_6(x_1, x_2, \ldots, x_n) = \frac{1}{3} \prod_{cyclic} \sqrt[n]{x_1 + x_2 + x_3}$$

and $\overline{D_i}(x_1, x_2, \ldots, x_n) = \frac{1}{D_i\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ $(i \in \{4, 5, 6\})$, for which we obtains the following new refinements:

Theorem 2.2. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$ then*

$$H \le \overline{D_6} \le \overline{D_5} \le \overline{D_4} \le G \le D_4 \le D_5 \le D_6 \le A.$$

Proof.

$$\prod_{k=1}^{n} x_k = \prod_{cyclic} \sqrt[3]{x_1 x_2 x_3} \le \prod_{cyclic} \frac{1}{2} \sqrt[3]{C_m(x_1, x_2, x_3)} \le$$

$$\le \prod_{cyclic} \frac{1}{2} \sqrt[3]{(x_1 + x_2)(x_2 + x_3)(x_3 + x_1)} \le \prod_{cyclic} \frac{x_1 + x_2 + x_3}{3} \le \left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)^n.$$

Theorem 3. *If $x, y, z > 0$, then*

1) $\frac{(x+y)(y+z)(z+x)}{xyz} \ge \max\left\{\left(1 + \frac{x+y}{\sqrt{xy}}\right)^2; \left(1 + \frac{y+z}{\sqrt{yz}}\right)^2; \left(1 + \frac{z+x}{\sqrt{zx}}\right)^2\right\} - 1 \ge 8$

2) $\frac{(x+y)(y+z)(z+x)}{xyz} \ge \frac{1}{3}\left(1 + \frac{x+y}{\sqrt{xy}}\right)^2 + \frac{1}{3}\left(1 + \frac{y+z}{\sqrt{yz}}\right)^2 + \frac{1}{3}\left(1 + \frac{z+x}{\sqrt{zx}}\right)^2 - 1 \ge 8$

Proof. $\left(\frac{x}{z} + \frac{z}{y}\right) + \left(\frac{y}{z} + \frac{z}{x}\right) \ge 2\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{x}}$ or $\sqrt{(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)} \ge 1 + \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}$ or $\sqrt{1 + \frac{(x+y)(y+z)(z+x)}{xyz}} \ge 1 + \frac{x+y}{\sqrt{xy}}$, therefore $\frac{(x+y)(y+z)(z+x)}{xyz} \ge \left(1 + \frac{x+y}{\sqrt{xy}}\right)^2 - 1 \ge 8$ etc.

We introduce the following notations:

$$F_1(x, y, z) = xyz \max\left\{\left(1 + \frac{x + y}{\sqrt{xy}}\right)^2; \left(1 + \frac{y + z}{\sqrt{yz}}\right)^2; \left(1 + \frac{z + x}{\sqrt{zx}}\right)^2\right\} - xyz,$$

$$F_2(x, y, z) = \frac{xyz}{3}\left(\left(1 + \frac{x + y}{\sqrt{xy}}\right)^2 + \left(1 + \frac{y + z}{\sqrt{yz}}\right)^2 + \left(1 + \frac{z + x}{\sqrt{zx}}\right)^2\right) - xyz,$$

so we have the following inequalities

$$8xyz \le \{F_1(x,y,z);F_2(x,y,z)\} \le (x+y)(y+z)(z+x).$$

Application 3.1. In all triangle ABC holds
1) $32sRr \le \{F_1(a,b,c);F_2(a,b,c)\} \le 2s\left(s^2+r^2+2Rr\right)$
2) $8sr^2 \le \{F_1(s-a,s-b,s-c);F_2(s-a,s-b,s-c)\} \le 4sRr$
3) $8s^2r \le \{F_1(r_a,r_b,r_c);F_2(r_a,r_b,r_c)\} \le 4s^2R$
4) $\frac{8r}{s} \le \left\{F_1\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right);F_2\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)\right\} \le \frac{4R}{s}$
5) $\frac{8s}{r} \le \left\{F_1\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right);F_2\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)\right\} \le \frac{4sR}{r}$
6) $\frac{r^2}{2R^2} \le \left\{F_1\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right);F_2\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\right\} \le \frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{32R^3}$
7) $\frac{s^2}{2R^2} \le \left\{F_1\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right);F_2\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\right\} \le \frac{(4R+r)^3+s^2(2R+r)}{32R^3}$
which are new refinements for Euler's and Gerretsen's inequalities.

In following we introduce the new means

$$R(x_1,x_2,\ldots,x_n) = \frac{1}{2n}\sum_{cyclic}\left\{\sqrt[3]{F_1(x_1,x_2,x_3)};\ \sqrt[3]{F_2(x_1,x_2,x_3)}\right\}$$

and $\overline{R}(x_1,x_2,\ldots,x_n) = \frac{1}{R\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$, then we have the following new refinements:

Theorem 4.1. *If $x_k > 0$ $(k=1,2,\ldots,n)$ then $H \le \overline{D_3} \le \overline{R} \le \overline{D_1} \le G \le D_1 \le R \le D_3 \le A$.*

