

# Nash-Equilibrium Conditions for Strategic Form Games <sup>1</sup>

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**Abstract.** The notion of the Lagrange vector-function is considered for strategic form games. Nash equilibrium conditions are formulated and proved. Pareto-Nash equilibrium conditions are formulated for multi-criteria strategic form games.

**Keywords:** Noncooperative game, strategic form game, Nash equilibrium (NE), Nash equilibria set (NES), graph of best response mapping, intersection, method, algorithm, Lagrange vector-function, necessary and sufficient conditions for Nash equilibrium, multi-criteria game, Pareto-Nash equilibrium, multi-criteria Nash equilibrium.

**Mathematics Subject Classification (2000):** 91A06, 91A10, 91A44, 90C25, 90C25, 90C31, 90C90.

## 1 Introduction

The problem of Nash equilibrium identification in strategic form games remains actual over a half century. In this paper Nash equilibrium conditions are constructed as an extension of some classical theorems such as von Neumann theorem for matrix games or Kuhn-Tucker optimality conditions for mathematical programming problems. Equilibria principles for multi-criteria strategic games are constructed, also.

Consider the noncooperative game

$$\Gamma = \langle N, \{X_p\}_{p \in N}, \{f_p(x)\}_{p \in N} \rangle,$$

where

- $N = \{1, 2, \dots, n\}$  is a set of players,
- $X_p = \{x^p \in R^{n_p} : g_i^p(x^p) \leq 0, i = \overline{1, m_p}, h_i^p(x^p) = 0, i = \overline{1, l_p}, x^p \in M_p\}$  is a set of strategies of player  $p \in N$ ,
- $l_p, m_p, n_p < +\infty, p \in N$ ,

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<sup>1</sup>The research is supported in part by ICIAM07 to attend the 6th International Congress on Industrial and Applied Mathematics in Zurich and Award CERIM-1006-06 of the Moldovan Research and Development Association (MRDA) and U.S. Civilian Research Development Foundation for the Independent States of the Former Soviet Union (CRDF).

- $g_i^p(x^p), i = \overline{1, m_p}, h_i^p(x^p), i = \overline{1, l_p}$  are functions defined on  $M_p, p \in N$
- and  $f_p(x)$  is a  $p^{\text{th}}$  player payoff function defined on the Cartesian product  $X = \times_{p \in N} X_p$ .

Elements  $x = (x^1, x^2, \dots, x^n) \in X$  are named outcomes of the game (global strategies, situations or strategy profiles).

Without loss of generality suppose that the all players minimize the values of their cost functions.

**Definition [5].** The outcome  $\hat{x} \in X$  of the game is a (global, absolute) Nash equilibrium if

$$f_p(x^p, \hat{x}^{-p}) \geq f_p(\hat{x}^p, \hat{x}^{-p}), \forall x^p \in X_p, \forall p \in N,$$

where

$$\begin{aligned} \hat{x}^{-p} &= (\hat{x}^1, \hat{x}^2, \dots, \hat{x}^{p-1}, \hat{x}^{p+1}, \dots, \hat{x}^n), \\ \hat{x}^{-p} \in X_{-p} &= X_1 \times X_2 \times \dots \times X_{p-1} \times X_{p+1} \times \dots \times X_n, \\ \hat{x}^p || \hat{x}^{-p} &= (\hat{x}^p, \hat{x}^{-p}) = (\hat{x}^1, \hat{x}^2, \dots, \hat{x}^{p-1}, \hat{x}^p, \hat{x}^{p+1}, \dots, \hat{x}^n) = \hat{x} \in X. \end{aligned}$$

**Definition.** The outcome  $\hat{x} \in X$  of the game is a local (relative) Nash equilibrium if there exists such  $\varepsilon > 0$  that:

$$f_p(x^p, \hat{x}^{-p}) \geq f_p(\hat{x}^p, \hat{x}^{-p}), \forall x^p \in X_p \cap V_\varepsilon(\hat{x}^p), \forall p \in N,$$

where

$$V_\varepsilon(\hat{x}^p) = \{x^p \in R^{n_p} : \|x^p - \hat{x}^p\| \leq \varepsilon\}.$$