Proof. Using the Theorem 3 and the AM-GM inequality, we obtain

$$2\sqrt[3]{x_1x_2x_3} \le \left\{\sqrt[3]{F_1(x_1,x_2,x_3)};\ \sqrt[3]{F_2(x_1,x_2,x_3)}\right\} \le$$

$$\le \sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)} \le \frac{2}{3}(x_1,x_2,x_3),$$

therefore

$$G(x_1,x_2,\ldots,x_n) \le D_1(x_1,x_2,\ldots,x_n) \le \frac{1}{2n}\sum_{cyclic}\left\{\sqrt[3]{F_1(x_1,x_2,x_3)};\ \sqrt[3]{F_2(x_1,x_2,x_3)}\right\} =$$

$$= R(x_1,x_2,\ldots,x_n) \le \frac{1}{2n}\sum_{cyclic}\sqrt[3]{(x_1+x_2)(x_2+x_3)(x_3+x_1)} =$$

$$= D_3(x_1,x_2,\ldots,x_n) \le A(x_1,x_2,\ldots,x_n).$$

If $x_k \to \frac{1}{x_k}$ $(k=1,2,\ldots,n)$ then holds the another inequalities:
Now we introduce the following new means:

$$D_7(x_1,x_2,\ldots,x_n) = \frac{1}{2}\prod_{cyclic}\left\{\sqrt[3n]{F_1(x_1,x_2,x_3)};\ \sqrt[3n]{F_2(x_1,x_2,x_3)}\right\}$$

and $\overline{D_7}(x_1, x_2, \ldots, x_n) = \frac{1}{D_7\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following new refinements:

Theorem 4.2. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$ *then* $H \le \overline{D_6} \le \overline{D_5} \le \overline{D_7} \le G \le D_7 \le D_5 \le D_6 \le A$.

Theorem 5. *If* $x, y, z > 0$, *then* $(x + y)(y + z)(z + x) \ge$

$$\ge \left(2\sqrt{xy} + \left(\frac{\sqrt{x} - \sqrt{y}}{2}\right)^2\right)\left(2\sqrt{yz} + \left(\frac{\sqrt{y} - \sqrt{z}}{2}\right)^2\right)\left(2\sqrt{zx} + \left(\frac{\sqrt{z} - \sqrt{x}}{2}\right)^2\right) \ge xyz.$$

Proof. $\prod_{cyclic}(x + y) \ge \prod_{cyclic}\left(2\sqrt{xy} + \left(\frac{\sqrt{x}-\sqrt{y}}{2}\right)^2\right)$ (see [2]).

If $K_1(x, y, z) = \prod_{cyclic}\left(2\sqrt{xy} + \left(\frac{\sqrt{x}-\sqrt{y}}{2}\right)^2\right)$, then we have the following

Application 5.1. In all triangle ABC holds

1) $32sRr \le K_1(a, b, c) \le 2s\left(s^2 + r^2 + 2Rr\right)$
2) $8sr^2 \le K_1(s - a, s - b, s - c) \le 4sRr$
3) $8s^2r \le K_1(r_a, r_b, r_c) \le 4s^2R$
4) $\frac{8r}{s} \le K_1\left(tg\frac{A}{2}, tg\frac{B}{2}, tg\frac{C}{2}\right) \le \frac{4R}{s}$
5) $\frac{8s}{r} \le K_1\left(ctg\frac{A}{2}, ctg\frac{B}{2}, ctg\frac{C}{2}\right) \le \frac{4sR}{r^2}$
6) $\frac{r^2}{2R^2} \le K_1\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \le \frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{32R^3}$
7) $\frac{s^2}{2R^2} \le K_1\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \le \frac{(4R+r)^3+s^2(2R+r)}{32R^3}$

which are new refinements for Euler's and Gerretsen's inequalities.

Now we introduce the following new mean: $K_2(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sum_{cyclic}\sqrt[3]{K_1(x_1, x_2, x_3)}$ and $\overline{K_2}(x_1, x_2, \ldots, x_n) = \frac{1}{K_2\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we obtain the following refinements:

Theorem 6.1. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$ *then* $H \le \overline{D_3} \le \overline{K_2} \le \overline{D_1} \le G \le D_1 \le K_2 \le D_3 \le A$.

Proof. Using the Theorem 3 and the AM-GM inequality, we obtain $G(x_1, x_2, \ldots, x_n) \le D_1(x_1, x_2, \ldots, x_n) \le K_2(x_1, x_2, \ldots, x_n) \le D_3(x_1, x_2, \ldots, x_n) \le A(x_1, x_2, \ldots, x_n)$. If in this we take $x_k \to \frac{1}{x_k}$, then we obtain the others inequalities.

Now, we introduce the following new means: $K_3(x_1, x_2, \ldots, x_n) = \frac{1}{2}\prod_{cyclic}\sqrt[3n]{K_1(x_1, x_2, x_3)}$ and $\overline{K_3}(x_1, x_2, \ldots, x_n) = \frac{1}{K\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ for which we obtain the following new refinements:

Theorem 6.2. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$ *then* $H \le \overline{D_6} \le \overline{D_5} \le \overline{K_3} \le G \le K_3 \le D_5 \le D_6 \le A$.

Theorem 7. *If* $x, y, z > 0$, *then* $(x + y)(y + z)(z + x) \ge$

$$\ge 8xyz\left(1 + \frac{\left(\sqrt{x} - \sqrt{y}\right)^2}{8\sqrt{xy}}\right)\left(1 + \frac{\left(\sqrt{y} - \sqrt{z}\right)^2}{8\sqrt{yz}}\right)\left(1 + \frac{\left(\sqrt{z} - \sqrt{x}\right)^2}{8\sqrt{zx}}\right) \ge 8xyz.$$

Proof. $\prod_{cyclic}(x+y)\geq\prod_{cyclic}2\sqrt{xy}\left(1+\frac{(\sqrt{x}-\sqrt{y})^2}{8\sqrt{xy}}\right)$ (see [2]).

If $P_1(x,y,z)=\prod_{cyclic}2\sqrt{xy}\left(1+\frac{(\sqrt{x}-\sqrt{y})^2}{8\sqrt{xy}}\right)$, then we have the following:

Application 7.1. In all triangle ABC holds
1) $32sRr\leq P_1(a,b,c)\leq 2s\left(s^2+r^2+2Rr\right)$
2) $8sr^2\leq P_1(s-a,s-b,s-c)\leq 4sRr$
3) $8s^2r\leq P_1(r_a,r_b,r_c)\leq 4s^2R$
4) $\frac{8r}{s}\leq P_1\left(tg\frac{A}{2},tg\frac{B}{2},tg\frac{C}{2}\right)\leq\frac{4R}{s}$
5) $\frac{8s}{r}\leq P_1\left(ctg\frac{A}{2},ctg\frac{B}{2},ctg\frac{C}{2}\right)\leq\frac{4sR}{r^2}$
6) $\frac{r^2}{2R^2}\leq P_1\left(\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right)\leq\frac{(2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2}{32R^3}$
7) $\frac{s^2}{2R^2}\leq P_1\left(\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right)\leq\frac{(4R+r)^3+s^2(2R-r)}{32R^3}$
which are new refinements for Euler's and Gerretsen's inequality.

Now, we introduce the following new mean $P_2(x_1,x_2,\ldots,x_n)=\frac{1}{2n}\sum_{cyclic}\sqrt[3]{P_1(x_1,x_2,x_3)}$ and $\overline{P_2}(x_1,x_2,\ldots,x_n)=\frac{1}{P_1\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ for which we obtain the following refinements:

Theorem 8.1. *If $x_k>0$ $(k=1,2,\ldots,n)$ then $H\leq\overline{D_3}\leq\overline{P_2}\leq\overline{D_1}\leq G\leq D_1\leq P_2\leq D_3\leq A$.*

Proof. See the proof of Theorem 7.

Now, we introduce the following new means $P_3(x_1,x_2,\ldots,x_n)=\frac{1}{2}\prod_{cyclic}\sqrt[3]{P_1(x_1,x_2,x_3)}$ and $\overline{P_3}(x_1,x_2,\ldots,x_n)=\frac{1}{P_3\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ for which we obtain the following refinements:

Theorem 8.2. *If $x_k>0$ $(k=1,2,\ldots,n)$ then $H\leq\overline{D_6}\leq\overline{D_5}\leq\overline{P_3}\leq G\leq P_3\leq D_5\leq D_6\leq A$.*

Conjecture 1. If $x_k>0$ $(k=1,2,\ldots,n)$ then

$$(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1)\leq(x_1+x_3)(x_2+x_4)\ldots(x_n+x_2).$$

If $M(x,y,z)\in\{C_n(x,y,z);F_1(x,y,z);F_2(x,y,z);K_1(x,y,z);P_1(x,y,z);Q_1(x,y,z)\}$, then we have the following

Conjecture 2. If $x_k>0$ $(k=1,2,\ldots,n)$ then

$$2^n\prod_{k=1}^n x_k\leq\prod_{cyclic}\sqrt[3]{M(x_1,x_2,x_3)}\leq(x_1+x_2)(x_2+x_3)\ldots(x_n+x_1).$$

Partial proof. The inequality $2^n\prod_{k=1}^n x_k\leq\prod_{cyclic}\sqrt[3]{M(x_1,x_2,x_3)}$ is immediately from Theorems 1, 3, 5, 7 but from this theorems for the right side results olny $\prod_{cyclic}M(x_1,x_2,x_3)\leq\prod_{cyclic}(x_1+x_2)^2(x_1+x_3)$, we must prove that $\prod_{cyclic}M(x_1,x_2,x_3)\leq\prod_{cyclic}(x_1+x_2)^3$.

2 Extension to the convex functions

Theorem 9. *If $F : R \to R$ is a convex function then*

$$\sum_{k=1}^{n} F\left(S-(n-1)a_k\right) - \sum_{k=1}^{n} F\left(a_k\right) \geq (n-2)\left(\sum_{k=1}^{n} F\left(a_k\right) - nF\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\right) \geq 0,$$

where $a_k \in R$ $(k = 1, 2, \ldots, n)$ and $S = \sum\limits_{k=1}^{n} a_k$.

Proof. We start with the generalized T. Popoviciu's inequality (see [3])

$$\sum_{k=1}^{n} F\left(y_k\right) + n(n-2)F\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) \geq (n-1)\sum_{k=1}^{n} F\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} y_i - y_k\right)\right)$$

or

$$\sum_{k=1}^{n} F\left(y_k\right) - \sum_{k=1}^{n} F\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} y_i - y_k\right)\right) \geq$$

$$\geq (n-2)\sum_{k=1}^{n} F\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} y_i - y_k\right)\right) - n(n-2)F\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) =$$

$$= (n-2)\left(\sum_{k=1}^{n} F\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} y_i - y_k\right)\right) - nF\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right)\right) \geq 0$$

because from Jensen's inequality holds:

$$\sum_{k=1}^{n} F\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} y_i - y_k\right)\right) \geq nF\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\sum\limits_{i=1}^{n} y_i - y_k}{n-1}\right)\right) = nF\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right).$$

Denote $a_k = \frac{1}{n-1}\left(\sum\limits_{i=1}^{n} y_i - y_k\right)$ $(k = 1, 2, \ldots, n)$, then result $\sum\limits_{i=1}^{n} y_i - y_k = (n-1)a_k$ and $\sum\limits_{i=1}^{n} y_i = \sum\limits_{i=1}^{n} a_i$, so $y_k = \sum\limits_{i=1}^{n} a_i - (n-1)a_k$ $(k = 1, 2, \ldots, n)$ and

$$\sum_{k=1}^{n} F\left(S-(n-1)a_k\right) - \sum_{k=1}^{n} F\left(a_k\right) \geq (n-2)\left(\sum_{k=1}^{n} F\left(a_k\right) - nF\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\right) \geq 0.$$

This is a refinement of inequality $\sum\limits_{k=1}^{n} F\left(S-(n-1)a_k\right) \geq \sum\limits_{k=1}^{n} F\left(a_k\right)$ doe to M.S. Klamkin (see [5]).