There are diverse alternative formulations of the NE (see, e.g., [4]): *as a fixed point of the best response correspondence, as a fixed point of a function, as a solution of a nonlinear complementarity problem, as a solution of a stationary point problem, as a minimum of a function on a polytope, as an element of semi-algebraic set.* The NES may be considered as an *intersection of the graphs of best response multivalued mappings*  $\text{Arg} \min_{\mathbf{x}^p \in X_p} f_p(\mathbf{x}^p, \mathbf{x}^{-p}) :$

$X_{-p} \multimap X_p, p = \overline{1, n}$ :

$$\begin{aligned} NES(\Gamma) &= \bigcap_{p \in N} Gr_p, \\ Gr_p &= \left\{ (\mathbf{x}^p, \mathbf{x}^{-p}) \in X : \begin{array}{l} \mathbf{x}^{-p} \in X_{-p}, \\ \mathbf{x}^p \in \text{Arg} \min_{\mathbf{x}^p \in X_p} f_p(\mathbf{x}^p, \mathbf{x}^{-p}) \end{array} \right\}, p \in N \end{aligned}$$

[7, 10, 11, 12, 13]. This article is a part of the series of works [7, 10, 11, 12, 13, 14, 15, 16, 17, 18] dedicated to equilibria principles investigation. Remark that the following theorem proofs

are in most similarly to the proofs of the optimality conditions theorems for mathematical programming problems (see chapter 7) in [9].

Remember the well known result that the all convex continuous compact games have NE [5], *i.e.*  $NES(\Gamma) \neq \emptyset$ .

Let  $L_p(x, u^p, v^p) = f_p(x) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p) + \sum_{i=1}^{l_p} v_i^p h_i^p(x^p)$  denotes the Lagrange function for the player  $p \in N$ , where  $u^p = (u_1^p, u_2^p, \dots, u_{m_p}^p)$ ,  $v^p = (v_1^p, v_2^p, \dots, v_{l_p}^p)$  are the Lagrange multipliers. Denote by

$$L(x, u, v) = (L_1(x, u^1, v^1), L_2(x, u^2, v^2), \dots, L_n(x, u^n, v^n))$$

the Lagrange vector function of the game  $\Gamma$ .

**Definition.** The point

$$(\hat{x}, \hat{u}, \hat{v}) = (\hat{x}, \hat{u}^1, \hat{v}^1, \dots, \hat{u}^n, \hat{v}^n) \in X \times R_{\geq}^{m_1} \times R^{l_1} \times \dots \times R_{\geq}^{m_n} \times R^{l_n}$$

is a saddle point of the  $\Gamma$ 's Lagrange vector function if

$$L_p(\hat{x}, u^p, v^p) \leq L_p(\hat{x}, \hat{u}^p, \hat{v}^p) \leq L_p(x^p, \hat{x}^{-p}, \hat{u}^p, \hat{v}^p)$$

for every  $x^p \in M_p$ ,  $u^p \in R_{\geq}^{m_p}$ ,  $v^p \in R^{l_p}$ ,  $p \in N$ .

## 2 The Saddle Point and The Nash Equilibrium General Sufficient Condition

**Theorem 1.** *If  $(\hat{x}, \hat{u}^1, \hat{v}^1, \dots, \hat{u}^n, \hat{v}^n)$  is a saddle point of the  $\Gamma$ 's Lagrange vector function  $L(x, u, v)$ , then  $\hat{x}$  is a Nash equilibrium for  $\Gamma$ .*

*Proof.* Assume  $(\hat{x}, \hat{u}^1, \hat{v}^1, \dots, \hat{u}^n, \hat{v}^n)$  is a saddle point of the Lagrange vector function  $L(x, u, v)$ . From the saddle point definition's left inequality it follows

$$\begin{aligned} f_p(\hat{x}) + \sum_{i=1}^{m_p} u_i^p g_i^p(\hat{x}^p) + \sum_{i=1}^{l_p} v_i^p h_i^p(\hat{x}^p) &= L_p(\hat{x}, u^p, v^p) \leq \\ &\leq L_p(\hat{x}, \hat{u}^p, \hat{v}^p) = f_p(\hat{x}) + \sum_{i=1}^{m_p} \hat{u}_i^p g_i^p(\hat{x}^p) + \sum_{i=1}^{l_p} \hat{v}_i^p h_i^p(\hat{x}^p). \end{aligned}$$