Application 9.1. If $a_k > 0$ $(k = 1, 2, \ldots, n)$ and $S = \sum\limits_{k=1}^{n} a_k$, $S > (n-1)a_k$ $(k = 1, 2, \ldots, n)$ then $\prod\limits_{k=1}^{n} a_k \geq \prod\limits_{k=1}^{n}\left(S-(n-1)a_k\right)\left(\frac{\left(\frac{1}{n}\sum\limits_{k=1}^{n} a_k\right)^n}{\prod\limits_{k=1}^{n} a_k}\right)^{n-2} \geq \prod\limits_{k=1}^{n}\left(S-(n-1)a_k\right)$, which is a refinement of inequality $\prod_{k=1}^{n} a_k \geq \prod_{k=1}^{n} (S-(n-1)a_k)$, doe to D.S. Mitrinovic and Adamovic (see [4]).

Proof. In Theorem 9, we take $F(x) = -\ln x$.

Application 9.2. (A refinement of Cesaro's inequality for n variables). If $x_k > 0$ $(k = 1, 2, \ldots, n)$ then

$$\prod_{k=1}^{n}(x_1+\ldots+x_{k-1}+x_{k+1}+\ldots+x_n) \geq (n-1)^n \sqrt[n-1]{\prod_{k=1}^{n} x_k \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^{\frac{n(n-2)}{n-1}}} \geq (n-1)^n \prod_{k=1}^{n} x_k.$$

Proof. In Application 9.1 we take $a_k = \frac{S-x_k}{n-1}$ $(k = 1, 2, \ldots, n)$, $S = \sum_{k=1}^{n} x_k$.

If $Q_1(x, y, z) = \sqrt{64xyz\left(\frac{x+y+z}{3}\right)^3}$, then we have the following

Application 9.3. If $x, y, z > 0$, then $8xyz \leq Q_1(x, y, z) \leq (x+y)(y+z)(z+x)$.

Proof. In Application 9.1 we take $n = 3$.

Application 9.4. In all triangle ABC holds

1) $27\left(s^2+r^2+2Rr\right)^2 \geq 512s^2Rr$
2) $3\sqrt{3}R \geq 2s$
3) $27s^2\left(s^2+r^2+2Rr\right)^2 \geq 16\left(s^2+r^2+4Rr\right)^3$
4) $27s^2R^2 \geq 4r(4R+r)^3$
5) $27\left((2R-r)\left(s^2+r^2-8Rr\right)-2Rr^2\right)^2 \geq 8192r^2(2R-r)^3$
6) $27\left((4R+r)^3+s^2(2R+r)\right)^2 \geq 8192s^2R(4R+r)^3$

which are new refinements of Euler's and Gerretsen's inequality.

Proof. In Application 9.1 we take

$$(x, y, z) \in \Big\{ (a, b, c); (s-a, s-b, s-c); (h_a, h_b, h_c); (r_a, r_b, r_c);$$
$$\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right); \left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right)\Big\}.$$

Now, we introduce the following means: $Q_2(x_1, x_2, \ldots, x_n) = \frac{1}{2n}\sqrt[3]{Q_1(x_1, x_2, x_3)}$ and $\overline{Q}_2(x_1, x_2, \ldots, x_n) = \frac{1}{Q_2\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$, for which we obtain the following refinements:

Theorem 10. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$ *then* $H \leq \overline{D_3} \leq \overline{Q_2} \leq \overline{D_1} \leq G \leq D_1 \leq Q_2 \leq D_3 \leq A$.

Proof. See the proof of Theorem 8.

Theorem 10. (A generalization of T. Popoviciu's inequality). *If* $f : A \to R$ $(A \subseteq R)$ *is a convex function, and* $x_k \in A$ $(k = 1, 2, \ldots, n)$, *then*

$$\sum_{k=1}^{n} f(x_k) + n(n-2) f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \geq (n-1)\sum_{k=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_k\right)\right).$$

Proof. We use the Hardy-Littlewood-Pólya's inequality. Let be $x_1 \geq x_2 \geq \cdots \geq x_n$ or $\frac{1}{n-1}\left(\sum\limits_{i=1}^{n} x_i - x_n\right) \geq \frac{1}{n-1}\left(\sum\limits_{i=1}^{n} x_i - x_{n-1}\right) \geq \cdots \geq \frac{1}{n-1}\left(\sum\limits_{i=1}^{n} x_i - x_1\right)$ and $x_1 \geq \frac{1}{n}\sum\limits_{k=1}^{n} x_k \geq x_n$, therefore we have the following cases:

1) If $x_1 \geq \frac{1}{n}\sum\limits_{k=1}^{n} x_k \geq x_2$, then we take $a_1 = x_1, a_2 = a_3 = a_4 = \cdots = a_{n^2-2n+1} = \frac{1}{n}\sum\limits_{k=1}^{n} x_k$ and $a_{n^2-2n+2} = x_2, a_{n^2-2n+3} = x_3, \ldots, a_{n^2-2n+k} = x_k, \ldots, a_{n^2-2n+n} = x_n$ but $a_{n^2-2n+n} = a_{n(n-1)}$ and $b_1 = b_2 = \cdots = b_{n-1} = \frac{1}{n-1}\left(\sum\limits_{k=1}^{n} x_k - x_n\right), b_n = b_{n+1} = \cdots = b_{n+(n-2)} = \frac{1}{n-1}\left(\sum\limits_{k=1}^{n} x_k - x_{n-1}\right), \ldots, b_{n^2-2n+2} = b_{n^2-2n+3} = \cdots = b_{n(n-1)} = \frac{1}{n-1}\left(\sum\limits_{k=1}^{n} x_k - x_1\right)$ so $a_1 + a_2 + \cdots + a_k \geq b_1 + b_2 + \cdots + b_k,\ k \in \left\{1, 2, \ldots, n^2 - n - 1\right\}$ and $a_1 + a_2 + \cdots + a_{n^2-n} = b_1 + b_2 + \cdots + b_{n^2-n}$, therefore from Hardy-Littlewood-Pólya's inequality, holds $f(a_1) + f(a_2) + \cdots + f(a_{n^2-n}) \geq f(b_1) + f(b_2) + \cdots + f(b_{n^2-n})$ and from this result the desired inequality. In general case k.) we have $x_k \geq \frac{1}{n}\sum\limits_{i=1}^{n} x_i \geq x_{k+1}$ and we take $a_1 = x_1, a_2 = x_2, \ldots, a_k = x_k, a_{k+1} = a_{k+2} = \cdots = a_{k+n^2-n} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, $a_{k+n^2-n+1} = x_{k+1}, \ldots, a_{k+n^2-n+n} = x_{k+n}, a_{n^2-n} = x_n$ and $b_1 = b_2 = \cdots = b_{n-1} = \frac{1}{n-1}\left(\sum\limits_{i=1}^{n} x_i - x_n\right), b_n = b_{n+1} = \cdots = b_{n+k-2} = \frac{1}{n-1}\left(\sum\limits_{i=1}^{n} x_i - x_{n-1}\right), \ldots, b_{n^2-2n+2} = b_{n^2-2n+3} = \cdots = b_{n^2-n} = \frac{1}{n-1}\left(\sum\limits_{i=1}^{n} x_i - x_1\right)$, finally we obtain (like in case 1) the desired inequality.

Application 10.1. If $x_k > 0$, then $\prod\limits_{k=1}^{n}(x_1 + x_2 + \cdots + x_{k-1} + x_{k+1} + \cdots + x_n) \geq$

$$\geq (n-1)^n \sqrt[n-1]{\prod_{k=1}^{n} x_k \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^{\frac{n(n-2)}{n-1}}} \geq (n-1)^n \prod_{k=1}^{n} x_k.$$

Proof. In Theorem 10 we take $f(x) = -\ln x$.

Theorem 11. (A generalization of T. Popoviciu's inequality). *If* $f: A \to R\ (k=1,2,\ldots,n)$, *then* $(n-2)\sum\limits_{k=1}^{n} f(x_k) + nf\left(\frac{1}{n}\sum\limits_{k=1}^{n} x_k\right) \geq 2\sum\limits_{1\leq i<j\leq n} f\left(\frac{x_i+x_j}{2}\right)$.

Proof. By induction. For $n = 3$ we have the Popoviciu's inequality (see [4]). We suppose true for $n-1$ so

$(*)\ (n-3)\sum\limits_{k=1}^{n-1} f(x_k) + (n-1)f\left(\frac{1}{n-1}\sum\limits_{k=1}^{n-1} x_k\right) \geq 2\sum\limits_{1\leq i<j\leq n-1} f\left(\frac{x_i+x_j}{2}\right)$

If we write all permutations of inequalities (*), then after addition we obtain

$$(n-3)(n-1)\sum_{k=1}^{n} f(x_k) + (n-1)\sum_{k=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_k\right)\right) \geq 2(n-2)\sum_{1\leq i<j\leq n} f\left(\frac{x_i+x_j}{2}\right).$$

Using the inequality from Theorem 10, holds

$$(n-3)(n-1)\sum_{k=1}^{n} f(x_k)+(n-1)\sum_{k=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_k\right)\right) \leq (n-3)(n-1)\sum_{k=1}^{n} f(x_k)+$$

$$+\left(\sum_{k=1}^{n} f(x_k) + n(n-2) f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right) = (n-2)\left((n-2)\sum_{k=1}^{n} f(x_k) + nf\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right)$$

or after simplification $(n-2)\sum\limits_{k=1}^{n} f(x_k) + nf\left(\frac{1}{n}\sum\limits_{k=1}^{n} x_k\right) \geq 2\sum\limits_{1\leq i<j\leq n} f\left(\frac{x_i+x_j}{2}\right)$.

Application 11.1. If $x_k > 0$, then

$$\prod_{1\leq i<j\leq n}(x_i+x_j) \geq 2^{\frac{n(n-1)}{2}}\left(\prod_{k=1}^{n} x_k\right)^{\frac{n-2}{2}}\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^{\frac{n}{2}} \geq^{\frac{n(n-1)}{2}}\left(\prod_{k=1}^{n} x_k\right)^{\frac{n-1}{2}}.$$

This is a new refinement of Cesaro's inequality.

Proof. From Theorem 11, we take $f(x) = -\ln x$.

Theorem 12. *If $f : A \to R$ $(A \subseteq R)$ is a convex function, $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $2 \leq k \leq n-1$, then*

$$\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) \geq k\sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{x_{i_1}+x_{i_2}+\cdots+x_{i_k}}{k}\right).$$

This inequality generalize the T. Popoviciu's inequality and is doe to Stankovici (see [4]*).*

Proof. By induction. For $n = 2$ we have the T. Popoviciu's inequality. We suppose true for $n-1$ so

(**) $\binom{n-3}{k-2}\left(\frac{n-1-k}{k-1}\sum\limits_{i=1}^{n-1} f(x_i) + (n-1) f\left(\frac{1}{n-1}\sum\limits_{i=1}^{n-1} x_i\right)\right) \geq k\sum\limits_{1\leq i_1<\cdots<i_k\leq n-1} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)$

We write all permutations of inequality (**) and after the addition holds:

$$\binom{n-3}{k-2}\left(\frac{n-1-k}{k-1}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i) - f(x_j)\right) + (n-1)\sum_{j=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_j\right)\right)\right) \geq$$

$$\geq k\sum\sum f\left(\frac{x_{r_1}+\cdots+x_{r_k}}{k}\right)$$

but

$$\sum\sum f\left(\frac{x_{r_1}+\cdots+x_{r_k}}{k}\right) = (n-k)k\sum f\left(\frac{x_{p_1}+\cdots+x_{p_k}}{k}\right)$$

and

$$(n-1)\sum_{j=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_j\right)\right) \leq n\,(n-2)\, f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f\left(x_i\right)$$