This inequality is equivalent to

$$f_p(\hat{x}) + \sum_{i=1}^{m_p} (u_i^p - \hat{u}_i^p) g_i^p(\hat{x}^p) + \sum_{i=1}^{l_p} (v_i^p - \hat{v}_i^p) h_i^p(\hat{x}^p) \leq 0,$$

which it is true for every  $u^p \in R_{>}^{m_p}$ ,  $v^p \in R^{l_p}$ ,  $p \in N$ . From these last  $n$  inequalities it follows that  $\hat{x} \in X$ . Furthermore, also it follows that

$$\hat{u}_i^p g_i^p(\hat{x}^p) = 0, \forall p \in N.$$

By this equalities and the definition right inequality it follows that

$$\begin{aligned} f_p(\hat{x}^p, \hat{x}^{-p}) &= L_p(\hat{x}, \hat{u}^p, \hat{v}^p) \leq \\ &\leq L_p(x^p, \hat{x}^{-p}, \hat{u}^p, \hat{v}^p) = f_p(x^p, \hat{x}^{-p}) + \sum_{i=1}^{m_p} \hat{u}_i^p g_i^p(x^p) + \sum_{i=1}^{l_p} \hat{v}_i^p h_i^p(x^p) \leq \\ &\leq f_p(x^p, \hat{x}^{-p}) \end{aligned}$$

for every  $x^p \in X_p$ ,  $p \in N$ . This proves that  $\hat{x}$  is the Nash equilibrium for  $\Gamma$ .  $\square$

### 3 Necessary and Sufficient Conditions for Convex Strategic Games

Consider the strategic form convex game  $\Gamma$  with strategy sets

$$X_p = \{x^p \in R^{n_p} : g_i^p(x^p) \leq 0, i = \overline{1, m_p}\}$$

where  $g_i^p(x^p)$ ,  $i = \overline{1, m_p}$  are convex on  $R^{n_p}$ ,  $p = \overline{1, n}$  and the cost function  $f_p(\mathbf{x}^p, \mathbf{x}^{-p})$  is convex on  $X_p$ , for every fixed  $x^{-p} \in X_{-p}$ ,  $p = \overline{1, n}$ .

The Lagrange regular vector function  $L(x, u)$  for this convex games has components

$$L_p(x, u^p) = f_p(x) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p), \quad p = \overline{1, n}.$$

**Theorem 2.** *Let every strategy set  $X_p$ ,  $p = \overline{1, n}$  satisfies the Slater regularity condition. The outcome  $\hat{x} \in X$  is a Nash equilibrium for  $\Gamma$  if and only if there exist the Lagrange multipliers  $\hat{u}^p \geq 0$ ,  $p = \overline{1, n}$  such that  $(\hat{x}, \hat{u}^1, \dots, \hat{u}^n)$  is a saddle point for the Lagrange vector function  $L(x, u)$  of the  $\Gamma$ .*

*Proof.* The sufficiency's truth follows as a corollary of the theorem 1.

*Necessity.* Suppose that  $\hat{x} \in X$  is a Nash equilibrium. In view of the definition, this implies that for fixed  $\hat{x}^{-p} \in X_{-p}$  the  $\hat{x}^p$  is a global minimum point for  $f_p(x^p, \hat{x}^{-p})$ . Denote

$$\begin{aligned} Z &= \left\{ \begin{pmatrix} z_0 \\ z \end{pmatrix} \in R^{m_p+1} : \begin{pmatrix} z_0 \\ z \end{pmatrix} < \begin{pmatrix} f_p(\hat{x}^p, \hat{x}^{-p}) \\ 0 \end{pmatrix} \right\}, \\ Y &= \left\{ \begin{pmatrix} y_0 \\ y \end{pmatrix} \in R^{m_p+1} : \exists x^p, \begin{pmatrix} f_p(x^p, \hat{x}^{-p}) \\ g^p(x^p) \end{pmatrix} \leq \begin{pmatrix} y_0 \\ y \end{pmatrix} \right\}. \end{aligned}$$