(See Theorem 10), so

$$(n-1)\binom{n-3}{k-2}\sum_{j=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_j\right)\right) \leq$$

$$\leq n\,(n-2)\binom{n-3}{k-2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \binom{n-3}{k-2}\sum_{i=1}^{n} f\left(x_i\right)$$

and adding to this the identity

$$\binom{n-3}{k-2}\frac{n-1-k}{k-1}\sum_{j=1}^{n}\left(\sum_{i=1}^{n} f\left(x_i\right) - f\left(x_j\right)\right) = \binom{n-3}{k-2}\frac{n-1-k}{k-1}\,(n-1)\sum_{i=1}^{n} f\left(x_i\right)$$

we obtain

$$(n-1)\binom{n-3}{k-2}\sum_{j=1}^{n} f\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_j\right)\right) + \binom{n-3}{k-2}\frac{(n-k-1)\,(n-1)}{k-1}\sum_{i=1}^{n} f\left(x_i\right) \leq$$

$$\leq \binom{n-3}{k-2} n\,(n-2)\, f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \binom{n-3}{k-2}\sum_{i=1}^{n} f(x_i) + \binom{n-3}{k-2}\frac{(n-k-1)\,(n-1)}{k-1}\sum_{i=1}^{n} f(x_i) =$$

$$= n\,(n-k)\binom{n-2}{k-2} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \left(\binom{n-3}{k-2}\frac{(n-1)\,(n-k-1)}{k-1}\binom{n-3}{k-2}\right)\sum_{i=1}^{n} f\left(x_i\right)$$

but

$$\binom{n-3}{k-2} + \frac{(n-1)\,(n-k-1)}{k-1}\binom{n-3}{k-2} = \frac{(n-k)^2}{k-1}\binom{n-2}{k-2}$$

so

$$k\,(n-k)\sum_{1\leq i_1<\cdots<i_k\leq n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \leq$$

$$\leq (n-k)\binom{n-2}{k-2}\left(nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) + \frac{n-k}{k-1}\sum_{i=1}^{n} f\left(x_i\right)$$

and after division by $n-k$ we obtain the desired inequality.

Application 12.1. If $x_i > 0$ $(i = 1, 2, \ldots, n)$ and $2 \leq k \leq n-1$, then

$$\prod_{1\leq i_1<\cdots<i_k\leq n}\left(x_{i_1}+\cdots+x_{i_k}\right) \geq k^{\binom{n}{k}}\left(\prod_{i=1}^{n} x_i\right)^{\frac{n-k}{k(k-1)}\binom{n-2}{k-2}}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^{\frac{n}{k}\binom{n-2}{k-2}} \geq k^{\binom{n}{k}}\left(\prod_{i=1}^{n} x_i\right)^{\frac{n-1}{k-1}}.$$

This is a new generalization and a new refinement of Cesaro's inequality.

Proof. In Theorem 12 we take $f(x) = -\ln x$.

3 Another refinements and generalizations

Theorem 13. *If $x, y, z > 0$, then*

$$\left(\frac{x+y+z}{3}\right)^3 \geq \frac{1}{8}(x+y)(y+z)(z+x) \geq \frac{1}{27}(x+\sqrt{xy}+y)(y+\sqrt{yz}+z)\left(z+\sqrt{zx}+x\right) \geq$$

$$\geq \frac{1}{64}\left(\sqrt{x}+\sqrt{y}\right)^2\left(\sqrt{y}+\sqrt{z}\right)^2\left(\sqrt{z}+\sqrt{x}\right)^2 \geq \frac{1}{27}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)^3 \geq xyz.$$

Proof. $\frac{27}{8}\prod_{cyclic}(x+y) = 27\prod_{cyclic}\frac{x+y}{2} \leq \left(\sum_{cyclic}\frac{x+y}{2}\right)^3 = \left(\sum_{cyclic}x\right)^3$. If $x, y \geq 0$, then $\frac{3}{2}(x+y) \geq x+\sqrt{xy}+y \geq \frac{3}{4}\left(\sqrt{x}+\sqrt{y}\right)^2$ etc. Let be $A = \sqrt{x}+\sqrt{y}+\sqrt{z}$, $B = \sqrt{xy}+\sqrt{yz}+\sqrt{zx}$, $C = \sqrt{xyz}$. From $\sum_{cyclic}\sqrt{x}\left(\sqrt{y}-\sqrt{z}\right)^2 \geq 0$ holds $AB \geq 9C$ and from $\sum_{cyclic}\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$ we have $A^2 \geq 3B$, therefore $27(AB-C)^2 \geq 27\left(AB-\frac{1}{9}AB\right)^2 = \frac{64}{3}A^2B^2 \geq 64B^3$ or $27\prod_{cyclic}\left(\sqrt{x}+\sqrt{y}\right)^2 \geq 64\left(\sum_{cyclic}\sqrt{xy}\right)^3$.

Now we introduce the following new means

$$U_1(x_1, x_2, \ldots, x_n) = \frac{1}{3n}\sum_{cyclic}\sqrt[3]{(x_1+\sqrt{x_1x_2}+x_2)(x_2+\sqrt{x_2x_3}+x_3)(x_3+\sqrt{x_3x_1}+x_1)},$$

$$U_2(x_1, x_2, \ldots, x_n) = \frac{1}{4n}\sum_{cyclic}\sqrt[3]{\left(\sqrt{x_1}+\sqrt{x_2}\right)^2\left(\sqrt{x_2}+\sqrt{x_3}\right)^2\left(\sqrt{x_3}+\sqrt{x_1}\right)^2},$$

$$U_3(x_1, x_2, \ldots, x_n) = \sum_{cyclic}\sqrt{x_1x_2}$$

and

$\overline{U_i}(x_1, x_2, \ldots, x_n) = \frac{1}{U_i\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ $(i = 1, 2, 3)$, for which we obtain the following refinements:

Theorem 14.1. *If $x_k > 0$ $(k = 1, 2, \ldots, n)$, then $H \leq \overline{D_3} \leq \overline{U_1} \leq \overline{U_2} \leq \overline{U_3} \leq \overline{D_1} \leq G \leq D_1 \leq U_3 \leq U_2 \leq U_1 \leq D_3 \leq A$.*

Proof. See the proof of Theorem 8.