The sets  $Z$  and  $Y$  are convex. Because  $\hat{x}^p$  is the global minimum point for  $f_p(x^p, \hat{x}^{-p})$ , the sets  $Z$  and  $Y$  don't have common points. In conformity with the hyperplane separation theorem there exist  $(c_0, c) \in R^{m_p+1}$ ,  $(c_0, c) \neq 0$ , such that

$$(c_0, c^T) \begin{pmatrix} z_0 \\ z \end{pmatrix} \leq (c_0, c^T) \begin{pmatrix} y_0 \\ y \end{pmatrix}$$

for every  $(z_0, z) \in Z, (y_0, y) \in Y$ .

From this inequality and the definition of the set  $Z$  it follows that  $(c_0, c) \geq 0$ .

Because  $(f_p(\hat{x}^p, \hat{x}^{-p}), 0)$  belongs to frontier of the set  $Z$ , it follows that:

$$c_0 f_p(\hat{x}^p, \hat{x}^{-p}) \leq (c_0, c^T) \begin{pmatrix} y_0 \\ y \end{pmatrix} \text{ for every } (y_0, y) \in Y.$$

As a consequence

$$c_0 f_p(\hat{x}^p, \hat{x}^{-p}) \leq c_0 f_p(x^p, \hat{x}^{-p}) + c^T g^p(x^p) \text{ for every } x^p. \quad (1)$$

The Slater regularity condition and  $c \geq 0$  implies that  $c_0 > 0$ .

By dividing inequality (1) with  $c_0$  it follows

$$\hat{u}^p \geq 0, \quad (2)$$

$$f_p(\hat{x}^p, \hat{x}^{-p}) \leq f_p(x^p, \hat{x}^{-p}) + (\hat{u}^p)^T g^p(x) \text{ for every } x^p, \quad (3)$$

where  $\hat{u}^p = \frac{1}{c_0} c$ .

By substitution  $x^p = \hat{x}^p$  in (3) we obtain  $(\hat{u}^p)^T g^p(x^p) \geq 0$ . Because  $\hat{u}^p \geq 0, g^p(x^p) \leq 0$ , we have  $(\hat{u}^p)^T g^p(\hat{x}^p) \leq 0$ . The last two relations imply

$$(\hat{u}^p)^T g^p(\hat{x}^p) = 0.$$

As well  $g^p(\hat{x}^p) \leq 0$ , it follows

$$f_p(\hat{x}^p, \hat{x}^{-p}) \geq f_p(\hat{x}^p, \hat{x}^{-p}) + u^p g^p(\hat{x}^p), \quad (4)$$

for every  $u^p \geq 0$ .

From (2)–(4) it follows that  $(\hat{x}, \hat{u}^1, \dots, \hat{u}^n)$  is a saddle point for the Lagrange vector function of the  $\Gamma$ .  $\square$

**Theorem 3.** *Let the all functions  $f_p(x), g_i^p(x^p), i = \overline{1, m_p}, p = \overline{1, n}$  are differentiable on  $\hat{x}$  and every strategy set  $X_p$  satisfies the Slater regularity conditions. The outcome  $\hat{x} \in X$  is a Nash equilibrium for  $\Gamma$  if and only if there exist the Lagrange multipliers  $\hat{u}^p \geq 0, p = \overline{1, n}$  such that the following conditions are verified*

$$\begin{aligned} \frac{\partial L_p(\hat{x}, \hat{u}^p)}{\partial x_j^p} &= 0, \quad j = \overline{1, n_p}, p = \overline{1, n}, \\ \hat{u}_i^p g_i^p(\hat{x}^p) &= 0, \quad i = \overline{1, m_p}, p = \overline{1, n}. \end{aligned}$$

*Proof. Necessity.* Let  $\hat{x} \in X$  is a Nash equilibrium. In view of the definition, this implies that for fixed  $\hat{x}^{-p} \in X_{-p}$  the  $\hat{x}^p$  is a minimum point for  $f_p(x^p, \hat{x}^{-p})$ .

Associate with the  $\Gamma$  the equivalent game  $\Gamma'$  with the strategy sets

$$X_p' = \{(x^p, s^p) \in R^{n_p + m_p} : g_i^p(x^p) + s_i^2 = 0, i = \overline{1, m_p}\}$$

and the same cost functions.

The Lagrange vector function of the  $\Gamma'$  has components

$$\Lambda_p(x, u^p, s^p) = L_p(x, u^p) + s_i^2, \quad p = \overline{1, n}.$$

The Lagrange principle may be applied to  $\Gamma'$  and we obtain the system

$$\frac{\partial \Lambda_p(\hat{x}, u^p, s^p)}{\partial x_j^p} = \frac{\partial L_p(\hat{x}, u^p)}{\partial x_j^p} = 0, \quad j = \overline{1, n_p}, \quad p = \overline{1, n},$$

$$\frac{\partial \Lambda_p(\hat{x}, u^p, s^p)}{\partial s_i^p} = u_i^p s_i^p = 0 \iff u_i^p g_i^p(\hat{x}^p) = 0, \quad i = \overline{1, m_p}, \quad p = \overline{1, n},$$

$$\frac{\partial \Lambda_p(\hat{x}, u^p, s^p)}{\partial u_i^p} = 0 \iff g_i^p(\hat{x}^p) \leq 0, \quad i = \overline{1, m_p}, \quad p = \overline{1, n},$$

which is obligatory verified for some  $\hat{u}^p \geq 0, p = \overline{1, n}$ .

*Sufficiency.* Suppose that  $\hat{x}$  and  $\hat{u}^p \geq 0, p = \overline{1, n}$  verify the theorem's conditions. Because  $L_p(x^p, x^{-p}, u^p)$  are convex by  $x^p$  for fixed  $x^{-p}$  and  $u^p$ , then

$$L_p(x^p, \hat{x}^{-p}, \hat{u}^p) \geq L_p(\hat{x}, \hat{u}^p) + (x^p - \hat{x}^p)^T \frac{\partial L_p(\hat{x}, \hat{u}^p)}{\partial x^p}.$$

It follows that

$$L_p(\hat{x}, \hat{u}^p) \leq L_p(x^p, \hat{x}^{-p}, \hat{u}^p) \quad (5)$$

for every  $x^p$ .

Because  $L_p(x, u^p)$  is linear by  $u^p$  we have

$$L_p(\hat{x}, u^p) = L_p(\hat{x}, \hat{u}^p) + (u^p - \hat{u}^p)^T \frac{\partial L_p(\hat{x}, \hat{u}^p)}{\partial u^p}.$$

From this and the strategy set definition it follows that

$$L_p(\hat{x}, u_p) \leq L_p(\hat{x}, \hat{u}^p), \quad u^p \geq 0, p = \overline{1, n}. \quad (6)$$

Inequalities (5)–(6) prove that  $(\hat{x}, \hat{u}^1, \dots, \hat{u}^n)$  is the saddle point for the Lagrange vector function of the  $\Gamma$ . Consequently, on the base of the theorem 2 it follows that  $\hat{x}$  is the Nash equilibrium for  $\Gamma$ . □

Theorem 2 and 3 may be formulated with some modifications for the cases when strategy sets include equations and sign restrictions. Thus, *e.g.*, if

$$X_p = \{x^p \in R^{n_p} : g_i^p(x^p) \leq 0, i = \overline{1, m_p}, x_j^p \geq 0, j = \overline{1, n_p}\}$$

where  $g_i^p(x^p), i = \overline{1, m_p}$  are convex on  $R^{n_p}$  then may be proved