Now we introduce the following new means

$$U_4(x_1, x_2, \ldots, x_n) = \frac{1}{3}\prod_{cyclic}\sqrt[3n]{\sqrt{x_1x_2}+\sqrt{x_2x_3}+\sqrt{x_3x_1}},$$

$$U_5(x_1, x_2, \ldots, x_n) = \frac{1}{4}\prod_{cyclic}\sqrt[3n]{\left(\sqrt{x_1}+\sqrt{x_2}\right)^2\left(\sqrt{x_2}+\sqrt{x_3}\right)^2\left(\sqrt{x_3}+\sqrt{x_1}\right)^2},$$

$$U_6(x_1,x_2,\ldots,x_n)=\frac{1}{3}\sum_{cyclic}\sqrt[3n]{\left(x_1+\sqrt{x_1x_2}+x_2\right)\left(x_2+\sqrt{x_2x_3}+x_3\right)\left(x_3+\sqrt{x_3x_1}+x_1\right)},$$

$\overline{U_i}(x_1,x_2,\ldots,x_n)=\frac{1}{U_i\left(\frac{1}{x_1},\frac{1}{x_2},,\ldots,\frac{1}{x_n},\right)}$ $(i\in\{4,5,6\})$, for which we obtain the following refinements:

Theorem 14.2. *If* $x_k>0$ $(k=1,2,\ldots,n)$, *then* $H\leq\overline{D_6}\leq\overline{D_5}\leq\overline{U_6}\leq\overline{U_5}\leq\overline{U_4}\leq G\leq U_4\leq U_5\leq U_6\leq D_5\leq D_6\leq A.$

Theorem 15. *If* $x_k>0$ $(k=1,2,\ldots,n)$, *then*

$$\left(\sum_{k=1}^{n}x_k^{n-1}\right)^n\geq\left(\frac{n}{2}\right)^n\prod_{cyclic}\left(x_1^{n-1}+x_2^{n-1}\right)\geq\prod_{cyclic}\left(x_1^{n-1}+x_1^{n-2}x_2+\cdots+x_1x_2^{n-2}+x_2^{n-1}\right)\geq$$

$$\geq\left(\frac{n}{2^{n-1}}\right)^n\prod_{cyclic}\left(x_1+x_2\right)^{n-1}\geq n^n\prod_{k=1}^{n}x_k^{n-1}$$

(A generalization of Cesaro's inequality).

Proof. $\left(\sum\limits_{k=1}^{n}x_k^{n-1}\right)^n=\left(\sum\limits_{cyclic}\frac{x_1^{n-1}+x_2^{n-1}}{2}\right)^n\geq\left(\frac{n}{2}\right)^n\prod\limits_{cyclic}\left(x_1^{n-1}+x_2^{n-1}\right).$

Now, we prove the following inequalities:

$$\frac{x^{n-1}+y^{n-1}}{2}\geq\frac{x^{n-1}+x^{n-2}y+\cdots+cy^{n-2}+y^{n-1}}{n}\geq\left(\frac{x+y}{2}\right)^{n-1}$$

for all $x,y\geq 0$. For $n=2$ and $n=3$ its true. We suppose true for $n-2$ and we prove for $n-1$. For this we starting from:

$$\left(\frac{x^{n-2}+x^{n-3}y+\cdots+xy^{n-3}+y^{n-2}}{n-1}\right)\left(\frac{x+y}{2}\right)\geq\left(\frac{x+y}{2}\right)^{n-1}$$

or

$$(n-2)\left(x^{n-1}+y^{n-1}\right)\geq 2\sum_{k=1}^{n-2}x^{n-1-k}y^k$$

but this result from

$$x^{n-1}+y^{n-1}\geq x^ky^{n-1-k}+x^{n-1-k}y^k$$

which is equivalent with $\left(x^k-y^k\right)\left(x^{n-1-k}-y^{n-1-k}\right)\geq 0$, where $k\in\{1,2,\ldots,n-1\}$. If $k\in\{1,2,\ldots,n-1\}$, then from pondered AM-GM inequality results that $kx^{n-1}+(n-k)y^{n-1}\geq(n-1)x^ky^{n-k}$. After addition we obtain:

$$\frac{x^{n-1}+y^{n-1}}{2}\geq\frac{x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}}{n}.$$

Finally we have

$$\left(\frac{n}{2}\right)^n\prod_{cyclic}\left(x_1^{n-1}+x_2^{n-1}\right)=n^n\prod_{cyclic}\frac{x_1^{n-1}+x_2^{n-1}}{2}\geq$$

$$\geq \prod_{cyclic} \left(x_1^{n-1} + x_1^{n-2}x_2 + \cdots + x_1x_2^{n-2} + x_2^{n-1}\right) \geq$$

$$n^n \prod_{cyclic} \left(\frac{x_1+x_2}{2}\right)^{n-1} = \left(\frac{n}{2^{n-1}}\right)^n \prod_{cyclic} (x_1+x_2)^{n-1} \geq n^n \prod_{k=1}^{n} x_k^{n-1}.$$