**Theorem 4.** *Let the all functions  $f_p(x), g_i^p(x^p), i = \overline{1, m_p}, p = \overline{1, n}$  are differentiable on  $\hat{x}$  and every strategy set  $X_p$  satisfies the Slater regularity conditions. The outcome  $\hat{x} \in X$  is a Nash equilibrium for  $\Gamma$  if and only if there exist the Lagrange multipliers  $\hat{u}^p \geq 0, p = \overline{1, n}$  such that the following conditions are verified*

$$\begin{aligned} \frac{\partial L_p(\hat{x}, \hat{u}^p)}{\partial x_j^p} &\geq 0, & j = \overline{1, n_p}, p = \overline{1, n}, \\ x_j^p \frac{\partial L_p(\hat{x}, \hat{u}^p)}{\partial x_j^p} &= 0, & j = \overline{1, n_p}, p = \overline{1, n}, \\ \hat{u}_i^p g_i^p(\hat{x}^p) &= 0, & i = \overline{1, m_p}, p \in N. \end{aligned}$$

Evidently, if the convexity requirement misses in  $\Gamma$  statement, then theorems 3 and 4 must be formulated only as necessary conditions for local Nash equilibrium.

Generally, from the practical and algorithmic points of view, equilibrium conditions from theorem 4 require to solve  $2^{n_1+\dots+n_p+m_1+\dots+m_p}$  systems of nonlinear equalities and inequalities. This is a difficult problem to solve already for modest games dimensions.

For various game statements other equilibrium conditions may be formulated and proved.

## 4 Equilibrium Principles and Conditions for Multi-Criteria Strategic Games

Suppose that in game  $\Gamma$  every player  $p$  has  $k_p$  cost functions  $f_i^p(x), i = \overline{1, k_p}, p = \overline{1, n}$  to minimize and the strategy sets are

$$X_p = \{x^p \in R^{n_p} : g_i^p(x^p) \leq 0, i = \overline{1, m_p}\}$$

where  $g_i^p(x^p), i = \overline{1, m_p}$  are defined on  $R^{n_p}, p = \overline{1, n}$ .

Based on optimality notions for multi-criteria optimization problems (see, *e.g.*, [6, 7]), diverse equilibria notions may be defined [1, 8, 2, 3, 19].

**Definition.** The outcome  $\hat{x} \in X$  of the multi-criteria game is a Pareto-Nash (multi-criteria Nash) equilibrium if doesn't exist  $p \in N$  and  $x^p \in X_p$  such that

$$f^p(x^p, \hat{x}^{-p}) \leq f^p(\hat{x}^p, \hat{x}^{-p}).$$

**Definition.** The outcome  $\hat{x} \in X$  of the multi-criteria game is a weak Pareto-Nash (Slater-Nash, weak multi-criteria Nash) equilibrium if doesn't exist  $p \in N$  and  $x^p \in X_p$  such that

$$f^p(x^p, \hat{x}^{-p}) < f^p(\hat{x}^p, \hat{x}^{-p}).$$

The set of the all Pareto-Nash Equilibria (PNES) may be defined also as *intersection of the graphs of Pareto optimal response multivalued mappings*

$$\text{Arg } P \min_{\mathbf{x}^p \in X_p} f^p(\mathbf{x}^p, \mathbf{x}^{-p}) : X_{-p} \dashv X_p, p = \overline{1, n} :$$

$$PNES(\Gamma) = \bigcap_{p \in N} Gr_p,$$

$$Gr_p = \left\{ (\mathbf{x}^p, \mathbf{x}^{-p}) \in X : \begin{array}{l} \mathbf{x}^{-p} \in X_{-p}, \\ \mathbf{x}^p \in \text{Arg } P \min_{\mathbf{x}^p \in X_p} f^p(\mathbf{x}^p, \mathbf{x}^{-p}) \end{array} \right\}, p \in N.$$

Equivalent proposition for the weak Pareto-Nash equilibria set may be formulated.

Denote by

$$F_p(x, \rho^p) = \sum_{i=1}^{k_p} \rho_i^p f_i^p(x),$$

the  $p$  player synthesis function, where  $\rho_i^p \geq 0, i = \overline{1, k_p}, \sum_{i=1}^{k_p} \rho_i^p = 1, p \in N$ .