Now, we introduce the following new means:

$$V_1(x_1, x_2, \ldots, x_n) = \frac{1}{2} \prod_{cyclic} \sqrt[n]{x_1+x_2}, \quad V_2(x_1, x_2, \ldots, x_n) = \frac{1}{n} \prod_{cyclic} \sqrt[n]{\sum_{i=0}^{n-1} x_1^{\frac{n-1-i}{n-1}} x_2^{\frac{i}{n-1}}},$$

$$V_3(x_1, x_2, \ldots, x_n) = \frac{1}{2^{n-1}} \prod_{cyclic} \left(x_1^{\frac{1}{n-1}} + x_2^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}},$$

$\overline{V_i}(x_1, x_2, \ldots, x_n) = \frac{1}{V_i\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}$ $(i \in \{1,2,3\})$, then we obtain the following refinements:

Theorem 16. *If* $x_k > 0$ $(k = 1, 2, \ldots, n)$, *then* $H \leq \overline{V_1} \leq \overline{V_2} \leq \overline{V_3} \leq G \leq V_3 \leq V_2 \leq V_1 \leq A$.

Theorem 17. *If* $x_i > 0$ $(i = 1, 2, \ldots, n)$ *and* $m, k \in N^*$, *then*

$$\frac{1}{n}\sum_{i=1}^{n} x_i^m \geq \frac{1}{n}\sum_{cyclic} \frac{1}{\binom{m+k-1}{k-1}} \sum_{m_1+\cdots+m_k=m} x_1^{m_1}\ldots x_k^{m_k} \geq \frac{1}{n}\sum_{cyclic}\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)^m \geq$$

$$\geq \frac{1}{n}\sum_{cyclic}\left(\sqrt[k]{\prod_{i=1}^{k} x_i}\right)^m \geq \left(\prod_{i=1}^{n} x_i\right)^m.$$

Proof. In [6] and [7] is proved that

$$\frac{1}{k}\sum_{i=1}^{k} x_i^m \geq \frac{1}{\binom{m+k-1}{k-1}} \sum_{m_1+\cdots+m_k=m} x_1^{m_1}\ldots x_k^{m_k} \geq \left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)^m,$$

therefore

$$\frac{1}{n}\sum_{i=1}^{n} x_i^m = \frac{1}{n}\sum_{cyclic}\left(\frac{1}{k}\sum_{i=1}^{k} x_i^m\right) \geq \frac{1}{n}\sum_{cyclic} \frac{1}{\binom{m+k-1}{k-1}} \sum_{m_1+\cdots+m_k=m} x_1^{m_1}\ldots x_k^{m_k} \geq$$

$$\geq \frac{1}{n}\sum_{cyclic}\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)^m \geq \frac{1}{n}\sum_{cyclic}\left(\sqrt[k]{\prod_{i=1}^{k} x_i}\right)^m \geq \left(\prod_{i=1}^{n} x_i\right)^m.$$

Now, we introduce the following new means:

$$W_1(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{cyclic} \frac{1}{\binom{m+k-1}{k-1}} \sum_{m_1+\cdots+m_k=m} \left(x_1^{m_1}\ldots x_k^{m_k}\right)^{\frac{1}{m}},$$

$$W_2(x_1,x_2,\ldots,x_n)=\frac{1}{n}\sum_{cyclic}\left(\frac{1}{k}\sum_{i=1}^{k}x_i^{\frac{1}{m}}\right)^m,\quad W_3(x_1,x_2,\ldots,x_n)=\frac{1}{n}\sum_{cyclic}\sqrt[k]{\prod_{i=1}^{k}x_i},$$

$\overline{W_j}(x_1,x_2,\ldots,x_n)=\frac{1}{W_j\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ $(j\in\{1,2,3\})$, for which we obtain the following refinements:

Theorem 18.1. *If* $x_i>0$ $(i=1,2,\ldots,n)$, *then* $H\le\overline{W_1}\le\overline{W_2}\le\overline{W_3}\le G\le W_3\le W_2\le W_1\le A$.

Now, we introduce the following new means: $W_4(x_1,x_2,\ldots,x_n)=\frac{1}{k}\prod_{cyclic}\left(\sum_{i=1}^{k}x_i\right)^{\frac{1}{n}}$,

$$W_5(x_1,x_2,\ldots,x_n)=\prod_{cyclic}\left(\frac{1}{\binom{m+k-1}{k-1}}\sum_{m_1+\cdots+m_k=m}\left(x_1^{m_1}\ldots x_k^{m_k}\right)^{\frac{1}{m}}\right)^{\frac{1}{n}},$$

$$W_6(x_1,x_2,\ldots,x_n)=\prod_{cyclic}\left(\frac{1}{k}\sum_{i=1}^{k}x_i^{\frac{1}{m}}\right)^{\frac{m}{n}},$$

$\overline{W_j}(x_1,x_2,\ldots,x_n)=\frac{1}{W_j\left(\frac{1}{x_1},\frac{1}{x_2},\ldots,\frac{1}{x_n}\right)}$ $(j\in\{1,2,3\})$ for which we obtain the following refinements: (for the Cesaro's inequality too.)

Theorem 18.2. *If* $x_i>0$ $(i=1,2,\ldots,n)$ *and* $m,k\in N^*$ *then* $H\le\overline{W_4}\le\overline{W_5}\le\overline{W_6}\le G\le W_6\le W_5\le W_4\le A$.

Conjecture 3. If $f:A\to R$ $(A\subseteq R)$ is convex, $x_i\in A$ $(i=1,2,\ldots,n)$, $1\le k\le n$, then $\frac{1}{n}\sum_{i=1}^{n}f(x_i)\ge\frac{1}{\binom{n+k-1}{k-1}}\sum_{n_1+\cdots+n_k=n}f\left(x_1^{\frac{n_1}{n}}\right)\ldots f\left(x_k^{\frac{n_k}{n}}\right)\ge f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)$.

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