**Theorem 5.** *Let the all functions  $f_i^p(x^p, x^{-p}), i = \overline{1, k_p}, g_i^p(x^p), i = \overline{1, m_p}, p = \overline{1, n}$  are convex on  $R^{n_p}$  for fixed  $x^{-p} \in X_{-p}$ . If the outcome  $\hat{x} \in X$  is a Pareto-Nash equilibrium for  $\Gamma$ , then there exist multipliers  $\hat{\rho}^p \geq 0, \sum_{i=1}^{k_p} \rho_i^p = 1, p \in N$  such that the  $\hat{x}^p$  is the solution of the optimization problem*

$$F_p(x^p, \hat{x}^{-p}, \hat{\rho}^p) \rightarrow \min, \quad x^p \in X_p. \quad (7)$$

The proof is based on theorem 1 (p. 181) from [7].

Let us examine the game  $\Gamma(\rho)$  which is distinguished from  $\Gamma$  by players cost functions  $F_p(x, \rho^p), p \in N$ .

**Theorem 6.** *Let the all functions  $f_i^p(x^p, x^{-p}), i = \overline{1, k_p}, g_i^p(x^p), i = \overline{1, m_p}, p = \overline{1, n}$  are convex on  $R^{n_p}$  for fixed  $x^{-p} \in X_{-p}$ . If the outcome  $\hat{x} \in X$  is a Pareto-Nash equilibrium for  $\Gamma$ , then there exist multipliers  $\hat{\rho}^p \geq 0, \sum_{i=1}^{k_p} \rho_i^p = 1, p \in N$  such that the  $\hat{x}$  is the Nash equilibrium for  $\Gamma(\rho)$ .*

Theorem 6 is a corollary of theorem 5.

**Theorem 7.** *Let the all functions  $f_i^p(x^p, x^{-p}), i = \overline{1, k_p}, g_i^p(x^p), i = \overline{1, m_p}, p = \overline{1, n}$  are convex on  $R^{n_p}$  for fixed  $x^{-p} \in X_{-p}$ , are differentiable on  $\hat{x}$  and every strategy set  $X_p$  satisfies the Slater regularity conditions.. If the outcome  $\hat{x} \in X$  is a Pareto-Nash equilibrium for  $\Gamma$ , then there exist  $\hat{\rho}^p \geq 0, \sum_{i=1}^{k_p} \rho_i^p = 1, p \in N$  and  $\hat{u} \geq 0$  such that the following conditions are verified*

$$\begin{aligned} \frac{\partial L_p(\hat{x}, \hat{\rho}^p, \hat{u}^p)}{\partial x_j^p} &= 0, \quad j = \overline{1, n_p}, p = \overline{1, n}, \\ \hat{u}_i^p g_i^p(\hat{x}^p) &= 0, \quad i = \overline{1, m_p}, p \in N, \end{aligned}$$

where  $L_p(\hat{x}, \hat{\rho}^p, \hat{u}^p) = F_p(x, \rho^p) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p), p = \overline{1, n}$ .

Theorem 7 follows as corollary from theorems 5–6 and 3.

Remark that theorems 5–7 formulate only necessary equilibrium conditions.

An affirmation similar with the theorem 4 remark is valid for equilibrium conditions of the theorem 7.

**Theorem 8.** *If  $\hat{x}$  is the solution of the problems (7) for some  $\rho^p > 0, p \in N$ , then  $\hat{x}$  is the Pareto-Nash equilibrium in  $\Gamma$ .*

*If  $\hat{x}$  is the unique solution of the problems (7) for some  $\rho^p \geq 0, p \in N$ , then  $\hat{x}$  is the Pareto-Nash equilibrium in  $\Gamma$ .*

The proof is based on theorem 1' (p. 183) from [7].

Other different Pareto-Nash, weak Pareto-Nash etc. equilibrium conditions may be formulated and proved for multi-criteria strategic game  $\Gamma$ .

## 5 Conclusions

Traditionally, von Neumann theorem for matrix games is considered a particular case of the Nash theorem. In this work, by theorems 1–4 we illustrate that the well known von Neumann theorem for matrix games may be extended on strategic form games. Theorems 3–4 are Kuhn-Tucker type theorems for strategic form games.

Theorems 5–8 are extensions on multi-criteria strategic form games of the well known theorems for multi-criteria optimization problems.

## References

- [1] Blackwell D., *An analog of the minimax theorem for vector payoffs.* – Pacific Journal of Mathematics, Vol. 6, 1956, pp. 1-8.
- [2] Borm P., Tijs S. and van der Aarssen J., *Pareto equilibria in multiobjective games.* – Methods of Operations Research, Vol. 60, 1988, pp. 302-312.
- [3] Ghose D. and Prasad U.R., *Solution Concepts in Two-person Multicriteria Games.* – Journal of Optimization Theory and Applications, Vol. 63, 1989, pp. 167-189.
- [4] McKelvey R.D. and McLennan A., *Computation of equilibria in finite games.* – Handbook of Computational Economics, Vol. 1, Elsevier, 1996, pp. 87-142.
- [5] Nash J.F., *Noncooperative game.* – Annals of Mathematics, Vol. 54, 1951, pp. 280-295.
- [6] Podinovskii V.V. and Nogin V.D., *Pareto-optimal solutions of the multi-criteria problems.* – Moscow, Nauka, 1982 (in Russian).
- [7] Sagaidac M. and Ungureanu V., *Operational research.* – Chişinău, CEP USM, 2004, 296 p. (in Romanian).
- [8] Shapley L.S., *Equilibrium Points in Games with Vector Payoffs.* – Naval Research Logistics Quarterly, Vol. 6, 1959, pp. 57-61.

- [9] Ungureanu V., *Mathematical programming*. – Chişinău, USM, 2001, 344 p. (in Romanian).
- [10] Ungureanu V. and Botnari A., *Nash equilibria sets in mixed extension of  $2 \times 2 \times 2$  games*. – Computer Science Journal of Moldova, Vol. 13, no. 1 (37), 2005, pp. 13-28.
- [11] Ungureanu V. and Botnari A., *Nash equilibria sets in mixed extended  $2 \times 3$  games*. – Computer Science Journal of Moldova, Vol. 13, no. 2 (38), 2005, pp. 136-150.
- [12] Ungureanu V., *A method for Nash equilibria set computing in multi-matrix extended games*. – Spanish-Italian-Netherlands Game Theory Meeting (SING2) and XVI Italian Meeting on Game Theory (XVI IMGTA), Foggia, Italy, 14-17 June 2006, Conference Book, 2006, p. 156.
- [13] Ungureanu V., *Nash equilibria set computing in finite extended games*. – Computer Science Journal of Moldova, Vol. 14, no. 3 (42), 2006, pp. 345–365.
- [14] Ungureanu V., *Nash Equilibrium Conditions for Normal Form Strategic Games*. – The XIV-th Conference on Applied and Industrial Mathematics dedicated to the 60-th anniversary of the foundation of the Faculty of Mathematics and Computer Science of Moldova State University (CAIM XIV), Satellite Conference of ICM 2006, Chisinau, august 17-19, 2006, p. 328-330.
- [15] Ungureanu V., *Nash Equilibrium Conditions — Extensions of Some Classical Theorems*. – CODE-2007: Conference de la SMAI sur l’Optimisation et la Decision, Colloque satellite de la conference SMAI-2007, Institut Henri Poincare, Paris 18-20 avril 2007, <http://www.ann.jussieu.fr/~plc/code2007/ungureanu.pdf>.
- [16] Ungureanu V., *Equilibrium Principles for Pareto-Nash-Stackelberg Games*. – 6th Congress of Romanian Mathematicians, June 28 - July 4, 2007, University of Bucharest, Romania, Abstracts, p. 164-165.
- [17] Ungureanu V., *Equilibrium Principles for Pareto-Nash-Stackelberg Games*. – Spain Italy Netherlands Meeting on Game Theory (SING3) and the 7th Spanish Meeting on Game Theory, 4-5-6 July, 2007, Facultad de Ciencias Matemáticas, Universidad Complutense, Madrid, Spain, Abstract Book, p. 81.
- [18] Ungureanu V., *Nash Equilibrium Conditions for Strategic Form Games*. – 6th International Congress on Industrial and Applied Mathematics, ETH and University of Zürich, Switzerland, 16-20 July 2007, Abstract Book, p. 436.
- [19] Wang S.Y., *Existence of Pareto Equilibrium*. – Journal of Optimization Theory and Applications, Vol. 79, 1993, pp. 373-384.